TO APPEAR IN THE V543N2 NOV 10, 2000 ISSUE OF THE ASTROPHYSICAL JOURNAL
Preprint typeset using LaTeX style emulateapj

THE PROFILE OF A NARROW LINE AFTER SINGLE SCATTERING BY MAXWELLIAN ELECTRONS: RELATIVISTIC CORRECTIONS TO THE KERNEL OF THE INTEGRAL KINETIC EQUATION

SERGEI Y. SAZONOV AND RASHID A. SUNYAEV
Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 86740 Garching bei München, Germany
Space Research Institute (IKI), Profsoyuznaya 84/32, Moscow 117810, Russia

To appear in the v543n2 Nov 10, 2000 issue of the Astrophysical Journal

ABSTRACT

The photon frequency distribution that results from single Compton scattering of monochromatic radiation on thermal electrons is derived in the mildly relativistic limit. Algebraic expressions are given for (1) the photon redistribution function, $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$, and (2) the spectrum produced in the case of isotropic incident radiation, $P(\nu\to\nu')$. The former is a good approximation for electron temperatures $kT_{\rm e}\lesssim 25$ keV and photon energies $h\nu\lesssim$ 50 keV, and the latter is applicable when $h\nu(h\nu/m_{\rm e}c^2)\lesssim kT_{\rm e}\lesssim 25$ keV, $h\nu\lesssim 50$ keV. Both formulae can be used for describing the profiles of X-ray and low-frequency lines upon scattering in hot, optically thin plasmas, such as present in clusters of galaxies, in the coronae of accretion disks in X-ray binaries and active galactic neclei (AGNs), during supernova explosions, etc. Both formulae can also be employed as the kernels of the corresponding integral kinetic equations (direction-dependent and isotropic) in the general problem of Comptonization on thermal electrons. The $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel, in particular, is applicable to the problem of induced Compton interaction of anisotropic low-frequency radiation of high brightness temperature with free electrons in the vicinity of powerful radiosources and masers. Fokker-Planck-type expansion (up to fourth order) of the integral kinetic equation with the $P(\nu\to\nu')$ kernel derived here leads to a generalization of the Kompaneets equation. We also present (1) a simpler kernel that is necessary and sufficient to derive the Kompaneets equation and (2) an expression for the angular function for Compton scattering in a hot plasma, which includes temperature and photon energy corrections to the Rayleigh angular function.

Subject headings: accretion, accretion disks — cosmic microwave background — intergalactic medium — line: profiles — masers — radiative transfer

1. INTRODUCTION

A monochromatic spectral line will be broadened after single Compton scattering on thermal electrons in an optically thin, hot plasma. The emergent spectrum will depend on the angle between the direction, $\mathbf{\Omega}$, from which the photons are supplied and the observer's direction, $\mathbf{\Omega'}$. A classical problem then arises of finding the redistribution function, $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$, which gives the probability for the incident photon $(\nu,\mathbf{\Omega})$ to be scattered in the direction $\mathbf{\Omega'}$ with a frequency of ν' (Dirac 1925).

In the case in which the incident radiation is isotropic and monochromatic, the spectrum resulting from single Compton scattering can be found by integrating $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ over the scattering angle ($\mu_{\rm s}=\mathbf{\Omega}\mathbf{\Omega'}$):

$$P(\nu\to\nu')=\int K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})\,d\mathbf{\Omega'}=2\pi\int K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})\,d\mu_{\rm s}. \quad (1)$$

Spectra described by equation (1) may in particular form when spherical symmetry is present in the system. Consider, for example, an isotropic source of monochromatic radiation that is either located at the center of a spherical cluster of galaxies with hot intergalactic gas or distributed spherically symmetrically across the cluster. The frequency distribution of photons that have experienced a single scattering within the cluster will then be precisely $P(\nu\to\nu')$ if the entire cluster is probed at the same time. If instead the spectrum is collected from a part of the cluster, then one must take into account the angular distribution of incident photons and make use of the direction-dependent $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ function.

Single-scattering line profiles, which can provide important information on the conditions (particularly the temperature) of the plasma in the source, can originate in various astrophysical environments. For the X-ray spectral band, these include intracluster gas, the coronae of accretion disks around black holes in binary stellar systems and active galactic nuclei (AGNs), and plasma streams outflowing from the neutron star during super-Eddington X-ray bursts in bursters.

High-quality spectroscopic X-ray observations necessary for the detection and measurement of such lines will soon become possible with the satellites *Chandra*, *XMM* (both already in orbit), and *Spectrum-X-Gamma*, and in a more distant outlook, with *Constellation-X* and *XEUS*. This was one of the primary motives for us in initiating the current study, in which we calculate analytically the functions $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ (eq. [7]) and $P(\nu\to\nu')$ (eqs. [19] and [31]) in the mildly relativistic limit.

1.1. *Integral Kinetic Equation*

Apart from the single-scattering problem outlined above, $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ and $P(\nu\to\nu')$ can be used as the kernels of the corresponding integrodifferential kinetic equations describing the Comptonization of photons on Maxwellian electrons. In the anisotropic problem, the kinetic equation in the case of an infinite homogeneous medium can be written as

$$\frac{\partial I(\nu,\mathbf{\Omega},\tau)}{\partial\tau}=-\int I(\nu,\mathbf{\Omega},\tau)K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})[1+n(\nu',\mathbf{\Omega'},\tau)]d\nu'd\mathbf{\Omega'}$$
$$+\int\frac{\nu}{\nu'}I(\nu',\mathbf{\Omega'},\tau)K(\nu',\mathbf{\Omega'}\to\nu,\mathbf{\Omega})[1+n(\nu,\mathbf{\Omega},\tau)]d\nu'd\mathbf{\Omega'}, \tag{2}$$

where $I(\nu,\mathbf{\Omega})$ is the specific intensity of the radiation, $n=c^2I/(2h\nu^3)$ is the occupation number in photon phase space, and $\tau=\sigma_{\rm T}N_{\rm e}ct$ is the dimensionless time ($\sigma_{\rm T}$ is the Thomson scattering cross section and $N_{\rm e}$ is the number density of electrons). It is easy to write down kinetic equations similar to equation (2) for any of the standard problems of radiation transfer.

In the isotropic case, the kinetic equation becomes

$$\frac{\partial I(\nu,\tau)}{\partial\tau}=-\int I(\nu,\tau)P(\nu\to\nu')[1+n(\nu',\tau)]d\nu'+\int\frac{\nu}{\nu'}I(\nu',\tau)P(\nu'\to\nu)[1+n(\nu,\tau)]d\nu'. \tag{3}$$

The integrodifferential equation (3) can in general be solved numerically if its kernel, $P(\nu\to\nu')$, is known. Alternatively, one can treat Comptonization problems using Monte Carlo methods (see, e.g., the review by Pozdnyakov et al. 1983). In the nonrelativistic limit, i.e. when the typical photon energy, $h\nu$, and the plasma temperature, $kT_{\rm e}$, are both negligibly small compared to the electron rest energy, $m_{\rm e}c^2$, the variation in intensity at a given frequency is largely governed by transitions in a narrow interval of a continuum spectrum near this frequency. If the initial distribution of the photons in frequency is smooth enough (in a problem with a small number of scatterings) or the formation of a spectrum by multiple scatterings is studied, it is possible to perform a Fokker-Planck-type expansion of the integral equation (3), thereby reducing it to a much simpler differential equation that describes the diffusion and flow of the photons in frequency space:

$$\frac{\partial n(\nu)}{\partial\tau}=\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\left\{-\nu^2n\langle\Delta\nu\rangle(1+n)+\frac{1}{2}\left[-\nu^2n\langle(\Delta\nu)^2\rangle\frac{\partial n}{\partial\nu}+(1+n)\frac{\partial}{\partial\nu}\nu^2n\langle(\Delta\nu)^2\rangle\right]\right.$$
$$+\frac{1}{6}\left[-\nu^2n\langle(\Delta\nu)^3\rangle\frac{\partial^2n}{\partial\nu^2}+\frac{\partial n}{\partial\nu}\frac{\partial}{\partial\nu}\nu^2n\langle(\Delta\nu)^3\rangle-(1+n)\frac{\partial^2}{\partial\nu^2}\nu^2n\langle(\Delta\nu)^3\rangle\right]$$
$$+\frac{1}{24}\left[-\nu^2n\langle(\Delta\nu)^4\rangle\frac{\partial^3n}{\partial\nu^3}+\frac{\partial^2n}{\partial\nu^2}\frac{\partial}{\partial\nu}\nu^2n\langle(\Delta\nu)^4\rangle-\frac{\partial n}{\partial\nu}\frac{\partial^2}{\partial\nu^2}\nu^2n\langle(\Delta\nu)^4\rangle\right.$$
$$\left.\left.+(1+n)\frac{\partial^3}{\partial\nu^3}\nu^2n\langle(\Delta\nu)^4\rangle\right]+\ldots\right\}. \tag{4}$$

The moments of the kernel that enter this equation are found from the formula

$$\langle(\Delta\nu)^n\rangle=\int P(\nu\to\nu')(\nu'-\nu)^nd\nu'. \tag{5}$$

Substituting the first two moments, $\langle\Delta\nu\rangle$ and $\langle(\Delta\nu)^2\rangle$, calculated to an accuracy of $kT_{\rm e}/m_{\rm e}c^2$ and $h\nu/m_{\rm e}c^2$ (using the kernel given by eq. [23] below) into the corresponding terms of equation (4) leads to the famous Kompaneets (1957) equation:

$$\frac{\partial n(\nu)}{\partial\tau}=\frac{h}{m_{\rm e}c^2}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4\left(n+n^2+\frac{kT_{\rm e}}{h}\frac{\partial n}{\partial\nu}\right). \tag{6}$$

The Kompaneets equation is valid in the nonrelativistic limit ($h\nu$, $kT_{\rm e}\ll m_{\rm e}c^2$). The last parenthesized term in equation (6) describes the frequency diffusion of photons due to the Doppler effect and the transfer of energy from the electrons to the radiation; the first term describes the downward photon flow along the frequency axis due to Compton recoil, and the second term, which also is owing to recoil, accounts for induced Compton scattering.

In studying the interaction between energetic photons and hot electrons ($h\nu$, $kT_{\rm e}\gtrsim 0.01m_{\rm e}c^2$), relativistic corrections to the kernel and the Kompaneets equation become important. In order to find the main corrections (of the order of $h\nu/m_{\rm e}c^2$ and $kT_{\rm e}/m_{\rm e}c^2$), it proves necessary to take into account in equation (4) all terms that depend on the first four moments of the kernel. The resultant generalization of the Kompaneets equation was found by Itoh et al. (1998) and Challinor & Lasenby (1998) when these authors examined the distortion of the spectrum of the cosmic microwave background (CMB) during interaction with the hot intergalactic gas in clusters of galaxies. Rephaeli (1995) pointed out the important role of corrections of the order of $kT_{\rm e}/m_{\rm e}c^2$ in this phenomenon. The kernel $P(\nu\to\nu')$ was not given in explicit form in the derivation of Itoh et al. (1998) and Challinor & Lasenby (1998). Instead, the Fokker-Planck operator was applied to the kinetic equation written in the most general form, where the amplitude appears for a transition from the initial state with given 4-momenta of the photon and electron into the final state with the corresponding 4-momenta. Earlier, Ross et al. (1978) and Illarionov et al. (1979) added to the Kompaneets equation a dispersion term associated with Compton recoil and a term that takes into account, in a first approximation, the transition from the Thomson cross section to the Klein-Nishina one. Their equation, which describes much better than the Kompaneets equation the scattering of energetic photons ($h\nu\sim 0.1m_{\rm e}c^2$) by sufficiently cold electrons ($kT_{\rm e}\ll h\nu$), is a particular case of the more general formula of Itoh et al. (1998) and Challinor & Lasenby (1998).

Despite the attractiveness of employing the Fokker-Planck approximation in treating Comptonization problems, its scope is limited. For example, in considering the effect of Comptonization from a small number of scatterings on the profiles of narrow spectral lines, the initial radiation spectrum cannot be represented as a finite Taylor series in terms of the frequency variation, and, therefore, Fokker-Planck-type equations are not applicable. For this kind of problem, it is necessary to make use of the integral kinetic equations (2) or (3), which requires knowledge of their kernels, $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ and $P(\nu\to\nu')$.

1.2. *The Kernel*

Dirac (1925) has given an approximate expression for the direction-dependent kernel, $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$. The Doppler shift was taken into account to within the first order in v/c, but Compton recoil was totally neglected. Integration of Dirac's $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ function over the scattering angle results in the zero-order approximation for the isotropic kernel $P(\nu\to\nu')$ (Hummer & Mihalas 1967; see also Weymann 1970). Being symmetric in frequency variation $[P(\nu\to\nu+\Delta\nu)=P(\nu\to\nu-\Delta\nu)]$, this kernel does not take into account the average Doppler increment in the photon energy: $\langle\Delta\nu/\nu\rangle=4kT_{\rm e}/m_{\rm e}c^2$. With the zero-order approximation for the kernel, it is not possible to describe many important astrophysical phenomena, such as:

(a) distortion of the spectrum of the CMB in the direction of galaxy clusters (Sunyaev & Zeldovich 1972; see the recent review by Birkinshaw 1999),

(b) The y (Zeldovich & Sunyaev 1969) and Bose-Einstein μ (Sunyaev & Zeldovich 1970) distortions of the CMB spectrum resulting from energy release in the early universe (see the reviews by Danese & de Zotti 1977; Sunyaev & Zeldovich 1980; the book by Peebles 1993; and the strict constraints on these distortions placed by the Far-Infrared Absolute Spectrophotometer on board COBE, Fixsen et al. 1996),

(c) the formation of hard power-law tails in the emission spectra of the famous X-ray source Cygnus X–1 (Sunyaev & Truemper 1979) and other stellar-mass black hole candidates, AGNs, and quasars; and in the spectra of accreting neutron stars (e.g., Shapiro et al. 1976; Sunyaev & Titarchuk 1980; Pozdnyakov et al. 1983; Tanaka & Shibazaki 1996; Poutanen & Svensson 1996; Narayan et al. 1998; Zdziarski et al. 1998).

Babuel-Peyrissac & Rouvillois (1970, hereafter BR70) have obtained a more accurate expression for the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel, by taking into account not only the average Doppler and recoil frequency shifts (the latter is $\langle\Delta\nu/\nu\rangle=-h\nu/m_{\rm e}c^2$), but also, in the first order, Klein-Nishina corrections to the scattering cross section. BR70 also attempted to find relativistic corrections due to high electron velocities ($kT_{\rm e}\sim 0.1m_{\rm e}c^2$), but did not include all relevant terms (as we show in the present paper), and therefore their expression is valid for nonrelativistic electrons only. The allowance for Compton recoil is also made in the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel written down by Zeldovich et al. (1972). Integration of the BR70 kernel over the scattering angle ignoring recoil ($h\nu\ll kT_{\rm e}\ll m_{\rm e}c^2$) leads to the first-order approximation for the $P(\nu\to\nu')$ kernel (Sunyaev 1980). The formula of Sunyaev (1980), as opposed to the approximation of Hummer & Mihalas (1967), accounts for the asymmetry of the scattered line profile and enables us to describe (in the nonrelativistic limit) the above-mentioned phenomena related to the transfer of energy from hot electrons to radiation. Using this expression, one can derive the diffusion part of the Kompaneets operator (the last term of eq. [6]). The remaining two (flow) terms can be found by applying the thermodynamic principle that the desired equation must obey: a Planckian distribution of photons must remain unchanged during the interaction with electrons of the same temperature. This is the method followed by the authors of the original derivation of the Kompaneets equation for finding not only the term describing the downward motion of the photons along the frequency axis, but also the term describing the effect of induced scattering (report No. 336 of the Chemical Physics Institute of USSR Academy of Sciences 1950; Kompaneets 1957; Ya.B. Zeldovich 1968, private communication).

Kershaw et al. (1986) have derived a semianalytical relativistic formula for the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel that is valid for arbitrary values of photon energy and electron temperature. With this formula, the calculation of the kernel is reduced to computing a single integral over the electron Lorentz factor. Kershaw et al. (1986) further succeded in expanding their basic expression in powers of $kT_{\rm e}/m_{\rm e}c^2$, and thus derived an algebraic formula for $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ that is applicable in the low-temperature limit $kT_{\rm e}\lesssim 0.1m_{\rm e}c^2$, with no limitation imposed on the photon energy.

Other previous relativistic studies of the Compton scattering kernel also need to be mentioned. Aharonian & Atoyan (1981) were the first to derive the exact relativistic formula for the direction-dependent kernel, $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$, for an isotropic ensemble of monoenergetic electrons. Nagirner & Poutanen (1994) (see also references therein) derived the exact relativistic expression for the isotropic kernel, $P(\nu\to\nu')$, again for an ensemble of monoenergetic electrons. These formulae make it possible to solve efficiently very general Comptonization problems, i.e., with no constraints set on the parameters. However, their application always implies the need to perform one or two numerical integrations, usually over a specified distribution of electron energies (e.g., Maxwellian), and sometimes over the scattering angle. We also note that the scattering kernel has been the subject of a number of studies that employed numerical methods (e.g., Pomraning 1973; Illarionov et al. 1979; Pozdnyakov et al. 1979; Loeb et al. 1991; Molnar & Birkinshaw 1999).

In the present paper, we have obtained algebraic approximate expressions for the kernels $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ (eq. [7]) and $P(\nu\to\nu')$ (eqs. [19] and [31]), which take into account (1) relativistic effects due to high electron velocities and (2) quantum effects, namely, Compton recoil and Klein-Nishina corrections. The formula for the angle-dependent kernel, $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$, is a very good approximation in the parameter range $kT_{\rm e}\lesssim 25$ keV and $h\nu\lesssim 50$ keV. Moreover, in the case of nonrelativistic electrons, this formula without the temperature correction terms (eq. [10]) gives the exact result for arbitrary photon energies, including $h\nu\gg m_{\rm e}c^2$. We derived our expression in the same manner as BR70 derived theirs, but our expression is more accurate, since it includes the relativistic corrections consistently.

Our expression for $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ can be compared with the corresponding formula derived by Kershaw et al. (1986) (their eq. [41]). Both formulae are fully algebraic and valid in roughly the same electron temperature range ($kT_{\rm e}/m_{\rm e}c^2\lesssim 0.1$). The expression of Kershaw et al. (1986) is more general than our equation (7). It holds for any $h\nu$ in all its temperature terms, while in the case of our formula, this statement is true for the main, nonrelativistic temperature term only. Therefore, our expression should be derivable from the formula of Kershaw et al. (1986). The referee of the present paper informed us that both formulae indeed yield similar numerical results for the same parameter values. On the other hand, our expression is significantly simpler in structure and, more importantly, when $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ is written in this form, it can be integrated analytically (under some additional constraints on the parameters) over the scattering angle or the emergent photon energy. The first of these integrations leads to the algebraic formula for the $P(\nu\rightarrow\nu')$ kernel, while the second leads to an expression for the angular scattering function. We stress that it is this unique integrability that makes equation (7) of interest and useful for further applications.

We derived the expression for $P(\nu\rightarrow\nu')$ by integrating $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ over the scattering angle under the assumption $h\nu(h\nu/m_{\rm e}c^2)\ll kT_{\rm e}$. In this limit, it turns out to be possible to carry the term describing the effect of Compton recoil out of an exponential factor that enters the expression for $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ and implement the subsequent integration analytically. As a result, the range of applicability of our approximation for the isotropic kernel is narrower than that of our formula for the angle-dependent kernel: $h\nu\lesssim 50$ keV, $h\nu(h\nu/m_{\rm e}c^2)\lesssim kT_{\rm e}\lesssim 25$ keV; in this range of parameter values, the accuracy of the approximation is better than 98 per cent. The latter constraint means that the average recoil-induced frequency shift must be less than the typical Doppler broadening. The opposite case, $h\nu(h\nu/m_{\rm e}c^2)\gtrsim kT_{\rm e}$, corresponds to a situation in which the recoil effect is predominant. In this case, the single-scattering line profile is double-peaked, which is related to the Rayleigh scattering phase function (e.g., Pozdnyakov et al. 1979, 1983). It is also worth noting that when the temperature of the matter is sufficiently low, scattering of X-rays on neutral hydrogen and helium may become more important than Compton scattering on free electrons (see a discussion of the recoil profile arising in this problem in Sunyaev & Churazov 1996).

The paper is organized as follows. In §2 we report our analytical results. For the observational astrophysicist, reading this part of the paper should be sufficient for finding all the information necessary for application of the results. The main results are the formulae for the $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ and $P(\nu\rightarrow\nu')$ kernels — equations (7),(10), and equations (19),(31),(23), and (34), respectively. We thoroughly examine the properties of the kernels and determine the parameter ranges of applicability of the different approximations. In §2 we also: (1) discuss the properties of the angular function for Compton scattering in a hot plasma that results from our $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ kernel; (2) show that the Fokker-Planck expansion (to fourth order) of the kinetic equation (3) with the kernel equation (19) leads to the generalized Kompaneets equation (Itoh et al. 1998; Challinor & Lasenby 1998), equation (44); (3) derive mildly-relativistic formulae, equations (49) and (51), for the kernels, $K^{\rm ind}(\nu,\mathbf{\Omega};\nu',\mathbf{\Omega'})$ and $P^{\rm ind}(\nu;\nu')$, that correspond to problems in which a decisive role is played by induced Compton scattering; and (4) give the result of the convolution of a spectrum described by the step function with $P(\nu\rightarrow\nu')$ as an example of application of the kernel to Comptonization problems.

The subsequent sections of the paper provide the calculation details. The derivation procedure for the $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ kernel is described in §3. The angular scattering function is calculated directly [to a better accuracy than $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$] in §4. The formula for the $P(\nu\rightarrow\nu')$ kernel is derived by integrating $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ over the scattering angle in §5. An alternative (fully independent) method for deriving the $P(\nu\rightarrow\nu')$ kernel in the low-frequency case ($h\nu\ll kT_{\rm e}$) is presented in §6.

2. RESULTS

2.1. *The $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$ Kernel*

BR70 have made an attempt to take into account relativistic effects associated with high electron velocities for the kernel $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$. We have examined the derivation of these authors and found that not all relevant correction terms were included by them. In particular, the relativistic corrections to the Maxwellian velocity distribution function were neglected. The BR70 kernel is therefore of better accuracy [correct up to terms of order $(kT_{\rm e}/m_{\rm e}c^2)^{1/2}h\nu/m_{\rm e}c^2$] with respect to photon energy than with respect to electron temperature. Therefore, this kernel is strictly only valid for nonrelativistic electrons. In §3, we revise the derivation of BR70 and obtain the following approximate formula for $K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'})$, which is correct up to terms of order $(kT_{\rm e}/m_{\rm e}c^2)^{3/2}$, $(kT_{\rm e}/m_{\rm e}c^2)^{1/2}h\nu/m_{\rm e}c^2$, and $(h\nu/m_{\rm e}c^2)^2$:

$$K(\nu,\mathbf{\Omega}\rightarrow\nu',\mathbf{\Omega'}) = \nu^{-1}\frac{3}{32\pi}\sqrt{\frac{2}{\pi}}\eta^{-1/2}\frac{\nu'}{g}\left\{1+\mu_{\rm s}^2+\left(\frac{1}{8}-\mu_{\rm s}-\frac{63}{8}\mu_{\rm s}^2+5\mu_{\rm s}^3\right)\eta-\frac{\mu_{\rm s}(1+\mu_{\rm s})}{2}\epsilon^2\right.$$
$$\left.-\frac{3(1+\mu_{\rm s}^2)}{32(1-\mu_{\rm s})^2}\frac{\epsilon^4}{\eta}+\mu_{\rm s}(1-\mu_{\rm s}^2)\epsilon\frac{h\nu}{m_{\rm e}c^2}+\frac{1+\mu_{\rm s}^2}{8(1-\mu_{\rm s})}\frac{\epsilon^3}{\eta}\frac{h\nu}{m_{\rm e}c^2}+(1-\mu_{\rm s})^2\frac{h^2\nu\nu'}{m_{\rm e}^2c^4}\right\}\exp\left[-\frac{\epsilon^2}{4(1-\mu_{\rm s})\eta}\right], \tag{7a}$$

where

$$g = |\nu\mathbf{\Omega}-\nu'\mathbf{\Omega'}| = (\nu^2-2\nu\nu'\mu_{\rm s}+\nu'^2)^{1/2}, \tag{7b}$$

$$\epsilon = \frac{[2(1-\mu_{\rm s})]^{1/2}}{g}\left[\nu'-\nu+\frac{h\nu\nu'}{m_{\rm e}c^2}(1-\mu_{\rm s})\right], \tag{7c}$$

$$\eta = \frac{kT_{\rm e}}{m_{\rm e}c^2}, \tag{7d}$$

and $\mu_{\rm s}=\mathbf{\Omega}\mathbf{\Omega'}$. Note that equation (7) gives the probability of a scattering event per unit dimensionless time, $\tau=\sigma_{\rm T}N_{\rm e}ct$.

Let us look at the sequence of terms within the braces in equation (7a). The main term, $1+\mu_{\rm s}^2$, is just the Rayleigh angular scattering function. The last term, which is proportional to $(h\nu/m_{\rm e}c^2)^2$, describes the second-order Klein-Nishina correction to the scattering cross section. The remaining five terms become important when the electron velocities are high. These temperature-correction terms are either incorrect or absent in the kernel of BR70. The term of order $(h\nu/m_{\rm e}c^2)^2$ was not given by BR70 either.

Equation (7) is a good approximation (which will be supported below by a direct comparison with results of numerical calculations) to the kernel if both the photon energy and electron temperature are moderately relativistic. Furthermore, this formula without the temperature-correction terms (see the resulting eq. [10] below) describes the scattering of photons of arbitrary energy (including $h\nu \gg m_e c^2$) on nonrelativistic electrons ($kT_e \ll m_e c^2$) exactly.

The term $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ must obey the detailed balance principle (including induced effects), i.e., ensure conservation of a blackbody spectrum, $B_\nu = 2h\nu^3/c^2[\exp(h\nu/kT_e)-1]^{-1}$, in thermodynamic equilibrium:

$$K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'}) \left[1+\frac{c^2 B_\nu(\nu')}{2h\nu'^3}\right] \frac{B_\nu(\nu)}{h\nu} = K(\nu',\mathbf{\Omega'} \to \nu,\mathbf{\Omega}) \left[1+\frac{c^2 B_\nu(\nu)}{2h\nu^3}\right] \frac{B_\nu(\nu')}{h\nu'}. \tag{8}$$

Equation (8) reduces to

$$K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'}) = \left(\frac{\nu'}{\nu}\right)^2 \exp\left[\frac{h(\nu-\nu')}{kT_e}\right] K(\nu',\mathbf{\Omega'} \to \nu,\mathbf{\Omega}). \tag{9}$$

It is easily verified that our expression (7) does satisfy eq. (9).

2.1.1. $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ *for Nonrelativistic Electrons and Photons of Arbitrary Energy*

If the electrons are nonrelativistic ($\eta \ll 1$), equation (7) simplifies to

$$K_{\rm nr}(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'}) = \frac{3}{32\pi}\sqrt{\frac{2}{\pi}}\eta^{-1/2}\frac{\nu'}{\nu g}\left[1+\mu_s^2+(1-\mu_s)^2\frac{h^2\nu\nu'}{m_e^2c^4}\right]$$
$$\exp\left\{-\frac{1}{2\eta g^2}\left[\nu'-\nu+\frac{h\nu\nu'}{m_e c^2}(1-\mu_s)\right]^2\right\}, \tag{10}$$

where g and η are given by equations (7b) and (7d), respectively.

Since equation (10) fully takes into account the Klein-Nishina scattering cross section, it holds true for photons of arbitrary energy when the electrons are nonrelativistic.

If we are interested in the case in which the photons are of sufficiently low energy, $h\nu \lesssim 0.1 m_e c^2$, the second-order Klein-Nishina correction term in equation (10) becomes small and can be omitted. The result is

$$K_{\rm nr}(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'}) = \frac{3}{32\pi}\sqrt{\frac{2}{\pi}}\eta^{-1/2}\frac{\nu'}{\nu g}(1+\mu_s^2)\exp\left\{-\frac{1}{2\eta g^2}\left[\nu'-\nu+\frac{h\nu\nu'}{m_e c^2}(1-\mu_s)\right]^2\right\}. \tag{11}$$

Equation (11) includes the first-order Klein-Nishina correction and takes into account Compton recoil; both effects are essentially described in the exponential factor (see the derivation in §3). It also accounts for (to first order) the asymmetry of the scattered profile due to the Doppler effect (the preexponential factor ν'/ν in eq. [11] is important here), and therefore allows one to derive the Kompaneets differential equation.

It is worth noting that the simplified derivation of the $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ kernel described in an appendix to the BR70 paper makes it possible to obtain the exponential factor in equation (10), but not the preexponential factor ν'/ν, and therefore does not lead to the Kompaneets equation and does not describe the energy transfer from the electrons to the photons by scattering.

2.1.2. *Angular Scattering Function*

By integrating the $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ kernel over the photon frequency upon scattering, one can determine how many photons are scattered in a unit time through a given angle, i.e., the angular function for scattering (defined by eq. [70] in §4):

$$\frac{d\sigma}{d\mu_s} = 2\pi \int K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})\, d\nu'. \tag{12}$$

We have verified, through numerical computation, that the angular function that corresponds to the $K_{\rm nr}$ kernel given by equation (10) is exactly the Klein-Nishina formula, describing the cross section for Compton scattering on an electron at rest:

$$\left(\frac{d\sigma}{d\mu_s}\right)_{\rm nr} = \frac{3}{8}\left[1+(1-\mu_s)\frac{h\nu}{m_e c^2}\right]^{-2}\left\{1+\mu_s^2+(1-\mu_s)^2\left[1+\frac{h\nu}{m_e c^2}(1-\mu_s)\right]^{-1}\left(\frac{h\nu}{m_e c^2}\right)^2\right\}. \tag{13}$$

Two examples of the angular function that corresponds to the $K_{\rm nr}$ kernel and is described by equation (13) are shown in Figure 1a: one for the case $h\nu \ll m_e c^2$ and the other for $h\nu \gg m_e c^2$. We see the well-known Klein-Nishina pattern, namely, that more photons are scattered forward ($\mu_s = 1$) than backward ($\mu_s = -1$). This angular function corresponds to the case of nonrelativistic electrons ($kT_e \ll m_e c^2$).

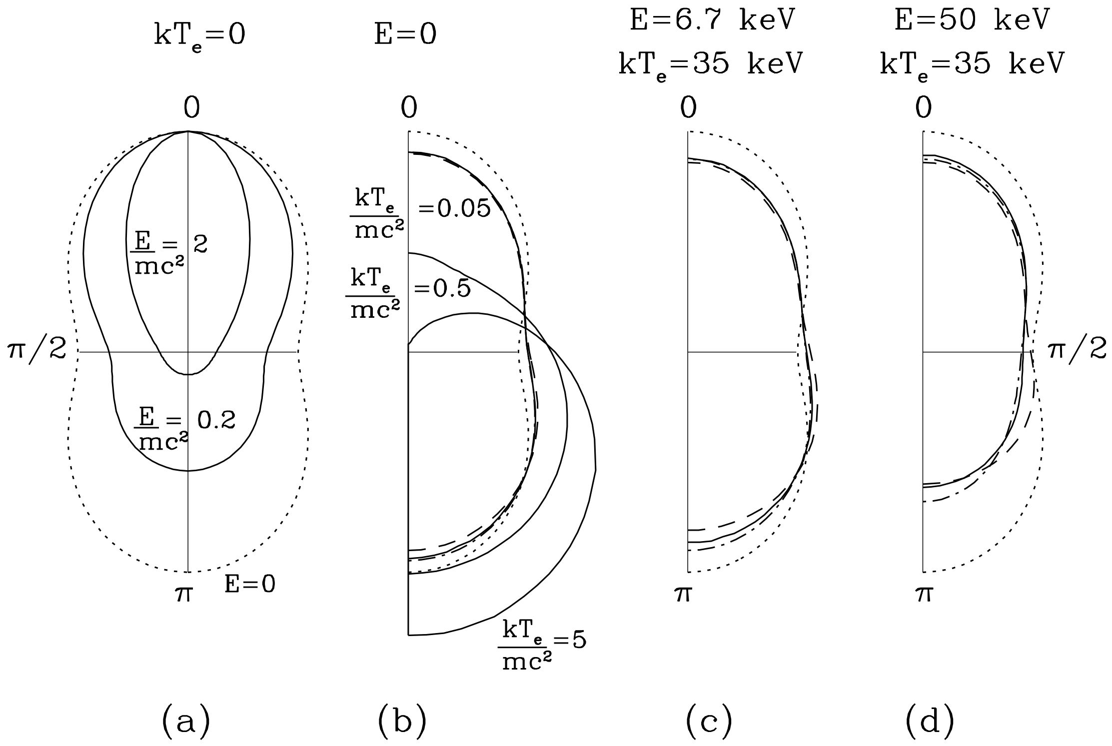

FIG. 1.— Angular function (in polar coordinates) for Compton scattering on thermal electrons, in various regimes set by the values of parameters: photon energy, $E = h\nu$, and electron temperature, $T_{\rm e}$. (a) Low-temperature case: $kT_{\rm e} \ll h\nu$, $kT_{\rm e} \ll m_{\rm e}c^2$. *Solid lines:* Angular functions corresponding to the $K_{\rm nr}$ angle-dependent kernel given by eq. (10). These patterns are described by the Klein-Nishina eq. (13). *Dotted line:* Rayleigh angular function (also in the other panels). (b) Low frequency case: $h\nu \ll kT_{\rm e}$, $h\nu \ll m_{\rm e}c^2$. *Solid lines:* Results of Monte Carlo simulations. For the mildly relativistic example, $kT_{\rm e}$ = 25 keV, also shown are the angular function calculated from eq. (14), which results from the K kernel (eq. [7]) (*dashed line*), and a more accurate approximation described by eq. (15) (*dash-dotted line*). (c) and (d): Both $h\nu$ and $kT_{\rm e}$ are mildly relativistic. The Monte Carlo results (*solid lines*) are compared with the results of eq. (14) (*dashed lines*) and eq. (15) (*dash-dotted lines*).

If the electrons are mildly relativistic, the $K_{\rm nr}$ kernel becomes inaccurate, and the temperature corrections included in the K kernel (eq. [7]) become important. Analytical integration of this kernel over the scattering angle is possible if an assumption is made that $h\nu(h\nu/m_{\rm e}c^2) \ll kT_{\rm e} \ll m_{\rm e}c^2$. In this limit, K can be written in the form given by equation (77) of §5, which leads to

$$\frac{d\sigma}{d\mu_{\rm s}} = \frac{3}{8}\left\{1+\mu_{\rm s}^2+\left[-2(1-\mu_{\rm s})(1+\mu_{\rm s}^2)\frac{h\nu}{m_{\rm e}c^2}+(1-\mu_{\rm s})^2(4+3\mu_{\rm s}^2)\left(\frac{h\nu}{m_{\rm e}c^2}\right)^2...\right]\right.$$
$$\left.+2(1-2\mu_{\rm s}-3\mu_{\rm s}^2+2\mu_{\rm s}^3)\frac{kT_{\rm e}}{m_{\rm e}c^2}+O((h\nu/m_{\rm e}c^2)(kT_{\rm e}/m_{\rm e}c^2),(kT_{\rm e}/m_{\rm e}c^2)^2)\right\}. \quad (14)$$

The expression in square brackets in equation (14) is an expansion series in powers of $h\nu/m_{\rm e}c^2$ resulting from equation (13) (only two leading terms are presented). More interesting is the correction term of order $kT_{\rm e}/m_{\rm e}c^2$. The origin of this term is related to Doppler aberration and has nothing to do with quantum effects. The terms that are given in implicit form in equation (14), i.e. $O(...)$, indicate the order of inaccuracy of our approximation for the kernel. Although equation (14) was obtained under the assumption that $h\nu(h\nu/m_{\rm e}c^2) \ll kT_{\rm e} \ll m_{\rm e}c^2$, it holds true for arbitrary proportions of $h\nu$ and $kT_{\rm e}$, which we have verified by a direct numerical calculation of the integral (12).

It is also possible to directly calculate the angular function, not using the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel. The corresponding derivation procedure, which is similar to but simpler than that for $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$, is described in §4. The result, which is more accurate than equation (14), is

$$\frac{d\sigma}{d\mu_{\rm s}} = \frac{3}{8}\left[1+\mu_{\rm s}^2-2(1-\mu_{\rm s})(1+\mu_{\rm s}^2)\frac{h\nu}{m_{\rm e}c^2}+2(1-2\mu_{\rm s}-3\mu_{\rm s}^2+2\mu_{\rm s}^3)\frac{kT_{\rm e}}{m_{\rm e}c^2}\right.$$
$$+(1-\mu_{\rm s})^2(4+3\mu_{\rm s}^2)\left(\frac{h\nu}{m_{\rm e}c^2}\right)^2+(1-\mu_{\rm s})(-7+14\mu_{\rm s}+9\mu_{\rm s}^2-10\mu_{\rm s}^3)\frac{h\nu}{m_{\rm e}c^2}\frac{kT_{\rm e}}{m_{\rm e}c^2}$$
$$\left.+(-7+22\mu_{\rm s}+9\mu_{\rm s}^2-38\mu_{\rm s}^3+20\mu_{\rm s}^4)\left(\frac{kT_{\rm e}}{m_{\rm e}c^2}\right)^2+...\right]. \quad (15)$$

We see that the first-order correction terms, $O(h\nu/m_{\rm e}c^2)$ and $O(kT_{\rm e}/m_{\rm e}c^2)$, and the second-order term proportional to $(h\nu/m_{\rm e}c^2)^2$ are the same as in equation (14). The last two terms in equation (15), $O((h\nu/m_ec^2)(kT_e/m_ec^2))$ and $O((kT_e/m_ec^2)^2)$, belong to the next-order approximation (with respect to K given by eq. [7]) to the kernel.

Integration of equation (15) over all scattering angles leads to the well-known expression that describes the total cross section for Compton scattering on Maxwellian electrons [defined as $\sigma = (\lambda N_e)^{-1}$, where λ is the photon mean free path] in the mildly relativistic limit (e.g., Pozdnyakov et al. 1983; Shestakov et al. 1988):

$$\sigma = \sigma_T\left[1-2\frac{h\nu}{m_ec^2}-5\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}+\frac{26}{5}\left(\frac{h\nu}{m_ec^2}\right)^2\right]. \tag{16}$$

Note that the pure temperature terms, $O((kT_e/m_ec^2)^n)$, in equation (15) give no contribution to σ. This is a well-known fact, which means that the total cross section for low-energy photons in a hot plasma is equal to the Thomson cross section. At the same time, the angular function (eq. [15]) is different from the Rayleigh function.

In Figures 1b–1d, several examples of the angular function for Compton scattering on hot electrons are presented; the results of Monte Carlo simulations are compared with the results of the calculation by the approximate formulae (14) and (15).

Figure 1b demonstrates scattering of low-frequency photons: $h\nu \ll kT_e$, $h\nu \ll m_ec^2$. The angular function in this case is totally different from the Klein-Nishina one — compare with Figure 1a. Let us first consider the mildly relativistic temperature range $kT_e \lesssim 25$ keV, within which equation (14) (with the terms containing $h\nu/m_ec^2$ vanishing) is a good approximation. Scattering is somewhat suppressed (compared to the Rayleigh angular function, which corresponds to the K_{nr} kernel in this case) both in the forward and backward directions. There is, however, a noticeable enhancement in the number of photons scattered through intermediate angles, between 69° and 138°; these values are found by equating the correction term of order kT_e/m_ec^2 in equation (14) to zero. The temperature-relativistic correction to the Rayleigh angular function reaches a maximum of $12(\eta/0.05)$% at an angle of 105°. The relativistic reduction of the angular function is maximal, $10(\eta/0.05)$%, for the two extreme values of the scattering angle, 0 and π. It is evident from Figure 1b that the approximation represented by equation (15) is more accurate that that given by equation (14). The inclusion of the correction term of order $(kT_e/m_ec^2)^2$ is particularly important for very large scattering angles (close to π). Equation (15) proves to be a good approximation to the actual scattering angular function in the range $kT_e \lesssim 35$ keV.

As the temperature becomes significantly relativistic (see the patterns for $kT_e = 0.5m_ec^2$ and $5m_ec^2$ in Fig. 1b), the scattering angular function modifies further and becomes totally unlike the Rayleigh angular function. Only the results of Monte Carlo simulations are shown for these cases, because the convergence of the expansion series in powers of kT_e/m_ec^2 for the angular function becomes poor when kT_e exceeds ~ 40 keV. Scattering in the forward direction is now heavily suppressed (i.e., the plasma effectively screens itself from the radiation incident from outside), while more and more photons are scattered through angles between $\pi/2$ and π. In particular, at temperatures $kT_e \gtrsim 0.5m_ec^2$, the number of photons scattered through an angle of π is higher than in the case $kT_e = 0$. In the limit $kT_e \gg m_ec^2$, the angular function is described by the law $d\sigma/d\mu_s = (1-\mu_s)/2$ (Sazonov & Sunyaev 2000).

Figures 1c and 1d show two examples of the angular scattering function when both $h\nu$ and kT_e are mildly relativistic. The approximations described by equations (14) and (15) work well for photon energies $h\nu \lesssim 50$ keV within the temperature ranges quoted above.

The phenomenon of anisotropic (backward) Compton scattering by a hot plasma has been previously mentioned in astrophysical literature (Ghisellini et al. 1991; Titarchuk 1994; Poutanen & Svensson 1996; Gierlinski et al. 1999). In particular, Haardt (1993) has performed a semianalytical calculation (using Monte Carlo simulations) of the angular scattering function, as well as other related quantities, and obtained results for the case of low-frequency radiation ($h\nu \ll m_ec^2$) that are similar to those depicted in Figure 1b.

2.1.3. *The Moments of the Kernel*

Using equation (77) of §5 it is possible to calculate the moments of the $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ kernel, in a similar way as we derived above the angular function. Here we define the moments as

$$\langle(\Delta\nu)^n\rangle_{\mu_s} = 2\pi\int K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})(\nu'-\nu)^n d\nu', \tag{17}$$

so that the integration of $\langle(\Delta\nu)^n\rangle_{\mu_s}$ over μ_s gives the moments of the $P(\nu \to \nu')$ kernel for the isotropic problem, which were defined in equation (5).

The result for the first four moments is

$$\begin{aligned}
\langle\Delta\nu\rangle_{\mu_s} &= \frac{3\nu}{8}\left[(1-\mu_s)(1+\mu_s^2)\left(4\frac{kT_e}{m_ec^2}-\frac{h\nu}{m_ec^2}\right)+2(7-21\mu_s+5\mu_s^2+19\mu_s^3-10\mu_s^4)\left(\frac{kT_e}{m_ec^2}\right)^2\right.\\
&\left.+\frac{-37+81\mu_s-65\mu_s^2+41\mu_s^3-20\mu_s^4}{2}\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}+3(1-\mu_s)^2(1+\mu_s^2)\left(\frac{h\nu}{m_ec^2}\right)^2\right],\\
\langle(\Delta\nu)^2\rangle_{\mu_s} &= \frac{3\nu^2}{8}\left[2(1-\mu_s)(1+\mu_s^2)\frac{kT_e}{m_ec^2}+(37-81\mu_s+65\mu_s^2-41\mu_s^3+20\mu_s^4)\left(\frac{kT_e}{m_ec^2}\right)^2\right.\\
&\left.-18(1-\mu_s)^2(1+\mu_s^2)\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}+(1-\mu_s)^2(1+\mu_s^2)\left(\frac{h\nu}{m_ec^2}\right)^2\right],
\end{aligned}$$

$$\langle(\Delta\nu)^3\rangle_{\mu_s} = \frac{3\nu^3}{8}(1-\mu_s)^2(1+\mu_s^2)\left[36\left(\frac{kT_e}{m_ec^2}\right)^2 - 6\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}\right],$$

$$\langle(\Delta\nu)^4\rangle_{\mu_s} = \frac{3\nu^4}{8}12(1-\mu_s)^2(1+\mu_s^2)\left(\frac{kT_e}{m_ec^2}\right)^2. \tag{18}$$

The moments of higher degrees turn out to be at least of order η^3. The importance of including relativistic corrections in the kernel becomes clear from examining the terms proportional to η^2 and $\eta h\nu/m_ec^2$ in the expressions for the first two moments, which become comparable to the main terms already at moderate $\eta, h\nu/m_ec^2 \sim 0.02$ for large scattering angles. The first and second moments describe correspondingly the average frequency increment by scattering and the broadening of the scattered profile.

It should be noted that the accuracy, to within $O(\eta^2, \eta h\nu/m_ec^2, (h\nu/m_ec^2)^2)$, of the derived moments is better than the accuracy, to within $O(\eta^{3/2}, \eta^{1/2}h\nu/m_ec^2)$, of the kernel itself. The reason for this is that $K(\nu, \mathbf{\Omega} \to \nu', \mathbf{\Omega'})$ is multiplied by a small quantity, the frequency shift ($\sim \nu\eta^{1/2}$) raised to a certain power, when the moments are calculated.

2.1.4. *Comparison of the Analytical Formulae for the Kernel with Numerical Results*

We performed a series of Monte Carlo simulations (a modification of the code described by Pozdnyakov et al. 1983 was used) to evaluate the accuracy of our analytical expressions for the $K(\nu, \mathbf{\Omega} \to \nu', \mathbf{\Omega'})$ kernel. Figures 2–7 show examples of spectra that may form through single scattering (with a given angle) of a monochromatic line on thermal electrons in various scattering regimes that are set by the values of the parameters: kT_e, $h\nu$, μ_s. The numerical results are compared with the analytical kernels K (mildly relativistic) and K_{nr} (non-relativistic).

As inferred from Figures 2–7 and further from the entire set of results of our numerical calculations, the mildly relativistic equation (7) is a very good approximation for electron temperatures $kT_e \lesssim 25$ keV and photon energies $h\nu \lesssim 50$ keV. In this range of parameter values, the accuracy is better than 98%, except in the far wings of the scattered profile. The latter can be roughly defined as the regions where $|\epsilon| \gtrsim 0.5(1-\mu_s)^{1/2}$ (this size should be compared with the characteristic width of the line, which is much smaller: $[(1-\mu_s)\eta]^{1/2}$). For very large scattering angles, $\mu_s \lesssim -0.8$, relativistic corrections are particularly important. In this angular range, the accuracy quoted above is achieved in a more narrow temperature range, $kT_e \lesssim 15$ keV.

One can safely use the K_{nr} kernel when $kT_e \lesssim 5$ keV. The photon energy can take arbitrary values in this case because equation (10) is exact for nonrelativistic electrons, as we pointed out before. This is demonstrated by Figures 5–7, which illustrate the scattering of a line with $h\nu = 1238$ keV (which corresponds to one of the strongest γ lines produced through radioactive decay of ^{56}Co during

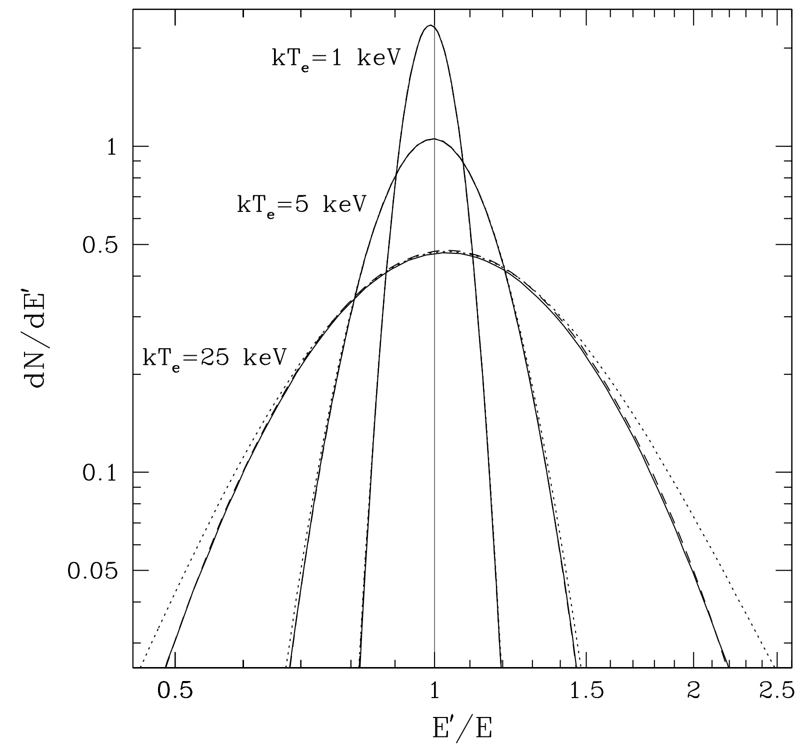

FIG. 2.— Photon number spectra resulting from the single scattering of a monochromatic line of energy, $h\nu = 6.7$ keV, on hot electrons with a scattering angle of $\pi/2$, for a number of values of electron temperature. The results of Monte Carlo simulations (*solid lines*) are compared with various approximations for the angle-dependent kernel: K_{nr} (eq. [10], *dotted lines*) and K (eq. [7], *dashed lines*). Note the increasing influence of relativistic corrections (included in the K kernel) on the spectrum as the electron temperature becomes higher.

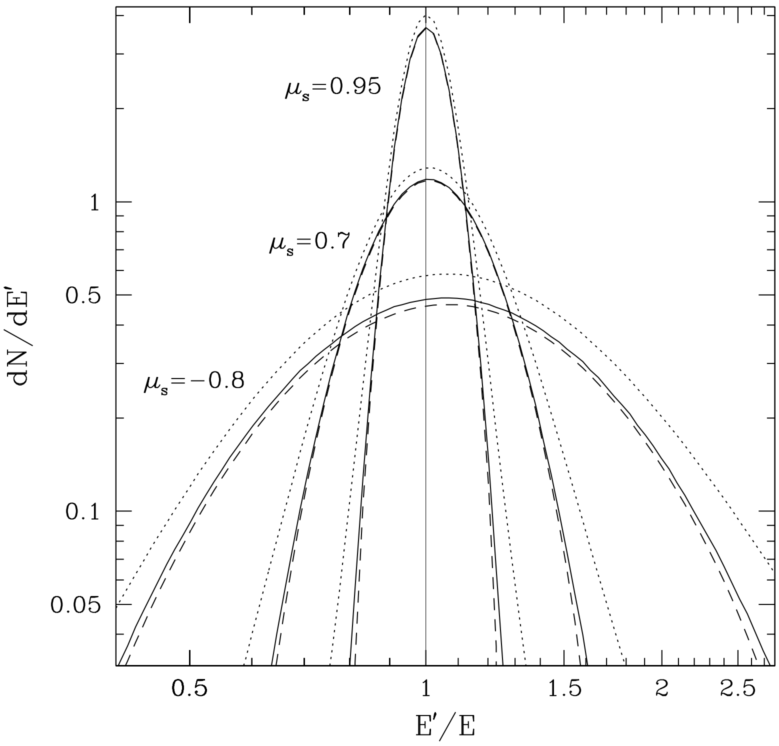

FIG. 3.— Same as Fig. 2, but for a fixed, mildly relativistic electron temperature, kT_e = 25 keV, and a varying scattering angle. Note that increasing the scattering angle for a given temperature has a similar broadening effect on the spectrum as increasing the temperature with the scattering angle fixed (compare with Fig. 2).

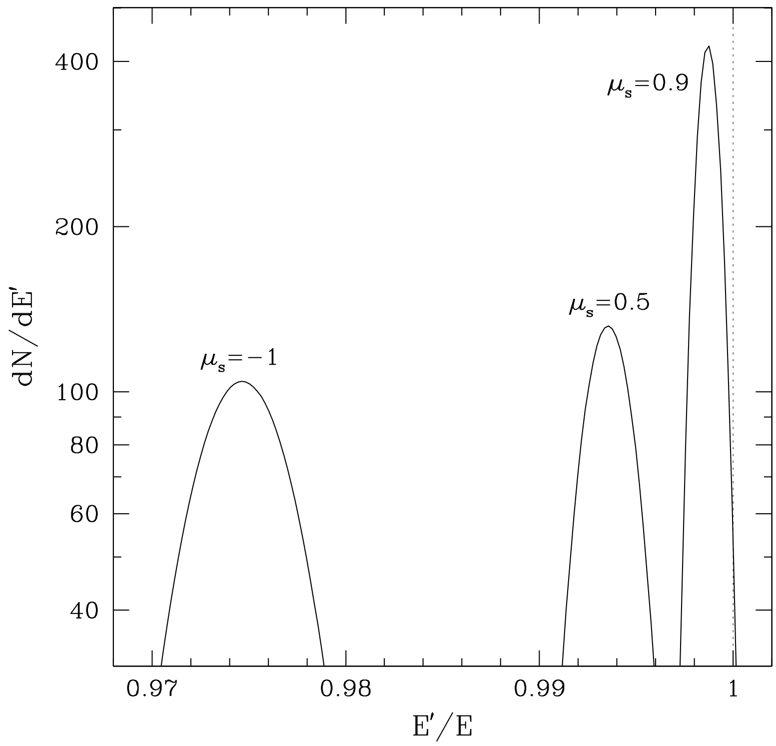

FIG. 4.— Spectra resulting from the single scattering of a monochromatic line of energy, $h\nu$ = 6.7 keV, on low-temperature electrons, $T_e = 10^4$ K, for a number of scattering angles. The corrections relevant for relativistic electrons are not important in this case. The presented spectra were calculated from eq. (10) for the K_{nr} kernel, which gives the exact result in the case of nonrelativistic electrons and leads to the same results as Monte Carlo simulations.

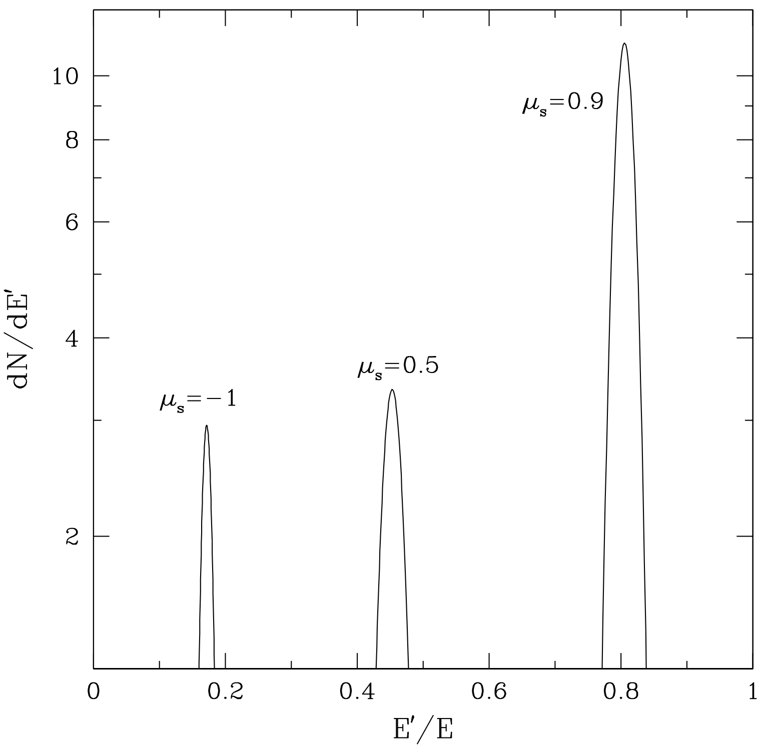

FIG. 5.— Same as Fig. 4, but for high-energy photons, $h\nu$ = 1238 keV, and electrons of kT_e = 1 keV. The K_{nr} kernel given by eq. (10) is still very accurate at this electron temperature and leads to practically identical results with those of Monte Carlo simulations. Note that the total number of scattered photons is drastically decreased on going from μ_s = 0.5 to μ_s = −1. This is due to the increasing effect of Klein-Nishina corrections on the scattering cross section (compare with the nonrelativistic case in Fig. 4).

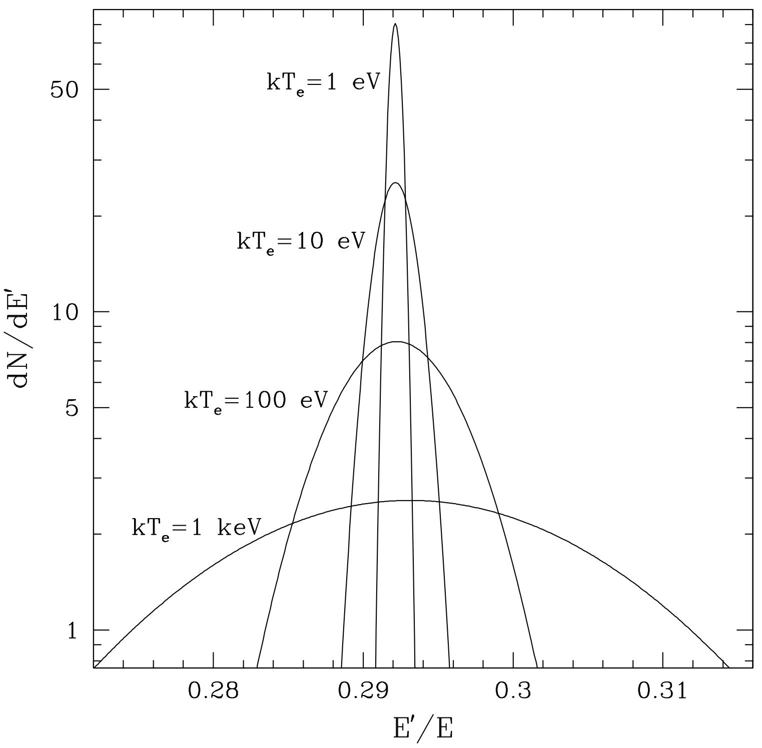

FIG. 6.— Same as Fig. 5 ($h\nu$ = 1238 keV), but for a fixed scattering angle, μ_s = 0, and a varying, nonrelativistic electron temperature.

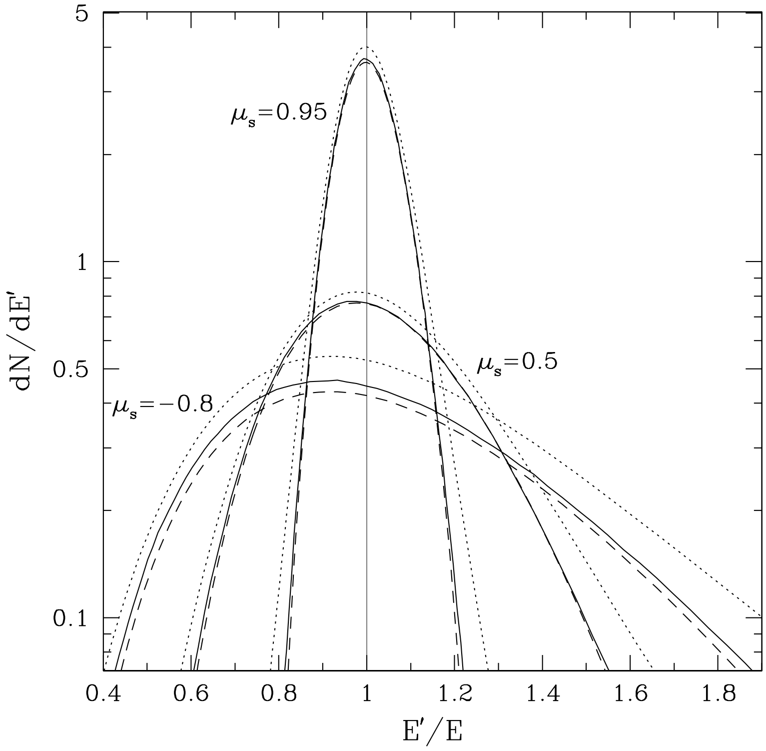

FIG. 7.— Spectra forming through single scattering of monochromatic radiation, $h\nu$ = 50 keV, on mildly relativistic thermal electrons, kT_e = 25 keV, for a set of scattering angles. The results of Monte Carlo simulations (*solid lines*) are compared with the different approximations for the angle-dependent kernel: K_{nr} (eq. [10], *dotted lines*) and K (eq. [7], *dashed lines*). Temperature relativistic corrections, which are included in the K kernel but not in the K_{nr} kernel, are of importance in this case.

supernova explosions) by thermal electrons, for a number of values of the electron temperature and scattering angle. Note that even for a temperature of $T_e = 10^4$ K, the Doppler width of the scattered line is $\sim$ 1 keV for a scattering angle of $\pi/2$. The Ge cryogenic detectors of the *International Gamma-Ray Astrophysical Observatory* (*INTEGRAL*), scheduled for launch in 2002, will be capable of measuring such broadening.

The line broadening arising from the Doppler effect depends on two parameters: the electron temperature, kT_e, and the scattering angle, μ_s. The width of the profile as calculated from equation (10) is strictly proportional to $[(1-\mu_s)\eta]^{1/2}$. Therefore, in the nonrelativistic limit, one can only determine this combination of the two parameters, rather than kT_e and μ_s separately, from a measurement of line broadening. This point is illustrated in Figures 2 and 3. One can see that identical profiles can be obtained by varying either the temperature or the scattering angle. However, if the electrons are mildly relativistic, the correction terms in equation (7) become important, with their own dependence on μ_s. This property makes it possible, in principle, to find both kT_e and μ_s from a measurement of a scattered line. However, in many situations, it would be easier to determine μ_s by measuring the recoil-induced shift of the line (i.e., the position of the line on the frequency axis, see Figs. 2–7) which is proportional to $(1-\mu_s)h\nu/m_ec^2$.

2.2. *The $P(\nu \to \nu')$ Kernel for the Isotropic Problem*

In the preceeding paragraph we presented the analytical formula for the $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ kernel, which can be used for describing spectra forming as a result of single Compton scattering of a monochromatic line, with a given angle between the direction from which photons are supplied ($\mathbf{\Omega}$) and the observer's direction ($\mathbf{\Omega'}$). However, in most astrophysical situations we are presented with some angular distribution of incident radiation. In such cases, one needs to convolve the $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ kernel with this initial distribution in order to determine the emergent spectrum.

If the initial distribution is isotropic, we arrive at the kinetic equation (3), with the kernel derivable by the integration given in equation (1). This integral can be performed analytically for the $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ kernel given by equation (7) if we assume that $h\nu(h\nu/m_ec^2) \ll kT_e$, so that the Doppler broadening prevails over the recoil shift. The details of the integration procedure are given in §5. The essence of this calculation is to write the argument of the exponential in equation (7) as a trinomial. The two terms dependent on the photon energy ($h\nu$) then turn out to be small with respect to the main term describing the Doppler broadening (because of our assumption that $h\nu \sim kT_e$), and therefore can be carried out of the exponential. The subsequent integration of K over the scattering angle becomes straightforward. The result is

$$P(\nu \to \nu') = \nu^{-1}\sqrt{\frac{2}{\pi}}\eta^{-1/2}\left\{\left[1+\sqrt{2}\delta\left(1-\frac{h\nu}{kT_e}\right)\eta^{1/2}-4\delta^2\frac{h\nu}{m_ec^2}\right.\right.$$

$$+2\sqrt{2}\delta^3\left(-2+\frac{1}{3}\left(\frac{h\nu}{kT_e}\right)^2\right)\frac{h\nu}{m_ec^2}\eta^{1/2}\Bigg]\, p_0+\left[1+\sqrt{2}\delta\left(1-\frac{h\nu}{kT_e}\right)\eta^{1/2}\right]p_t$$
$$+\left[1+\sqrt{2}\delta\left(3-\frac{h\nu}{kT_e}\right)\eta^{1/2}\right]p_r\Bigg\}, \tag{19a}$$

where

$$\begin{aligned}
p_0 &= \left(\frac{11}{20}+\frac{4}{5}\delta^2+\frac{2}{5}\delta^4\right)F+|\delta|\left(-\frac{3}{2}-2\delta^2-\frac{4}{5}\delta^4\right)G,\\
p_t &= \left[\left(-\frac{1091}{1120}-\frac{507}{560}\delta^2+\frac{57}{35}\delta^4+\frac{68}{35}\delta^6\right)F+|\delta|\left(\frac{9}{4}+\delta^2-\frac{26}{5}\delta^4-\frac{136}{35}\delta^6\right)G\right]\eta,\\
p_r &= \left[\left(-\frac{23}{280}+\frac{26}{35}\delta^2+\frac{34}{35}\delta^4+\frac{16}{35}\delta^6\right)F+|\delta|\left(-2\delta^2-\frac{12}{5}\delta^4-\frac{32}{35}\delta^6\right)G\right]\left(\frac{h\nu}{m_ec^2}\right)^2\eta^{-1},\\
F &= \exp(-\delta^2),\ G=\int_{|\delta|}^{\infty}\exp(-t^2)dt=0.5\pi^{1/2}Erfc(|\delta|),\ \delta=(2\eta)^{-1/2}\frac{\nu'-\nu}{\nu'+\nu}.
\end{aligned} \tag{19b}$$

The kernel given by equation (19) is a series in powers of $\eta^{1/2}$ (given that $h\nu\sim kT_e$), written up to the third order.

In the case of scattering of low-frequency radiation in a hot plasma ($h\nu\ll kT_e$), equation (19) simplifies significantly:

$$P_T=\nu^{-1}\sqrt{\frac{2}{\pi}}\eta^{-1/2}\left[1+\sqrt{2}\delta\eta^{1/2}\right](p_0+p_t). \tag{20}$$

Equation (20), which totally neglects Compton recoil, can be obtained independently from the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel, using a simple calculation procedure that employs a transition to the rest frame of the scattering electron (§6). As a matter of fact, we first obtained equation (20), and this formula allowed us to check (by applying the detailed balance principle; see below) some of the terms dependent on $h\nu/m_ec^2$ in the more general equation (19).

We have verified that the expression (19) obeys the detailed balance principle, satisfying (in all orders up to $\eta^{3/2}$) the equation

$$P(\nu\to\nu')=\left(\frac{\nu'}{\nu}\right)^2\exp\left[\frac{h(\nu-\nu')}{kT_e}\right]P(\nu'\to\nu), \tag{21}$$

which is the analog of the corresponding equation for the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel (eq. [9]).

2.2.1. *The Kernel Leading to the Kompaneets Equation*

The expression (19) is the sum of four leading terms of the series in powers of $\eta^{1/2}$ for the $P(\nu\to\nu')$ kernel. By retaining a smaller number of terms in this series, one can build up cruder approximations to the kernel. Below, we shall consider two such approximations.

The least accurate approximation uses the first term in the series given in equation (19a):

$$P_0(\nu\to\nu')=\nu^{-1}\sqrt{\frac{2}{\pi}}\eta^{-1/2}p_0, \tag{22}$$

with p_0 given by equation (19b).

The expression (22) is equivalent to the formula obtained by Hummer & Mihalas (1967). P_0 is symmetric in frequency shift: $p_0(-\delta)=p_0(\delta)$. Therefore, it only describes the Doppler (random-walk) broadening, not accounting for the average increase in the photon energy by the Doppler effect.

A more accurate approximation can be obtained by summing up two leading terms in the series given in equation (19a):

$$P_K(\nu\to\nu')=\nu^{-1}\sqrt{\frac{2}{\pi}}\eta^{-1/2}\left[1+\sqrt{2}\delta\left(1-\frac{h\nu}{kT_e}\right)\eta^{1/2}\right]p_0. \tag{23}$$

The P_K kernel is necessary and sufficient for deriving the Kompaneets equation (6). For this reason, we refer to it as the "Kompaneets equation kernel". When $h\nu\ll kT_e$, equation (23) is equivalent to the formula obtained by Sunyaev (1980). The P_K kernel already accounts for the asymmetry of the scattered line and the corresponding photon heating. The Kompaneets equation kernel also takes into account (to first order) Compton recoil.

2.2.2. *The Moments and Normalization of the Kernel*

Important information on the properties of the $P(\nu\to\nu')$ kernel is provided by its moments, which were defined in equation (5).

The required integration is readily performed for the kernel given by equation (19) after we express the differential $d\nu'$ appearing in equation (5) through $d\delta$,

$$d\nu' = 2\sqrt{2}\nu\eta^{1/2}(1+2\sqrt{2}\eta^{1/2}\delta+6\delta^2\eta+8\sqrt{2}\delta^3\eta^{3/2}+...)d\delta. \tag{24}$$

The quantity δ was introduced in equation (19b). The subsequent integration over δ should be carried out in the limits $-\infty$ to ∞.

Collecting terms up to $O(\eta^2, \eta h\nu/m_ec^2, (h\nu/m_ec^2)^2)$, we obtain

$$\begin{aligned}
\langle\Delta\nu\rangle &= \nu\left[4\frac{kT_e}{m_ec^2}-\frac{h\nu}{m_ec^2}+10\left(\frac{kT_e}{m_ec^2}\right)^2-\frac{47}{2}\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}+\frac{21}{5}\left(\frac{h\nu}{m_ec^2}\right)^2\right],\\
\langle(\Delta\nu)^2\rangle &= \nu^2\left[2\frac{kT_e}{m_ec^2}+47\left(\frac{kT_e}{m_ec^2}\right)^2-\frac{126}{5}\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}+\frac{7}{5}\left(\frac{h\nu}{m_ec^2}\right)^2\right],\\
\langle(\Delta\nu)^3\rangle &= \nu^3\left[\frac{252}{5}\left(\frac{kT_e}{m_ec^2}\right)^2-\frac{42}{5}\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}\right],\\
\langle(\Delta\nu)^4\rangle &= \nu^4\left[\frac{84}{5}\left(\frac{kT_e}{m_ec^2}\right)^2\right].
\end{aligned} \tag{25}$$

The moments of higher degrees turn out to be at least of order η^3. The values for the moments above reproduce the results of Itoh et al. (1998); Challinor & Lasenby (1998) (more general expressions for the first two moments are also available; see Shestakov et al. 1988; §4.3 in Nagirner & Poutanen 1994). It is important to mention that the equations (25) are valid for arbitrary values of the $h\nu/kT_e$ ratio, including the case $kT_e = 0$, in contrast to the kernel equation (19) itself, which holds in the limit $h\nu(h\nu/m_ec^2) \lesssim kT_e$ only (see the next paragraph for a discussion of the scope of the analytical approximations).

The moments (25) can also be obtained by integrating over the scattering angle the moments of the angle-dependent $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel (eq. [18]).

The normalization of the kernel equation (19) can be calculated similarly to its moments:

$$\int P\,d\nu' = 1-2\frac{h\nu}{m_ec^2}+\left[-\frac{53}{10}\left(\frac{kT_e}{m_ec^2}\right)^2-\frac{44}{5}\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}+\frac{63}{20}\left(\frac{h\nu}{m_ec^2}\right)^2\right]. \tag{26}$$

This expression should be compared with the known expansion series for the total cross section, equation (16). One can see that the terms in square brackets in equation (26) describe the inaccuracy of the approximation in equation (19), whereas the term $-2h\nu/m_ec^2$ corresponds to the actual first-order Klein-Nishina correction to the cross section (see eq. [16]). Note that equation (26) does not describe the second-order Klein-Nishina correction to the scattering cross section, although our expression for the angle-dependent kernel $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ (eq. [7]) does contain the corresponding term. This term was omitted when we integrated $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ over the scattering angle (in §5), because in the current limit ($h\nu \sim kT_e$), inclusion of this term in the formula for $P(\nu\to\nu')$ (eq. [19]) would be inconsistent with the absence of some (unknown) terms of the same order, such as $O(\eta^2)$ or $O((h\nu/m_ec^2)^3\eta^{-1})$, in this formula.

For the zero-order kernel, P_0 (eq. [22]), the first two moments are

$$\begin{aligned}
\langle\Delta\nu\rangle_0 &= 0,\\
\langle(\Delta\nu)^2\rangle_0 &= \nu^2\frac{kT_e}{m_ec^2},
\end{aligned} \tag{27}$$

and the normalization is

$$\int P_0\,d\nu' = 1+\frac{3}{2}\frac{kT_e}{m_ec^2}. \tag{28}$$

The corresponding relations for the Kompaneets equation kernel (23) are

$$\begin{aligned}
\langle\Delta\nu\rangle_k &= \nu\left(4\frac{kT_e}{m_ec^2}-\frac{h\nu}{m_ec^2}\right),\\
\langle(\Delta\nu)^2\rangle_k &= 2\nu^2\frac{kT_e}{m_ec^2},
\end{aligned} \tag{29}$$

and

$$\int P_K\,d\nu' = 1+\left(\frac{5}{2}\frac{kT_e}{m_ec^2}-\frac{h\nu}{m_ec^2}\right). \tag{30}$$

The terms in parentheses in equation (30) describe the inaccuracy of the P_K kernel. In particular, the Klein-Nishina reduction of the integral cross section is not covered by this approximation. The P_K kernel also neglects altogether the frequency diffusion of photons due to Compton recoil (as a result of the recoil-induced frequency shift depending on the scattering angle). The term of order

$(h\nu/m_ec^2)^2$ in the expansion series for $\langle(\Delta\nu)^2\rangle$, which is present in equation (25) and absent from equation (29), is responsible for this. The Kompaneets equation (6), which is a Fokker-Planck equation with its coefficients governed by the moments (29), does not include the corresponding dispersion term, the importance of which for the case $h\nu \gg kT_e$ was pointed out in (Ross et al. 1978; Illarionov et al. 1979).

Having calculated the normalization (26) for the P kernel and knowing the exact result for the total scattering cross section (eq. [16]), we can try to crudely take into account the terms of order η^2 for P, not calculating them. To this end, we have to renormalize the kernel as

$$P' = \left[1+\frac{53}{10}\left(\frac{kT_e}{m_ec^2}\right)^2+\frac{19}{5}\frac{h\nu}{m_ec^2}\frac{kT_e}{m_ec^2}+\frac{41}{20}\left(\frac{h\nu}{m_ec^2}\right)^2\right]P, \tag{31}$$

where P is given by equation (19). As a result, the normalization of the modified kernel P' is precisely the bracketed expression in equation (16).

In the case $h\nu \ll kT_e$, the kernel given by equation (20) can be renormalized similarly:

$$P'_T = \left[1+\frac{53}{10}\left(\frac{kT_e}{m_ec^2}\right)^2\right]P_T, \tag{32}$$

Finally, we may introduce the renormalized kernels P'_0 and P'_K:

$$P'_0 = \left(1-\frac{3}{2}\frac{kT_e}{m_ec^2}\right)P_0, \tag{33}$$

$$P'_K = \left(1-\frac{5}{2}\frac{kT_e}{m_ec^2}-\frac{h\nu}{m_ec^2}\right)P_K. \tag{34}$$

The assumed normalizations of P'_0 and P'_K are 1 and $1-2h\nu/m_ec^2$, respectively.

We should stress that the renormalized kernels are still approximations of the same order of uncertainty as the original kernels, but these kernels turn out to be more accurate than the original ones, as follows from a comparison with results of numerical calculations, which we present in the next paragraph. Note also that the moments are not affected by the renormalization procedure.

2.2.3. *Comparison of the Analytical Approximations for the Kernel with Numerical Results*

The case of $h\nu \ll kT_e \ll m_ec^2$. In this case, the profile of a Compton-scattered monochromatic line forms through the Doppler mechanism alone. The exact kernel can be computed numerically, using equation (89) of §6 or by means of Monte Carlo simulations. We employ both methods in our analysis. Figures 8 and 9 compare, for two values of the electron temperature (kT_e = 10 keV and 25 keV), the accurate spectra with the corresponding analytical dependences as calculated in different approximations for the kernel: P_T, P_0, and P_K (eqs. [20,22,23]).

As expected, the asymmetry of the line (domination of the right wing over the left) increases as the temperature grows. One can see that the zero approximation (Hummer & Mihalas 1967), P_0, which is symmetric in frequency shift, matches the line profile poorly. Therefore, we recommend its usage be restricted to the range $kT_e \lesssim 500$ eV, where P_0 is accurate to better than 98%, except in the far wings of the line. The latter can be roughly defined as the regions $|\nu'-\nu|/(\nu'+\nu) \gtrsim 1/4$.

The Kompaneets equation kernel, P_K, which was derived for the considered case ($h\nu \ll kT_e$) by Sunyaev (1980), works well when $kT_e \lesssim 5$ keV. At higher temperatures, it becomes important to take into account the relativistic correction terms, $(1+\sqrt{2}\delta\eta^{1/2})p_t$, in equation (20), i.e., to use our most accurate approximation, P_T. At $kT_e = 25$ keV, the (Sunyaev 1980) kernel overestimates the flux at the line peak by 10% (see Fig. 9). The flux turns out to be even more overestimated in the wings. The P_T kernel is accurate to better than 98% at this temperature.

At $kT_e \sim 50$ keV, the electrons are already strongly relativistic: $\langle(v/c)^2\rangle^{1/2} \sim \eta^{1/2} \sim 0.5$, and the $\eta^{1/2}$ series in equation (20) converges poorly, particularly in the wings of the line. At higher temperatures, one must use the exact kernel (which can be calculated from eq. [89]) and treat the problem numerically.

Figure 10 is the same as Figure 9, but presents the results for the modified kernels, P'_T, P'_0, and P'_K (eqs. [32]–[34]). It is seen that the renormalization has indeed appreciably improved the accuracy of the approximations.

Inclusion of quantum effects: $h\nu \neq 0$, $kT_e \ll m_ec^2$. The assumption $h\nu(h\nu/m_ec^2) \ll kT_e$, which we used when deriving equation (19), means that the recoil-induced downward shift in the photon frequency must be small compared to the Doppler line broadening. Using Monte Carlo simulations, we have established that P and P_K (eqs. [19] and [23]) remain good approximations up to $h\nu(h\nu/m_ec^2) \sim kT_e$, i.e., when the Doppler and recoil effects become comparable. Naturally, the zero-order kernel, P_0, is a very poor approximation in this case, because it totally neglects Compton recoil. This is demonstrated by Figures 11 and 12, which compare accurate (Monte Carlo) single-scattering spectra with the corresponding profiles computed from the renormalized kernels P', P'_0, and P'_K (eqs. [31], [33], and [34]).

The results of our simulations imply that the temperature limits quoted above for the case $h\nu \ll kT_e$ remain valid upon inclusion of quantum effects. We finally conclude that the P kernel can be used in the following range of parameter values: $h\nu(h\nu/m_ec^2) \lesssim kT_e \lesssim$ 25 keV, $h\nu \lesssim 50$ keV. The corresponding limits for the P_K kernel are $h\nu(h\nu/m_ec^2) \lesssim kT_e \lesssim 5$ keV, $h\nu \lesssim 50$ keV.

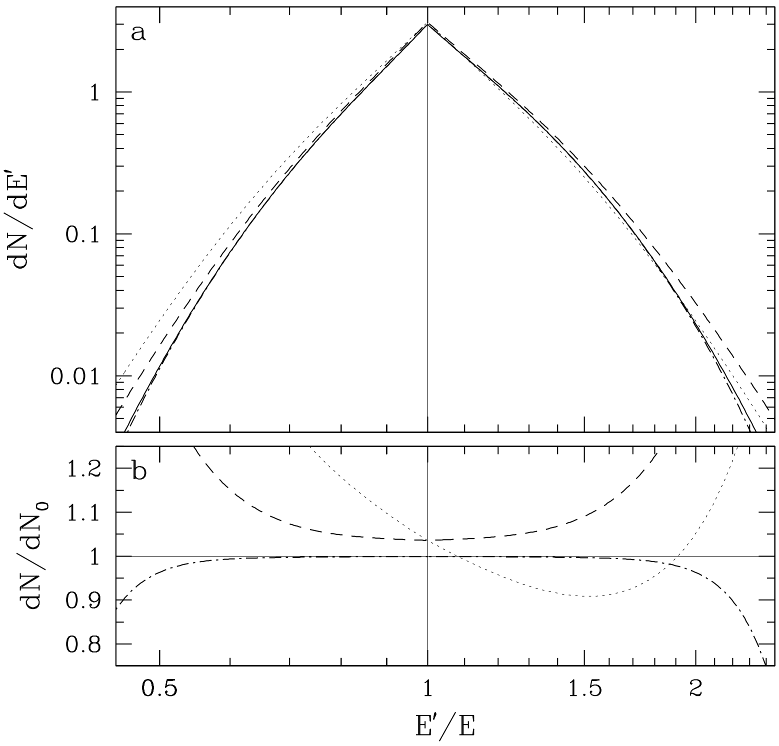

FIG. 8.— (a) Spectrum that forms through single scattering of isotropic monochromatic low-frequency ($h\nu \ll kT_e$) radiation on weakly relativistic thermal electrons, kT_e = 10 keV. The solid line shows the result of an accurate numerical calculation from eq. (89). Also shown are the approximations given by: eq. (22), the zero-order kernel P_0 (*dotted line*); eq. (23), the Kompaneets equation kernel P_K (*dashed line*); and eq. (20), the mildly relativistic kernel P_T (*dash-dotted line*). (b) Ratio of the approximate spectra shown in (a) to the accurate spectrum.

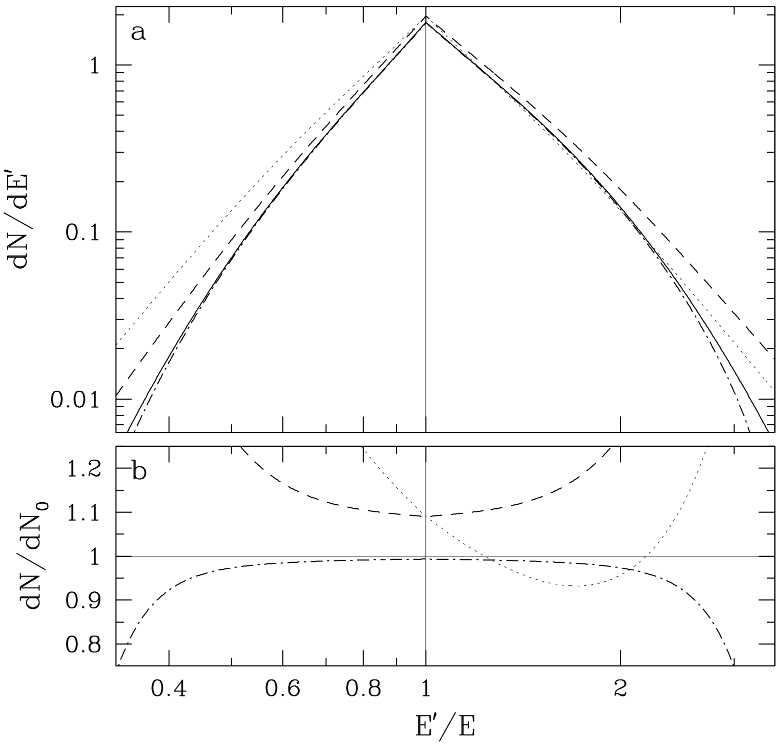

FIG. 9.— Same as Fig. 8, but for a higher temperature, kT_e = 25 keV.

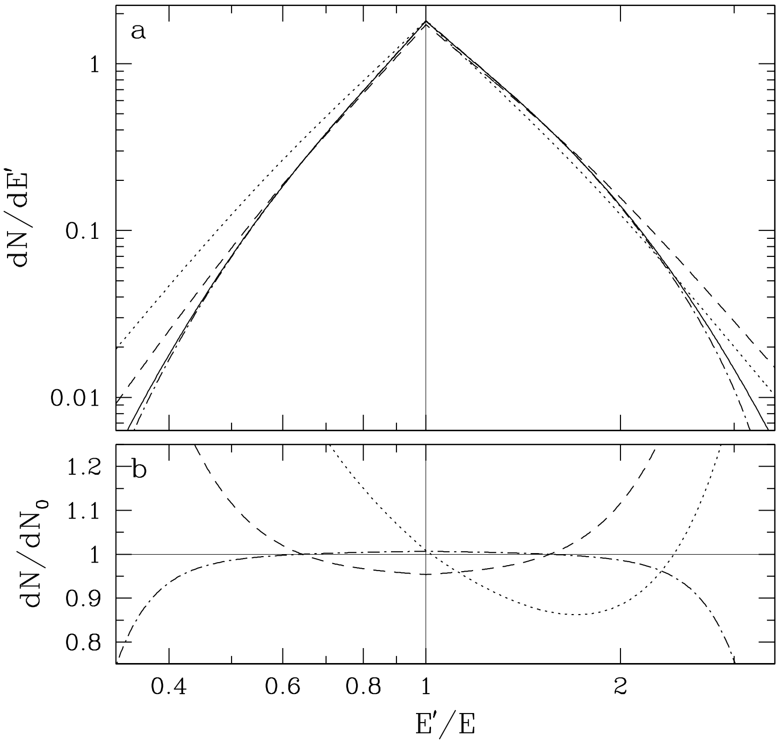

FIG. 10.— Same as Fig. 9, but the renormalized kernels P_0' (eq. [33]), P_K' (eq. [34]), and P_T' (eq. [32]), are used. One can see that the agreement between the approximations and the exact kernel is better than in Fig. 9.

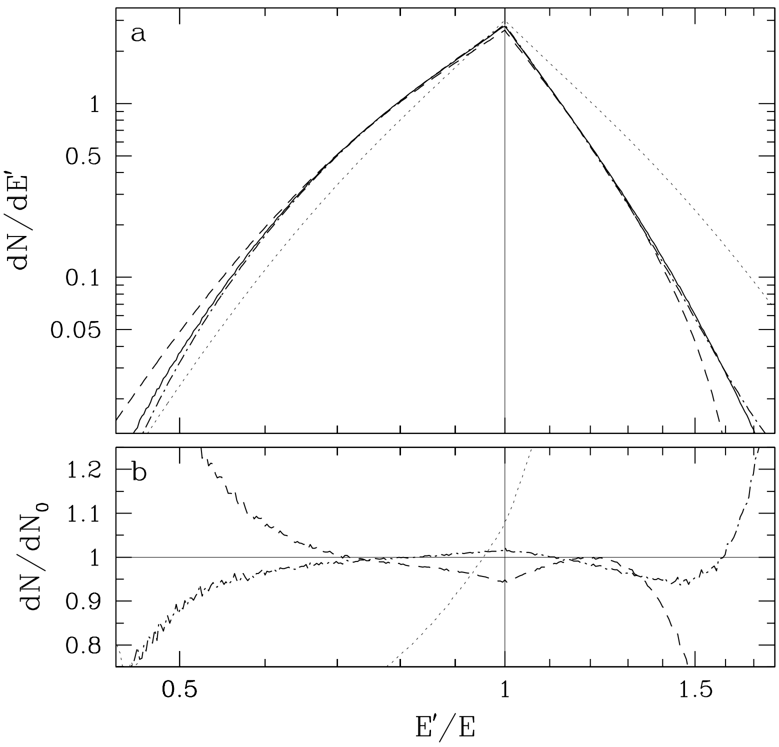

FIG. 11.— (a) Spectrum that forms through single scattering of isotropic monochromatic radiation of high energy, $h\nu$ = 50 keV, on weakly relativistic thermal electrons, kT_e = 10 keV. The result of a Monte Carlo simulation (*solid line*) is compared with the different approximations for the kernel: P_0' (eq. [33], *dotted line*), P_K' (eq. [34], *dashed line*), and P' (eq. [31], *dash-dotted line*). (b) Ratio of the approximate spectra shown in (a) to the accurate (Monte Carlo) spectrum.

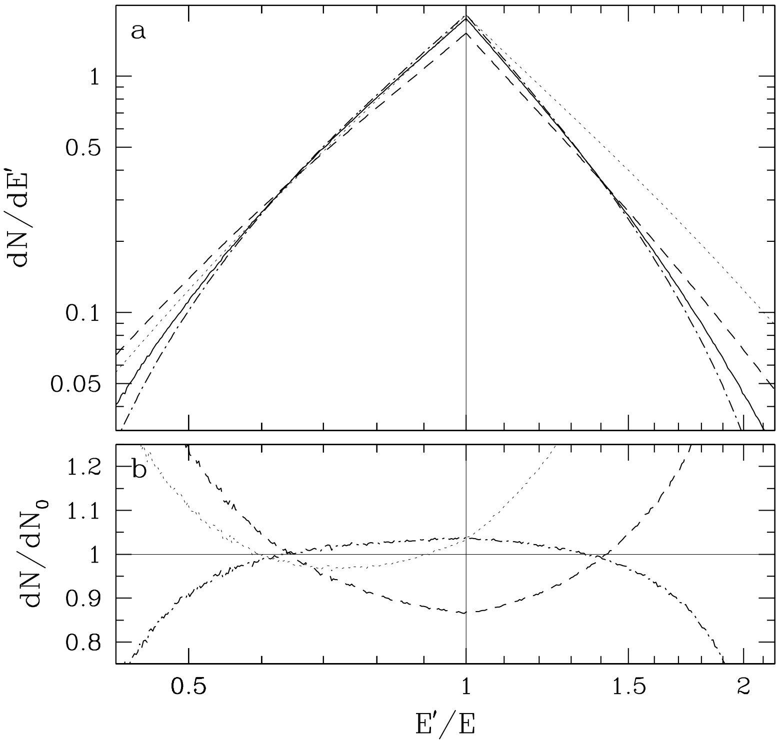

FIG. 12.— Same as Fig. 11 ($h\nu$ = 50 keV), but for a higher electron temperature, kT_e = 25 keV.

At $h\nu(h\nu/m_ec^2) \gtrsim kT_e$, recoil becomes more important than the Doppler effect. The single-scattered profile becomes double-peaked (see, e.g., the results of Monte Carlo simulations of Pozdnyakov et al. 1983). This case requires a special analytical treatment that is beyond the scope of this paper. Here we may only point out the principle mathematical diffuculty that makes it impossible to use our current method for calculating the $P(\nu \to \nu')$ kernel in this limit; the exponential entering the expression for K (eq. [7]) cannot be expanded in powers of $\eta^{1/2}$, as was done for the case $h\nu(h\nu/m_ec^2) \lesssim kT_e$.

Numerous examples of spectra forming by single Compton scattering, which may well be encountered in various astrophysical situations, and which are well approximated by our analytical kernel equation (19), are presented in Figures 13–15.

2.2.4. *Properties of the Single-Scattering Profile*

The line profile forming by single scattering of isotropic radiation on thermal electrons (see Figs. 8–15), which is approximated by equation (19), is unique in its properties. It therefore appears interesting to examine its characteristic features in some detail (following Pozdnyakov et al. 1983).

First, let us compare the line profile as calculated from equation (19) with the usual Gaussian profile that, e.g., may result from Doppler broadening of an emission line in the presence of thermal or turbulent motions of ions. To simplify this comparison, let us assume that $h\nu \ll kT_e$. For a given plasma temperature, it is natural to adopt $\Delta\nu_D = \nu(2\eta)^{1/2}$ for the Doppler shift[1], i.e. $N \sim \exp[-(\nu'-\nu)^2/\Delta\nu_D^2]$. The mean (rms) frequency shift, $\sqrt{\langle(\Delta\nu)^2\rangle}$, is $\nu\eta^{1/2}$ for the Doppler profile. The corresponding value for the Compton-scattered line is $\nu[2\eta(1+23.5\eta)]^{1/2}$ (as results from the value of the second moment given by eq. [25]). The width of the single-scattering profile at half-maximum (FWHM) is approximately equal to $2\nu[\ln 2\eta]^{1/2} = 1.66\nu\eta^{1/2}$. The corresponding value for the Doppler profile is $2\nu[2\ln 2\eta]^{1/2}$, i.e., $\sqrt{2}$ times more, which is opposite to the situation with the rms shift. Thus, in the case of the line forming by Compton scattering, relatively few photons appear in the upper part of the profile (above half-maximum), and an accordingly large fraction of the scattered radiation emerges in the wings of the line. It is also worth noting that the Doppler profile is symmetric, while the profile due to Compton scattering is not.

Now let us consider the peak of the single-scattering profile, a detail which makes it so peculiar. In the vicinity of the maximum ($|\nu'-\nu| \ll \nu\eta^{1/2}$), the spectrum is well approximated by the following expression, which results from equation (19):

$$P(\nu \to \nu')_{+,-} = \nu^{-1}\frac{11}{20}\sqrt{\frac{2}{\pi}}\eta^{-1/2}\left\{\left[1+\left(-\frac{1091}{616}-\frac{23}{154}\left(\frac{h\nu}{kT_e}\right)^2\right)\eta\right]\right.$$
$$\left.+\frac{\nu'/\nu-1}{\eta^{1/2}}\left[\mp\frac{15}{22}\sqrt{\frac{\pi}{2}}+\frac{1}{2}\left(1-\frac{h\nu}{kT_e}\right)\eta^{1/2}+\dots\right]+\dots\right\}, \quad (35)$$

[1]Note that the width used, $\Delta\nu_D$, is $(M/m_e)^{1/2} = 43(M/m_p)^{1/2}$ times the thermal width of lines of an ion of mass M.

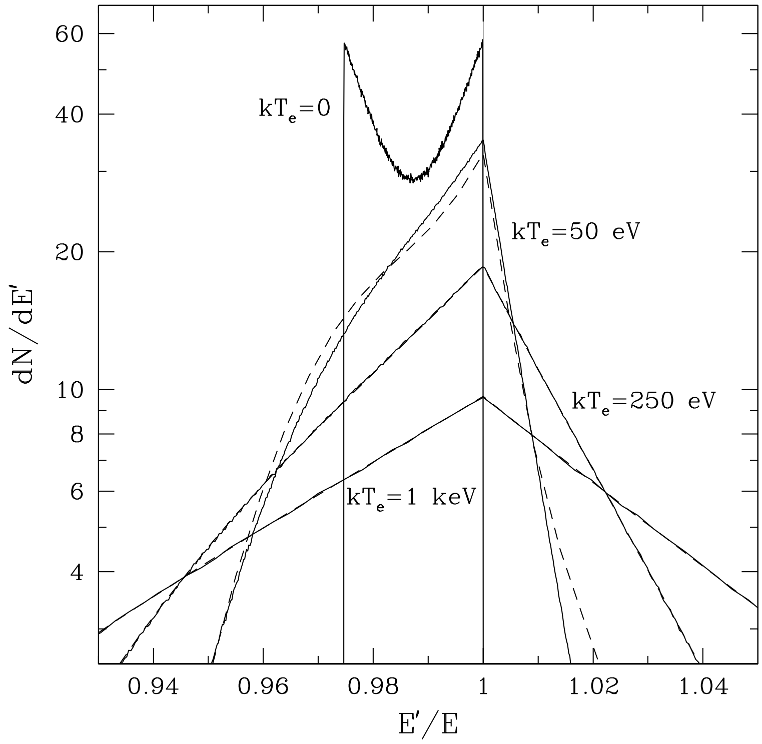

FIG. 13.— Spectra resulting from the single scattering of isotropic monochromatic radiation of energy $h\nu$ = 6.7 keV on low-temperature ($h\nu > 4kT_e$) thermal electrons, for different values of kT_e. In this case, the Compton-recoil shift is larger than the Doppler shift. The results of Monte Carlo simulations (*solid lines*) are compared with the results of the calculation by the approximate eq. (31) for the mildly relativistic kernel P' (*dashed lines*). For the case $kT_e = 0$ (cold electrons), only the Monte Carlo result (*double-peaked profile*) is shown, since our approximation for the kernel is not valid in this case.

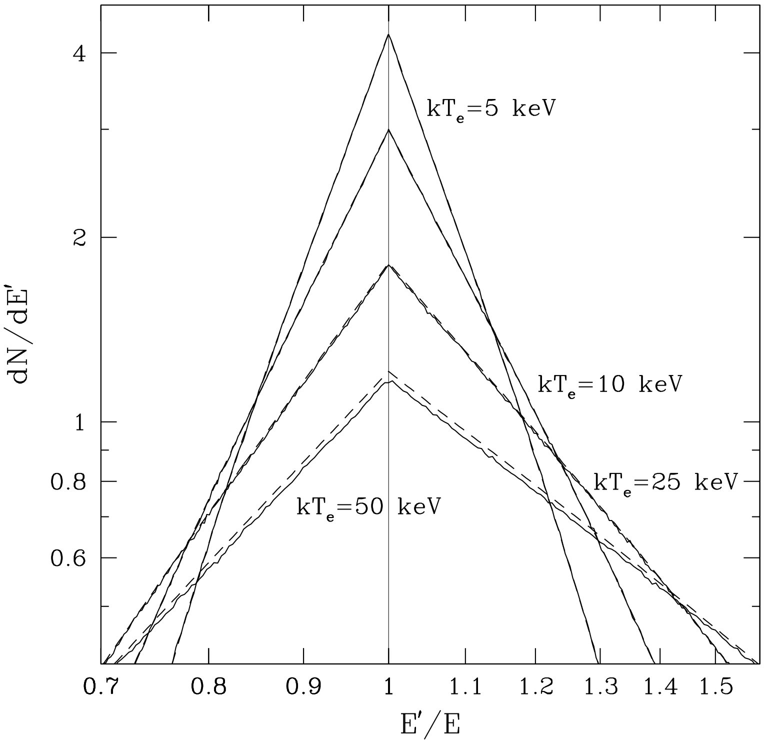

FIG. 14.— Same as Fig. 13 ($h\nu$ = 6.7 keV), but for high-temperature electrons, $h\nu < 4kT_e$. In this case, the Doppler shift is larger than the recoil shift.

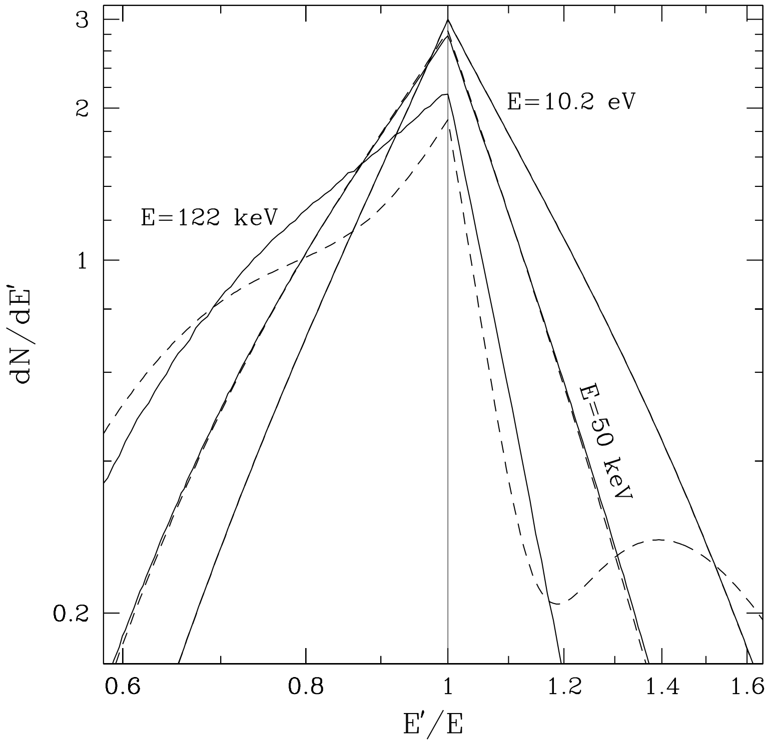

FIG. 15.— Spectra resulting from the single scattering of isotropic monochromatic radiation on weakly relativistic electrons, kT_e = 10 keV, for different photon energies. The results of Monte Carlo simulations (*solid lines*) are compared with the results of the calculation by the approximate eq. (31) for the mildly relativistic kernel P' (*dashed lines*). One can observe how the effect of Compton recoil on the spectrum increases as the photon energy becomes higher. The case of the 122 keV nuclear line produced by ^{57}Co is beyond the scope of our analytical approximation for the isotropic kernel.

where the indices plu and minus signs correspond to the right and left wings, respectively.

We see that the spectrum has a cusp at $\nu' = \nu$ (a break in the derivative occurs there). Near the cusp, on its both sides, the spectrum can be approximated as a power law, the slopes in the right and left wings [the coefficient at $(\nu'/\nu - 1)$ in eq. (35)] being significantly different:

$$\begin{aligned} \alpha_+ &= -\left[\frac{d\ln P}{d\ln(\nu'/\nu)}\right]_{\nu'=\nu+0} = \frac{15}{22}\sqrt{\frac{\pi}{2}}\eta^{-1/2} - \frac{1}{2} + \frac{1}{2}\frac{h\nu}{kT_e}, \\ \alpha_- &= \left[\frac{d\ln P}{d\ln(\nu'/\nu)}\right]_{\nu'=\nu-0} = \frac{15}{22}\sqrt{\frac{\pi}{2}}\eta^{-1/2} + \frac{1}{2} - \frac{1}{2}\frac{h\nu}{kT_e}, \qquad \alpha_- - \alpha_+ = 1 - \frac{h\nu}{kT_e} \end{aligned} \tag{36}$$

(in Pozdnyakov et al. 1983, $\alpha_- - \alpha_+ = 3$ when $h\nu = 0$, because they considered the energy spectrum that is the product of ν'/ν and the photon spectrum considered here).

It is interesting that when $h\nu = kT_e$, the line profile in the vicinity of the cusp is symmetric (in logarithmic coordinates) about $\nu' = \nu$ ($\alpha_+ = \alpha_-$). Let us give a few further examples. If $h\nu = 0$ and $\eta = 0.001$, 0.01, and 0.1, the slopes run: $\alpha_{+,-} = 27.0 \mp 0.5$, 8.5 ∓ 0.5, and 2.7 ∓ 0.5. For $h\nu/m_ec^2 = 0.1$ and $\eta = 0.02$ (the case of Figure 11): $\alpha_+ = 8.0$, $\alpha_- = 4.1$, and $\alpha_- - \alpha_+ = -3.9$ (for $h\nu = 0$ and $\eta = 0.02$, we would have $\alpha_+ = 5.5$, $\alpha_- = 6.6$, and $\alpha_- - \alpha_+ = 1.0$).

The asymmetry of the single-scattering profile can be demonstrated by comparing the fractions of the scattered radiation contained in the right ($\nu' > \nu$) and left ($\nu' < \nu$) wings of the line. Using the renormalized kernel given by equations (31) and (19), we find in terms of number of photons ($N = \int P'\,d\nu'$) and total energy ($W = \int h\nu' P'\,d\nu'$),

$$\begin{aligned} N_{+,-} = \frac{1}{2} \pm \eta^{1/2}\sqrt{\frac{2}{\pi}}\left[\frac{69}{70} - \frac{23}{70}\frac{h\nu}{kT_e}\right] - \frac{h\nu}{m_ec^2} \pm \eta^{3/2}\sqrt{\frac{2}{\pi}}\left[-\frac{1577}{1680} - \frac{4061}{1680}\frac{h\nu}{kT_e}\right. \\ \left. + \frac{43}{84}\left(\frac{h\nu}{kT_e}\right)^2 + \frac{43}{1260}\left(\frac{h\nu}{kT_e}\right)^3\right] + \eta^2\left[-\frac{5}{2}\frac{h\nu}{kT_e} + \frac{13}{5}\left(\frac{h\nu}{kT_e}\right)^2\right], \end{aligned} \tag{37}$$

$$W_{+,-} = \left\{\frac{1}{2} \pm \eta^{1/2}\sqrt{\frac{2}{\pi}}\left[\frac{23}{14} - \frac{23}{70}\frac{h\nu}{kT_e}\right] + \eta\left[2 - \frac{3}{2}\frac{h\nu}{kT_e}\right] \pm \eta^{3/2}\sqrt{\frac{2}{\pi}}\left[\frac{187}{48} - \frac{521}{80}\frac{h\nu}{kT_e}\right.\right.$$

$$+\frac{43}{6}\left(\frac{h\nu}{kT_e}\right)^2+\frac{43}{1260}\left(\frac{h\nu}{kT_e}\right)^3\Bigg]+\eta^2\left[5-\frac{57}{4}\frac{h\nu}{kT_e}+\frac{47}{10}\left(\frac{h\nu}{kT_e}\right)\right]\Bigg\}h\nu. \tag{38}$$

The renormalization is important here because it enables us to obtain the terms $O(\eta^2)$, $O(\eta h\nu/m_ec^2)$, and $O((h\nu/m_ec^2)^2)$ in equations (37) and (38). This procedure is strictly correct for the following reason. The terms of even orders in $\eta^{1/2}$ in the power series for the $P(\nu\to\nu')$ kernel (see eq. [19]), i.e. p_0, p_t, p_r and analogous (unknown) terms of higher orders, are symmetric in frequency variation. Therefore, if we know (and we indeed do) the contribution of such a term, say $O(\eta^2)$, to the normalization of the accurate kernel, we immediately know that the contribution of this term to both N_+ and N_- is equal to half this value. $W_{+,-}$ are then determined as $\int h(\nu+\nu'-\nu)P'\,d\nu' = N_{+,-}h\nu+\int h(\nu'-\nu)P'\,d\nu'$. The last integral is accurate to within η^2, $\eta h\nu/m_ec^2$, and $(h\nu/m_ec^2)^2$ because of the presence of the small factor $\nu'-\nu$.

An interesting conclusion can be drawn from both equation (38) and the above discussion: the left wing contributes to the total accumulation of energy by the photons exactly as much as the right wing.

2.2.5. *Single Scattering of a Step-Function Spectrum*

Although the present paper is mainly devoted to the case of single Compton scattering of narrow spectral lines, our basic formulae — equation (7) for the direction-dependent problem and equation (19) for the isotropic problem — describe the kernels of the corresponding integral kinetic equations (eqs. [2] and [3]) appearing in the general problem of Comptonization in thermal plasmas. Let us present a simple example of using the $P(\nu\to\nu')$ kernel. Consider the scattering of an isotropic photon distribution described by the step function, i.e., $dN_0/d\nu=1$ if $\nu\le\nu_0$ and $dN_0/d\nu=0$ if $\nu>\nu_0$, in an optically thin (so that multiple scatterings are not important), hot plasma. The spectrum of the scattered photons can be found by convolving the initial frequency distribution with the kernel:

$$\frac{dN(\nu)}{d\nu}=\tau\int\frac{dN_0(\nu')}{d\nu}P(\nu'\to\nu)\,d\nu', \tag{39}$$

where $\tau\ll 1$ is the Thomson optical depth of the scattering medium.

Figure 16 presents examples of spectra of the scattered component as calculated from equation (39) using different analytical approximations for the kernel: P_0' (eq. [33]), P_K' (eq. [34]), and P' (eq. [31]). The radiation is assumed to be low-frequency ($h\nu_0\ll kT_e$). In one case, $kT_e=1$ keV, a spectrum forming in the case of non-negligible photon energy ($h\nu_0=7.1$ keV) is also shown. The integral in equation (39) was performed numerically. Note that the actual observable spectrum will be the sum of the unscattered and scattered components, and will depend on τ, i.e., $(dN/d\nu)_{\rm total}=(1-\tau)dN_0/d\nu+\tau dN/d\nu$.

The spectra shown in Figure 16 allow analytical description. We present here only the result of the convolution of the step function with the Kompaneets equation kernel, P_K:

$$\frac{dN}{d\nu}=\begin{cases}(f_0+f_1)\tau, & \nu>\nu_0\\ (1-f_0+f_1)\tau, & \nu\le\nu_0,\end{cases} \tag{40a}$$

where

$$\begin{aligned}f_0&=\frac{1}{\sqrt{\pi}}\left[\left(-\frac{6}{5}\delta_0-\frac{13}{15}\delta_0^3-\frac{4}{15}\delta_0^5\right)F+\left(1+3\delta_0^2+2\delta_0^4+\frac{8}{15}\delta_0^6\right)G\right],\\ f_1&=\sqrt{\frac{2}{\pi}}\left[\left(\frac{23}{70}-\frac{27}{35}\delta_0^2-\frac{24}{35}\delta_0^4-\frac{8}{35}\delta_0^6\right)F+\left(2\delta_0^3+\frac{8}{5}\delta_0^5+\frac{16}{35}\delta_0^7\right)G\right]\eta^{1/2}\left(1-\frac{h\nu_0}{kT_e}\right),\\ F&=\exp(-\delta_0^2),\ G=\int_{\delta_0}^{\infty}\exp(-t^2)\,dt=0.5\pi^{1/2}Erfc(\delta_0),\ \delta_0=(2\eta)^{-1/2}\frac{|\nu-\nu_0|}{\nu_0+\nu}.\end{aligned} \tag{40b}$$

The main term in equation (40a) — $f_0\tau$ (if $\nu>\nu_0$) or $(1-f_0)\tau$ (if $\nu\le\nu_0$) — results from the zero-order kernel, P_0 (Hummer & Mihalas 1967).

The spectrum of the scattered component at frequencies $\nu>\nu_0$ has a quasi-power-law shape, with a slope that is approximately equal to the slope of the right wing of the kernel itself (see eq. [36]):

$$\alpha=\frac{11}{10}\sqrt{\frac{2}{\pi}}\eta^{-1/2}-\frac{253}{175\pi}\left(1-\frac{h\nu_0}{kT_e}\right). \tag{41}$$

As is the case with the kernel itself, the first-order temperature correction causes the spectrum of the scattered component to be flatter than results from the zero-order approximation. Compton recoil has an opposite effect on the slope and may cause a significant steepening of the spectrum when $h\nu_0\gg kT_e$. It is clear from Figure 16 and equation (41) that in the low-frequency case, it is possible to determine the temperature of the scattering plasma by measuring the slope of the spectrum of the scattered component.

Let us also give an expression for the total number of photons that have been redistributed from the frequency region $\nu\le\nu_0$ into the region $\nu>\nu_0$:

$$N=\left[\frac{23}{35}\sqrt{\frac{2}{\pi}}+\left(1-\frac{h\nu_0}{m_ec^2}\right)\eta^{1/2}\right]\nu_0\eta^{1/2}\tau. \tag{42}$$

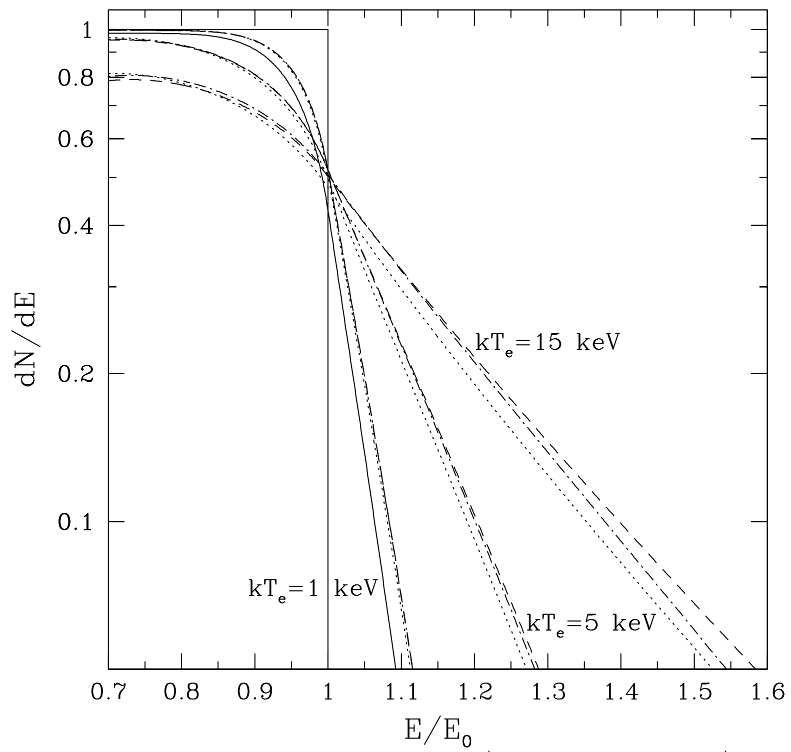

FIG. 16.— Single scattering of a spectrum described by the step function — $dN_0/dE = 1$ if $E \leq E_0$ and $dN_0/dE = 0$ if $E > E_0$ — on mildly relativistic thermal electrons, considered for various electron temperatures, assuming that the photon energy is negligible ($E_0 \ll kT_e$). The spectra resulting from the convolution (eq. [39]) of the step function with the following approximations for the isotropic kernel are shown: the zero-order kernel P'_0 (eq. [33], *dotted lines*), the Kompaneets equation kernel P'_K (eq. [34], *dashed lines*), and the mildly relativistic kernel P' (eq. [31], *dash-dotted lines*). For the case of $kT_e = 1$ keV, also shown is the spectrum (corresponding to the P' kernel) for a nonnegligible photon energy, $E_0 = 7.1$ keV (*solid line*). Note that the actual observable spectrum will be the sum of the nonscattered and scattered components, i.e., $(dN/dE)_{\rm total} = (1-\tau)dN_0/dE + \tau dN/dE$, where $\tau \ll 1$ is the Thomson optical depth of the scattering medium.

We can also imagine a situation in which a spectrum described by the left-side step function: $dN_0/d\nu = 1$ if $\nu \geq \nu_0$ and $dN_0/d\nu = 0$ if $\nu < \nu_0$, is being scattered. In this case, we arrive at a formula that is similar to equation (40):

$$\frac{dN}{d\nu} = \begin{cases} (f_0 - f_1)\tau, & \nu < \nu_0 \\ (1 - f_0 - f_1)\tau, & \nu \geq \nu_0. \end{cases} \tag{43}$$

We see that here the asymmetry of the kernel has the opposite effect on the scattered spectrum as compared with the previous situation, in which the right-side step function was considered (compare the signs of the term f_1 in eqs. [40a] and [43]), namely, the temperature correction causes the spectrum to steepen, while the recoil correction makes the slope flatter.

2.3. *Fokker-Planck Expansion of the Integral Kinetic Equation*

We can carry out a Fokker-Planck expansion (eq. [4]) of the kinetic equation (3) with the $P(\nu \to \nu')$ kernel derived in this paper (eq. [19]). The coefficients in equation (4) depend on the moments of the kernel, which are given by equation (25). As a result, we obtain the generalized (for the mildly relativistic case) Kompaneets equation:

$$\frac{\partial n(\nu)}{\partial \tau} = \frac{h}{m_e c^2} \frac{1}{\nu^2} \frac{\partial}{\partial \nu} \nu^4 \left\{ n(1+n) + \frac{kT_e}{h} \frac{\partial n}{\partial \nu} + \frac{7}{10} \frac{h\nu^2}{m_e c^2} \frac{\partial n}{\partial \nu} + \frac{kT_e}{m_e c^2} \left[\frac{5}{2} \left(n(1+n) + \frac{kT_e}{h} \frac{\partial n}{\partial \nu} \right) \right. \right.$$
$$\left. \left. + \frac{21}{5} \nu \frac{\partial}{\partial \nu} \left(n(1+n) + \frac{kT_e}{h} \frac{\partial n}{\partial \nu} \right) + \frac{7}{10} \nu^2 \left(-2 \left(\frac{\partial n}{\partial \nu} \right)^2 + 2(1+2n) \frac{\partial^2 n}{\partial \nu^2} + \frac{kT_e}{h} \frac{\partial^3 n}{\partial \nu^3} \right) \right] \right\}. \tag{44}$$

This equation was earlier obtained, in a different way, by Itoh et al. (1998) and Challinor & Lasenby (1998). Substituting the Planckian distribution that corresponds to the temperature of the electrons, $n = (e^{h\nu/kT_e} - 1)^{-1}$, into equation (44) yields $\partial n(\nu)/\partial \tau = 0$. This test confirms once more that equation (19) for the kernel consistently takes into account necessary corrections. One can also make sure that equation (44) without the terms responsible for induced scattering does not modify a Wien spectrum, $n = e^{-h\nu/kT_e}$. The fact that equation (44) conserves the total number of photons follows directly from its divergent form. Thus, as expected, the basic properties of the Kompaneets equation are retained in equation (44).

Equation (44) can be simplified in the case $h\nu \ll kT_{\rm e}$. Omitting the terms related to quantum effects and induced scattering, we find

$$\frac{\partial n(\nu)}{\partial \tau} = \frac{kT_{\rm e}}{m_{\rm e}c^2}\frac{1}{\nu^2}\frac{\partial}{\partial \nu}\nu^4\left[\frac{\partial n}{\partial \nu}+\frac{kT_{\rm e}}{m_{\rm e}c^2}\left(\frac{5}{2}\frac{\partial n}{\partial \nu}+\frac{21}{5}\nu\frac{\partial^2 n}{\partial \nu^2}+\frac{7}{10}\nu^2\frac{\partial^3 n}{\partial \nu^3}\right)\right]. \tag{45}$$

This equation allows one to derive the first-order relativistic corrections to the effect of distortion of the CMB spectrum in clusters of galaxies with hot gas (Itoh et al. 1998; Challinor & Lasenby 1998; see another way of finding these analytical corrections in Sazonov & Sunyaev 1998).

In another limit, $h\nu(h\nu/m_{\rm e}c^2) \gg kT_{\rm e}$, when the Doppler effect is not important, one obtains (ignoring induced-scattering terms)

$$\frac{\partial n(\nu)}{\partial \tau} = \frac{h}{m_{\rm e}c^2}\frac{1}{\nu^2}\frac{\partial}{\partial \nu}\nu^4\left(n+\frac{7}{10}\frac{h\nu^2}{m_{\rm e}c^2}\frac{\partial n}{\partial \nu}\right). \tag{46}$$

The second parenthesized term in this equation, which describes frequency diffusion of photons, was added to the Kompaneets equation by Ross et al. (1978) and Illarionov et al. (1979). This correction becomes especially important when studying the scattering of hard radiation on cold electrons in an optically thick medium. Such a situation takes place, for example, during a supernova explosion; an analytical solution to the corresponding diffusion problem was derived and employed in a calculation of the evolution of the X-ray spectrum of Supernova 1987A by Grebenev & Sunyaev (1987).

In the intermediate case of $h\nu \gg kT_{\rm e}$, $h\nu(h\nu/m_{\rm e}c^2) \ll kT_{\rm e}$, the dispersion term $(kT_{\rm e}/m_{\rm e}c^2)\nu^{-2}\partial/\partial\nu(\nu^4\partial n/\partial\nu)$ of the Kompaneets equation, which describes the diffusion of the photons due to the Doppler effect, must be added to equation (46).

2.4. *Kinetic Equation for Problems with a Decisive Role of Induced Compton Scattering*

If the conditions $n \gg 1$ and $n \gg kT_{\rm e}/h\nu$ are both satisfied, then equation (44) will simplify to (only induced-scattering terms have been retained)

$$\frac{\partial n(\nu)}{\partial \tau} = \frac{h}{m_{\rm e}c^2}\frac{1}{\nu^2}\frac{\partial}{\partial \nu}\nu^4\left\{n^2+\frac{kT_{\rm e}}{m_{\rm e}c^2}\left[\frac{5}{2}n^2+\frac{42}{5}\nu n\frac{\partial n}{\partial \nu}+\frac{14}{5}\nu^2 n\frac{\partial^2 n}{\partial \nu^2}-\frac{7}{5}\nu^2\left(\frac{\partial n}{\partial \nu}\right)^2\right]\right\}. \tag{47}$$

It is interesting that only correction terms that are proportional to $kT_{\rm e}/m_{\rm e}c^2$ appear in this equation. There is no term proportional to $h\nu/m_{\rm e}c^2$, although such a term is present in the diffusion equation describing the spontaneous scattering process, equation (46). This is a result of the joint operation of Compton recoil and Klein-Nishina corrections, which both contribute to the first two moments of the kernel (eq. [25]).

In the past, many phenomena caused by induced Compton scattering were investigated in the nonrelativistic approximation, using only the main term in equation (47). One such phenomenon is distortions in the low-frequency radiation spectra of radio sources (Sunyaev 1971), which become large if $kT_b = 0.5I_\nu\lambda^2 \gg m_{\rm e}c^2/\tau(1+\tau)$, where I_ν is the intensity of quasi-isotropic radiation at a wavelength λ and τ is the Thomson optical depth of the scattering cloud. Particularly interesting is the case of bright extragalactic radio sources, for which $kT_b \gg m_{\rm e}c^2/\tau$ even though $\tau \ll 1$, because $T_b \sim 10^{11} \div 10^{13}$ K. Other phenomena include plasma heating (Peyraud 1968; Zeldovich & Levich 1970; Levich & Sunyaev 1971; Vinogradov & Pustovalov 1972; Blandford & Scharlemann 1976; Illarionov & Kompaneets 1977) and induced light-pressure force (Levich et al. 1972) in the vicinity of astrophysical objects emitting low-frequency radiation with high T_b. Obviously, relativistic corrections, described by equation (47), could play an important role in such problems. In particular, these terms (although small) should play the role of viscosity for such phenomena as the formation of shock waves in the photon spectrum during Bose-condensation of photons (Zeldovich & Sunyaev 1972).

Induced Compton scattering may also lead to essentially anisotropic effects, such as narrowing or spreading (depending on the spectrum of the radiation) of a radiation beam traversing a plasma (Goldin et al. 1975; Zeldovich & Sunyaev 1976). It may also play a major role in the interaction of beams of maser radiation having narrow spectra ($\Delta\nu < \Delta\nu_D$, where $\Delta\nu_D = \nu[2(1-\mu_{\rm s})kT_{\rm e}/m_{\rm e}c^2]^{1/2}$ is the Doppler shift) (Zeldovich et al. 1972). Let us write (for an infinite homogeneous medium) the integral kinetic equation that arises in such problems,

$$\begin{aligned}\frac{\partial n(\nu,\mathbf{\Omega},\tau)}{\partial \tau} &= n(\nu,\mathbf{\Omega},\tau)\int n(\nu',\mathbf{\Omega'},\tau)\left[\left(\frac{\nu'}{\nu}\right)^2 K(\nu',\mathbf{\Omega'}\to\nu,\mathbf{\Omega})-K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})\right]d\nu' d\mathbf{\Omega'}\\ &= n(\nu,\mathbf{\Omega},\tau)\int n(\nu',\mathbf{\Omega'},\tau)K^{\rm ind}(\nu,\mathbf{\Omega};\nu',\mathbf{\Omega'})\,d\nu' d\mathbf{\Omega'},\end{aligned} \tag{48}$$

where $n(\nu)$ is the occupation number in photon phase space, and we have introduced the new kernel $K^{\rm ind}(\nu,\mathbf{\Omega};\nu',\mathbf{\Omega'})$. In the limit $h\nu \ll m_e c^2$, $h\nu \ll kT_{\rm e}$, which always takes place in the case of compact sources of low-frequency radiation, a particularly simple expression for this kernel can be given. Indeed, we can expand the exponential in the expression (7a) according to equation (72) of §5, and then, taking the second term in the resulting series out of the exponential, get

$$\begin{aligned}K^{\rm ind}(\nu,\mathbf{\Omega};\nu',\mathbf{\Omega'}) &= \frac{3}{16\pi}\sqrt{\frac{2}{\pi}}\eta^{-3/2}\frac{h\nu'^2(\nu'-\nu)}{m_{\rm e}c^2g^3}(1-\mu_{\rm s})\left[1+\mu_{\rm s}^2+\left(\frac{1}{8}-\mu_{\rm s}-\frac{63}{8}\mu_{\rm s}^2+5\mu_{\rm s}^3\right)\eta\right.\\ &\left.-\mu_{\rm s}(1-\mu_{\rm s}^2)\left(\frac{\nu'-\nu}{g}\right)^2-\frac{3(1+\mu_{\rm s}^2)}{8\eta}\left(\frac{\nu'-\nu}{g}\right)^4\right]\exp\left[-\frac{(\nu'-\nu)^2}{2g^2\eta}\right],\\ g &= |\nu\mathbf{\Omega}-\nu'\mathbf{\Omega'}| = (\nu^2-2\nu\nu'\mu_{\rm s}+\nu'^2)^{1/2}.\end{aligned} \tag{49}$$

Equation (49), like the formula in equation (7) for the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel, can be used when the plasma is mildly relativistic, $kT_{\rm e}\lesssim 0.1m_{\rm e}c^2$. In the case of nonrelativistic electrons, $kT_{\rm e}\ll m_{\rm e}c^2$, the correction terms in equation (49) can be neglected, and ν' can be replaced by ν everywhere except in the difference $\nu'-\nu$, which results in the following formula

$$K^{\rm ind}_{\rm nr}(\nu,\mathbf{\Omega};\nu',\mathbf{\Omega'})=\frac{3}{32}(\pi\eta)^{-3/2}\nu^{-1}\frac{h(\nu'-\nu)}{m_{\rm e}c^2}(1-\mu_{\rm s})^{-1/2}(1+\mu_{\rm s}^2)\exp\left[-\frac{(\nu'-\nu)^2}{4\nu^2(1-\mu_{\rm s})\eta}\right], \tag{50}$$

which was earlier obtained by Zeldovich et al. (1972).

Integrating the kernel (eq. [49]) over all scattering angles gives the corresponding kernel for the isotropic problem. This kernel is, however, more easily deduced from equation (19):

$$P^{\rm ind}(\nu;\nu')=\left(\frac{\nu'}{\nu}\right)^2P(\nu'\to\nu)-P(\nu\to\nu')=\sqrt{\frac{2}{\pi}}\eta^{-3/2}\frac{h(\nu^2+\nu'^2)(\nu'-\nu)}{m_{\rm e}c^2\nu^2(\nu+\nu')}(p_0+p_t), \tag{51}$$

where p_0 and p_t are given by equation (19b).

The kernel given in equation (51) is applicable in the limit $h\nu\ll kT_{\rm e}\lesssim 0.1m_{\rm e}c^2$ and makes possible to obtain the differential equation (47).

3. DERIVATION OF THE $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ KERNEL

Consider a photon of frequency ν that propagates in the direction $\mathbf{\Omega}$. We calculate the probability (per unit dimensionless time, τ) that the photon will be scattered by a Maxwellian distribution of electrons into a solid-angle interval $d\mathbf{\Omega'}$, with the emergent photon frequency falling in an interval $d\nu'$.

Our primary goal is to derive a formula that would be a good approximation for situations in which both the electrons and photons are mildly relativistic, i.e., when $\eta=kT_{\rm e}/m_{\rm e}c^2, h\nu/m_{\rm e}c^2\sim 0.1$. Therefore, we make an initial assumption that $\eta, h\nu/m_{\rm e}c^2\ll m_{\rm e}c^2$. The final formula will contain correction terms of order up to $\eta^{3/2}$, $\eta^{1/2}h\nu/m_{\rm e}c^2$, and $(h\nu/m_{\rm e}c^2)^2$. The term of order $(h\nu/m_{\rm e}c^2)^2$ originates directly from the Klein-Nishina formula for the scattering cross section. We retain this term in our final expression on purpose (although it causes the formula to be of a slighly better accuracy in terms of photon energy that in terms of electron temperature), because this expression without the temperature-correction terms then becomes the *exact formula* for the case of scattering of photons of arbitrary (including $h\nu\gg m_{\rm e}c^2$) energy on nonrelativistic electrons ($\eta\ll 1$).

Let us introduce the following system of reference: **Ox** points along $\mathbf{\Omega}$, **Oy** is in the ($\mathbf{\Omega}$, $\mathbf{\Omega'}$) plane, **Oz** is normal to this scattering plane. There are two basic equations. The first is the energy-conservation relation (e.g., Pozdnyakov et al. 1983):

$$\frac{\nu'}{\nu}=\frac{1-\mathbf{\Omega}\boldsymbol{\beta}}{1-\mathbf{\Omega'}\boldsymbol{\beta}+(h\nu/\gamma mc^2)(1-\cos\alpha)}, \tag{52}$$

where $c\boldsymbol{\beta}$ is the electron velocity and $\cos\alpha=\mathbf{\Omega}\mathbf{\Omega'}$. The second equation describes the differential cross section for Compton scattering (Jauch & Rohrlich 1976; Beresteskii et al. 1982):

$$\frac{d\sigma}{d\mathbf{\Omega'}}=\frac{3\sigma_{\rm T}}{16\pi\gamma^2}\frac{X}{(1-\mathbf{\Omega}\boldsymbol{\beta})^2}\left(\frac{\nu'}{\nu}\right)^2, \tag{53}$$

where

$$X=2-\frac{2(1-\cos\alpha)}{\gamma^2(1-\mathbf{\Omega}\boldsymbol{\beta})(1-\mathbf{\Omega'}\boldsymbol{\beta})}+\frac{(1-\cos\alpha)^2}{\gamma^4(1-\mathbf{\Omega}\boldsymbol{\beta})^2(1-\mathbf{\Omega'}\boldsymbol{\beta})^2}+\frac{\nu'}{\nu}\left(\frac{h\nu}{m_{\rm e}c^2}\right)^2\frac{(1-\cos\alpha)^2}{\gamma^2(1-\mathbf{\Omega}\boldsymbol{\beta})(1-\mathbf{\Omega'}\boldsymbol{\beta})}. \tag{54}$$

Introducing the components of the electron velocity (β_x, β_y, β_z), we find

$$\mathbf{\Omega}\boldsymbol{\beta}=\beta_x,\ \ \mathbf{\Omega'}\boldsymbol{\beta}=\beta_x\cos\alpha+\beta_y\sin\alpha. \tag{55}$$

Equation (52) imposes a link between the different $\boldsymbol{\beta}$ components for given ν'/ν and α:

$$\beta_y=\frac{1}{\sin\alpha}\left[\beta_x\left(\frac{\nu}{\nu'}-\cos\alpha\right)+1-\frac{\nu}{\nu'}+\frac{h\nu}{\gamma m_{\rm e}c^2}(1-\cos\alpha)\right], \tag{56}$$

with

$$\gamma^2=\frac{1}{1-(\beta_x^2+\beta_y^2+\beta_z^2)}. \tag{57}$$

In order to calculate $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$, we ought to carry out the following integration over electron velocities:

$$K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})\equiv\frac{dP}{d\tau\, d\Omega'\, d\nu'}=\frac{1}{\sigma_{\rm T}}\int\frac{d\sigma}{d\mathbf{\Omega'}}(1-\mathbf{\Omega}\boldsymbol{\beta})f_\beta(\beta_x,\beta_y,\beta_z)\left|\frac{\partial\beta_y}{\partial\nu'}\right|d\beta_x d\beta_z. \tag{58}$$

Here, the factor $(1-\mathbf{\Omega}\boldsymbol{\beta})$ allows for the relative velocity of the photon and electron along the direction of the former's motion (Landau & Lifshitz 1975), $f_\beta(\beta_x,\beta_y,\beta_z)$ is the electron velocity distribution function. In equation (58), one of the velocity components (say,

β_y) must be expressed through the other two. The interval, $d\nu'$, of the photon frequency after scattering can then be related with the corresponding interval of β_y; hence the appearence of the factor $|\partial\beta_y/\partial\nu'|$ in equation (58).

For f_β we substitute the relativistic Maxwellian distribution function,

$$f_\beta = (2\pi\eta)^{-3/2}\left(1+\frac{15}{8}\eta\right)^{-1}\left(1+\frac{5}{2}\beta^2-\frac{3}{8}\frac{\beta^4}{\eta}\right)\exp\left(-\frac{\beta^2}{2\eta}\right), \tag{59}$$

Here we have retained only correction terms of order η.

We can proceed with the integration in equation (58) on expanding the integrand in powers of β. The correct account of temperature terms of order β^2 and β^3 necessitates inclusion of the corresponding terms in equation (59) for the velocity distribution (contrary to the statement made in BR70). We also note that the terms of order β^3, $\beta h\nu/m_ec^2$, and $(h\nu/m_ec^2)^2$, which we are keep throughout, were neglected altogether in the derivation of BR70.

As follows from equation (56), the derivative $\partial\beta_y/\partial\nu'$ to the first order is

$$\frac{\partial\beta_y}{\partial\nu'} = \frac{\nu}{\nu'^2\sin\alpha}(1-\beta_x). \tag{60}$$

The last bracketed term in the expression (56) gives rise to additional terms of order $\beta h\nu/m_ec^2$ due to the presence of the factor $1/\gamma$. Using equation (57) we find

$$\frac{\partial(1/\gamma)}{\partial\nu'} \approx -\beta_y\frac{\partial\beta_y}{\partial\nu'} \approx -\beta_y\frac{\nu}{\nu'^2\sin\alpha}, \tag{61}$$

which finally yields

$$\frac{\partial\beta_y}{\partial\nu'} = \frac{\nu}{\nu'^2\sin\alpha}\left\{1-\beta_x-\frac{h\nu}{m_ec^2}\frac{1-\cos\alpha}{\sin^2\alpha}\left[\beta_x\left(\frac{\nu}{\nu'}-\cos\alpha\right)+1-\frac{\nu}{\nu'}+\frac{h\nu}{m_ec^2}(1-\cos\alpha)\right]\right\}. \tag{62}$$

Note that the main Klein-Nishina correction to the scattering cross section, which is of the order of $h\nu/m_ec^2$, is contained in the factor $(\nu'/\nu)^2$ in equation (53), rather than in the X function (eq. [54]), which describes Doppler aberration and the Klein-Nishina correction of order $(h\nu/m_ec^2)^2$. The reciprocal factor, ν/ν'^2, enters equation (62), so upon multiplication of $d\sigma/d\mathbf{\Omega'}$ and $|\partial\beta_y/\partial\nu'|$ in the integrand of equation (58), the presence of the $O(h\nu/m_ec^2)$ correction in $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ is not explicit.

Proceeding further with expansions, we obtain

$$\begin{aligned}\frac{1}{\sigma_\mathrm{T}}\frac{d\sigma}{d\mathbf{\Omega'}}(1-\mathbf{\Omega}\boldsymbol{\beta})\left|\frac{\partial\beta_y}{\partial\nu'}\right| = \frac{3}{16\pi}\frac{1}{\nu\sin\alpha}&\left\{1+\mu_\mathrm{s}^2+2s(-\mu_\mathrm{s}+\mu_\mathrm{s}^2)+2\beta_x(-\mu_\mathrm{s}+\mu_\mathrm{s}^2)\right.\\
+s^2(1-4\mu_\mathrm{s}+3\mu_\mathrm{s}^2)+2s\beta_x(1-3\mu_\mathrm{s}+2\mu_\mathrm{s}^2)+\beta_x^2(1-4\mu_\mathrm{s}+3\mu_\mathrm{s}^2)&+\beta^2(-1+2\mu_\mathrm{s}-3\mu_\mathrm{s}^2)\\
+2s^3(1-3\mu_\mathrm{s}+2\mu_\mathrm{s}^2)+2s^2\beta_x(2-5\mu_\mathrm{s}+3\mu_\mathrm{s}^2)+2s\beta_x^2(2-5\mu_\mathrm{s}+3\mu_\mathrm{s}^2)&+2\beta_x^3(1-3\mu_\mathrm{s}+2\mu_\mathrm{s}^2)\\
+2\beta^2s(-1+4\mu_\mathrm{s}-3\mu_\mathrm{s}^2)+2\beta^2\beta_x(-1+4\mu_\mathrm{s}-3\mu_\mathrm{s}^2)+\frac{h\nu}{m_ec^2}&\left(1-\frac{\nu}{\nu'}+\frac{h\nu}{m_ec^2}(1-\mu_\mathrm{s})\right)\frac{(1+\mu_\mathrm{s}^2)}{1+\mu_\mathrm{s}}\\
-\frac{h\nu}{m_ec^2}\beta_x\frac{(1-\mu_\mathrm{s})(1+\mu_\mathrm{s}^2)}{1+\mu_\mathrm{s}}&\left.+\frac{\nu'}{\nu}\left(\frac{h\nu}{m_ec^2}\right)^2(1-\mu_\mathrm{s})^2\right\},\end{aligned} \tag{63}$$

where $\mu_\mathrm{s}=\cos\alpha$,

$$s = \mathbf{\Omega'}\boldsymbol{\beta} = \beta_x\frac{\nu}{\nu'}+1-\frac{\nu}{\nu'}+\frac{h\nu}{m_ec^2}(1-\mu_\mathrm{s}), \tag{64}$$

$$\beta^2 \approx \tilde{\beta^2} = \beta_x^2+\beta_z^2+\frac{1}{1-\mu_\mathrm{s}^2}\left[\beta_x\left(\frac{\nu}{\nu'}-\mu_\mathrm{s}\right)+1-\frac{\nu}{\nu'}+\frac{h\nu}{m_ec^2}(1-\mu_\mathrm{s})\right]^2. \tag{65}$$

Here $\tilde{\beta^2}$ must be substituted for β^2 in equation (63) and in the terms preceeding the exponential in equation (59).

Note that in the situation of scattering of nonrelativistic electrons ($\eta \ll 1$), equation (63) reduces to a simple formula:

$$\frac{1}{\sigma_\mathrm{T}}\frac{d\sigma}{d\mathbf{\Omega'}}(1-\mathbf{\Omega}\boldsymbol{\beta})\left|\frac{\partial\beta_y}{\partial\nu'}\right| = \frac{3}{16\pi}\frac{1}{\nu\sin\alpha}\left[1+\mu_\mathrm{s}^2+\frac{\nu'}{\nu}\left(\frac{h\nu}{m_ec^2}\right)^2(1-\mu_\mathrm{s})^2\right], \tag{66}$$

which is valid for arbitrary values of photon energy, including $h\nu \gg m_ec^2$. Our final formula will consequently be exact in the limit $\eta \ll 1$, as we claimed at the beginning of this section.

Having made this remark, let us return to considering our main situation of interest, i.e,. with mildly relativistic electrons and photons. The accuracy of equation (65) turns out to be insuffiecent for describing the exponential factor of the distribution function (eq. [59]), in which β^2 is divided by η. In this factor, β^2 need be given accurately to the fifth order, which requires the inclusion of the factor $1/\gamma$ in equation (56) (similarly to the situation with $\partial\beta_y/\partial\nu'$ above). We consequently get

$$\exp\left(-\frac{\beta^2}{2\eta}\right) = \exp\left(-\frac{\tilde{\beta^2}+\Delta\beta^2}{2\eta}\right) \approx \left(1-\frac{\Delta\beta^2}{2\eta}\right)\exp\left(-\frac{\tilde{\beta^2}}{2\eta}\right), \tag{67}$$

where

$$\Delta\beta^2 = -\frac{1-\cos\alpha}{\sin\alpha}\frac{h\nu}{m_e c^2}\beta_y\tilde{\beta^2}, \tag{68}$$

with β_y and $\tilde{\beta^2}$ given by equations (56) and (65), respectively.

Now, having completed all the necessary preparations, we can implement the integration in equation (58). Integrals of the following type then appear:

$$\int \beta_x^k \beta_z^l \exp\left\{-\frac{1}{2\eta}[\beta_x^2+\beta_z^2+(a\beta_x+b)^2]\right\} d\beta_x\, d\beta_z,$$

where a and b are constants set by equation (65). Such integrals are readily done (see, e.g., BR70). It is natural to present the final result in the form of a series in terms of the quantity

$$\Delta = \nu' - \nu + \nu'\frac{h\nu}{m_e c^2}(1-\cos\alpha). \tag{69}$$

Indeed, consider the situation with $kT_e = 0$, when all scattered photons undergo the same decrement in frequency, owing to Compton recoil: $\nu/\nu' = 1+(1-\cos\alpha)h\nu/m_e c^2$, as follows from equation (52). This shift corresponds exactly to $\Delta = 0$. If we now allow a nonvanishing electron temperature ($kT_e \neq 0$), the scattered profile will be Doppler-broadened near the recoil-shifted peak of the line. The term Δ will then measure the frequency variation relative to this peak and therefore will always be of the order of $\nu\eta^{1/2}$, which ensures that the final expression will be convergent regardless of the proportion of $h\nu$ and kT_e (for comparison, we can consider a similar quantity $\nu'-\nu$, which is $\sim \nu\eta^{1/2}$ if the Doppler effect is dominant but $\sim -h\nu^2/m_e c^2$ if recoil is more important). Moreover, note that Δ enters as an entity in all the major expressions we obtained above (see eqs. [56], [62]–[65]). After lengthy calculations, we finally arrive at equation (7) given in §2.

4. CALCULATION OF THE ANGULAR SCATTERING FUNCTION

The angular function, $d\sigma/d\mu_s$, for Compton scattering on Maxwellian electrons can be directly calculated using the same formalism that we employed to derive the $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$ kernel in the preceeding section. In fact, the derivation of $d\sigma/d\mu_s$ is less tedious than that of $K(\nu,\mathbf{\Omega}\to\nu',\mathbf{\Omega'})$. For this reason, we here calculate the angular function to a better accuracy than that of our final formula for the kernel (eq. [7]). Namely, we are willing to obtain an expression that will contain correction terms of the order of $(kT_e/m_e c^2)^2$ and $(h\nu/m_e c^2)(kT_e/m_e c^2)$ to the Rayleigh angular function. The term of order $(h\nu/m_e c^2)^2$ will also be found, but this term already follows from equation (10) for the K_{nr} kernel, which, as we recall, is accurate for arbitrary photon energies in the case of nonrelativistic electrons ($kT_e \ll m_e c^2$).

In order to find the angular function, the integration over the electron velocity space needs to be done:

$$\frac{d\sigma}{d\mu_s} = \frac{2\pi}{\sigma_T}\int \frac{d\sigma}{d\mathbf{\Omega'}}(1-\mathbf{\Omega}\boldsymbol{\beta})f_\beta(\beta_x,\beta_y,\beta_z)\, d\beta_x d\beta_y d\beta_z. \tag{70}$$

Here the differential scattering crosscsection, $d\sigma/d\mathbf{\Omega'}$, for given $(\beta_x,\beta_y,\beta_z)$ and $h\nu/m_e c^2$ is given by equations (52) and (53), and the relativistic Maxwellian distribution function is represented by the series given in equation (59). The principal difference (which simplifies the calculation) of equation (70) from the similar equation (58) is that β_y is now a free parameter, like the other components of the electron velocity.

Next, the quantity $(1-\mathbf{\Omega}\boldsymbol{\beta})d\sigma/d\mathbf{\Omega'}$ needs to be expanded in powers of β_x, β_y, β_z (to fourth order) and $h\nu/m_e c^2$ (to second order), as we similarly did (to a worse accuracy) in the previous section (see eq. [63]). The resultant expression is rather cumbersome, so we do not give it here. For the f_β distribution function, the approximation of equation (59) is sufficient, because the next order terms, $O(\eta^2,\eta\beta^2,\beta^4,\beta^6/\eta,\beta^8/\eta^2)$, in the series f_β cancel upon integration.

The integration in equation (70) is connected to the calculation of standard integrals

$$\int \beta_x^k \beta_y^l \beta_z^m \exp\left\{-\frac{\beta_x^2+\beta_y^2+\beta_z^2}{2\eta}\right\} d\beta_x d\beta_y d\beta_z, \tag{71}$$

where k,l and m are even numbers [odd terms with respect to one of the $(\beta_x,\beta_y,\beta_z)$ components vanish upon integration].

The final result is equation (15).

5. DERIVATION OF THE $P(\nu \to \nu')$ KERNEL

Here we demonstrate how to perform the integral in equation (1) with $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ given by equation (7) when $h\nu(h\nu/m_ec^2) \ll kT_e$.

Write the exponential factor entering equation (7a) in the form of a polynomial:

$$\frac{\epsilon^2}{4(1-\mu_s)\eta} = \frac{(\nu'-\nu)^2}{2g^2\eta} + \frac{h\nu\nu'}{m_ec^2}\frac{(\nu'-\nu)(1-\mu_s)}{g^2\eta} + \left(\frac{h\nu\nu'}{m_ec^2}\right)^2\frac{(1-\mu_s)^2}{2g^2\eta} \tag{72}$$

where

$$g^2 = 2\nu\nu'(1-\mu_s)+(\nu'-\nu)^2. \tag{73}$$

One can see from equation (73) that $g^2 \to 0$ at $\nu' = \nu$ when $\mu_s \to 1$. The factor g^2 enters the denominator of the first member of the polynomial in equation (72), which describes the Doppler broadening. Knowing this property, which means that $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ is a δ-function for $\mu_s = 0$, one can immediately make a prediction that the $P(\nu \to \nu')$ kernel resulting from the integration in equation (1) over μ_s will have a cusp (a point where a break in the derivative occurs) at $\nu' = \nu$. Such a cusp is indeed present in our final expression for $P(\nu \to \nu')$, as we shall see below.

There is no singularity at $\mu_s = 1$ in the second and third members of the polynomial (72), which describe the frequency variation due to Compton recoil. This suggests that the cusp mentioned above will remain at the same position, $\nu' = \nu$, regardless of the initial photon energy ($h\nu$), which indeed proves to be the case (see the results of Monte Carlo simulations in Fig. 1 of the review by Pozdnyakov et al. 1983).

By assumption ($h\nu \sim kT_e$), the main contribution to the photon frequency increment by scattering comes from the Doppler effect. This implies that $\nu'-\nu$ is typically $\sim \nu\eta^{1/2}$. The second and third terms in equation (72) are thus infinitesimal of the order of $\eta^{1/2}$ and η, respectively. One can therefore take these terms out of the exponential, which will make possible analytical integration in equation (1).

It is convenient to present our final result for $K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'})$ in terms of the quantity

$$\delta = (2\eta)^{-1/2}\frac{\nu'-\nu}{\nu+\nu'}. \tag{74}$$

This new variable, describing the relative frequency shift, is similar to ϵ, in powers of which the original equation (7) is written, when $h\nu \to 0$ (see the main term of eq. [75] below). Moreover, the combination $(\nu'-\nu)/(\nu+\nu')$ arises in a natural way if one first calculates the analog of the $P(\nu \to \nu')$ kernel for a monoenergetic isotropic electron distribution and then convolves this quantity with the Maxwellian distribution function (as we do in §6). Indeed, for a given electron speed, βc, the photon frequency after scattering can take values in the range $|\nu'-\nu|/(\nu+\nu') \le \beta$. From this fact it becomes immediately clear that $P(\nu \to \nu')$ must (and indeed does) possess a certain symmetry if expressed in terms of δ given by equation (74).

After a few intermediate steps (see eq. [7]), which are

$$\begin{aligned}\frac{\epsilon^2}{\eta} \;=\;& 8\delta^2+4\sqrt{2}(1-\mu_s)\delta\frac{h\nu}{m_ec^2}\eta^{-1/2}-\frac{16(1+\mu_s)}{1-\mu_s}\delta^4\eta+8(1-\mu_s)\delta^2\frac{h\nu}{m_ec^2}+(1-\mu_s)^2\left(\frac{h\nu}{m_ec^2}\right)^2\eta^{-1}\\ &-8\sqrt{2}(1+\mu_s)\delta^3\frac{h\nu}{m_ec^2}\eta^{1/2}+2\sqrt{2}(1-\mu_s)^2\delta\left(\frac{h\nu}{m_ec^2}\right)^2\eta^{-1/2}+\dots,\end{aligned} \tag{75}$$

$$\frac{\nu'}{g} = [2(1-\mu_s)]^{-1/2}\left(1+\sqrt{2}\delta\eta^{1/2}-\frac{1+\mu_s}{1-\mu_s}\delta^2\eta-\frac{\sqrt{2}(1+\mu_s)}{1-\mu_s}\delta^3\eta^{3/2}\right)+\dots, \tag{76}$$

we finally obtain

$$\begin{aligned}K(\nu,\mathbf{\Omega} \to \nu',\mathbf{\Omega'}) =\;& \nu^{-1}\frac{3}{32\pi}[\pi(1-\mu_s)\eta]^{-1/2}\exp\left(-\frac{2\delta^2}{1-\mu_s}\right)\\ \times\Bigg\{&\left[1+\sqrt{2}\delta\left(1-\frac{h\nu}{kT_e}\right)\eta^{1/2}-4\delta^2\frac{h\nu}{m_ec^2}+2\sqrt{2}\delta^3\left(-2+\frac{1}{3}\left(\frac{h\nu}{kT_e}\right)^2\right)\frac{h\nu}{m_ec^2}\eta^{1/2}\right]k_0\\ &+\left[1+\sqrt{2}\delta\left(1-\frac{h\nu}{kT_e}\right)\eta^{1/2}\right]k_t+\left[1+\sqrt{2}\delta\left(3-\frac{h\nu}{kT_e}\right)\eta^{1/2}\right]k_r\Bigg\},\end{aligned} \tag{77a}$$

where

$$\begin{aligned}k_0 \;&=\; 1+\mu_s^2,\\ k_t \;&=\; \left[\frac{1}{8}-\mu_s-\frac{63}{8}\mu_s^2+5\mu_s^3+\frac{-1-5\mu_s-\mu_s^2+3\mu_s^3}{1-\mu_s}\delta^2+\frac{2(-1+2\mu_s-\mu_s^2+2\mu_s^3)}{(1-\mu_s)^2}\delta^4\right]\eta,\\ k_r \;&=\; (1+\mu_s^2)\left(-\frac{1-\mu_s}{4}+\delta^2\right)\left(\frac{h\nu}{m_ec^2}\right)^2\eta^{-1}.\end{aligned} \tag{77b}$$

Note that we have omitted in equation (77) the second-order, $O((h\nu/m_e c^2)^2)$, Klein-Nishina correction term, which was present in the original expression in equation (7). This is because in the limit we are currently working with ($h\nu \sim kT_e$), inclusion of this term would be inconsistent with the absence of (unknown) terms of the same order, such as $O(\eta^2)$ or $O((h\nu/m_e c^2)^3\eta^{-1})$, in the series given in equation (77a).

The term $K(\nu, \mathbf{\Omega} \to \nu', \mathbf{\Omega'})$ in the form (77) is easily integrated over the scattering angle using the change of variables ($\mu_s \to t$) $2\delta^2/(1-\mu_s) = t^2$, which results in the final equation (19) for the $P(\nu \to \nu')$ kernel for the isotropic problem.

6. DIRECT CALCULATION OF THE $P(\nu \to \nu')$ KERNEL IN THE CASE OF $H\nu \ll KT_e$

Given is an isotropic field of electromagnetic radiation of frequency $\tilde{\nu}$. Its spectrum (number of photons per unit solid angle, unit frequency interval, and unit detector area) is

$$\frac{dN(\nu)}{d\Omega\, d\nu} = \frac{\delta(\nu - \tilde{\nu})}{4\pi}. \tag{78}$$

Consider scattering of the radiation on an electron moving at a speed of $v = \beta c$. The energy of the photons is assumed to be low enough ($h\tilde{\nu} \ll m_e v^2$) that Compton recoil can be ignored. We consider electrons that are not too relativistic: $(h\tilde{\nu}/m_e c^2)\gamma \ll 1$, where $\gamma = (1-\beta^2)^{-1/2}$; thus Klein-Nishina relativistic corrections are not significant. For the moment, we ignore induced scattering. Consider the situation in the electron rest frame. In this frame, the spectral intensity of the incident radiation is direction dependent:

$$\frac{dN_0(\mu_0, \nu_0)}{d\Omega_0\, d\nu_0} = \left(\frac{\nu_0}{\nu}\right)^2 \frac{dN(\nu)}{d\Omega\, d\nu}, \tag{79}$$

where μ is the cosine of the angle between the velocity of the electron and the direction of propagation of the photon. Quantities that are measured in the electron rest frame (with subscript "0") are related to the corresponding quantities measured in the laboratory frame via the Lorentz transformations:

$$\mu_0 = \frac{\mu - \beta}{1 - \beta\mu},\ \nu_0 = \frac{\nu}{\gamma(1+\beta\mu_0)}. \tag{80}$$

This leads to

$$\frac{dN_0(\mu_0, \nu_0)}{d\Omega_0\, d\nu_0} = \frac{1}{4\pi\gamma^3(1+\beta\mu_0)^3}\delta\left(\nu_0 - \frac{\tilde{\nu}}{\gamma(1+\beta\mu_0)}\right). \tag{81}$$

Under the given assumptions, the scattering can be treated in the Thomson limit (the photon frequency does not change) in the electron rest frame. Therefore, the number of photons scattered into an interval $d\Omega_0$ of solid angle in a unit time is

$$\begin{aligned}\frac{dN_0(\mu_0, \nu_0)}{dt\, d\Omega_0\, d\nu_0} &= \frac{3\sigma_T c}{16}\int_{-1}^{1} d\mu_0'(3 + 3\mu_0^2\mu_0'^2 - \mu_0^2 - \mu_0'^2)\frac{dN(\mu_0', \nu_0)}{d\Omega_0'\, d\nu_0} \\ &= \frac{3\sigma_T \nu_0}{64\pi\gamma\beta\tilde{\nu}^2}\left[3 + \frac{3\mu_0^2}{\beta^2}\left(1 - \frac{\tilde{\nu}}{\gamma\nu_0}\right)^2 - \mu_0^2 - \frac{1}{\beta^2}\left(1 - \frac{\tilde{\nu}}{\gamma\nu_0}\right)^2\right].\end{aligned} \tag{82}$$

The reverse transition to the laboratory frame can be performed using equation (80) by the formula

$$\begin{aligned}\frac{dN(\mu, \nu)}{dt\, d\Omega\, d\nu} &= \frac{dt_0}{dt}\frac{d\mu_0}{d\mu}\frac{d\nu_0}{d\nu}\frac{dN_0(\mu_0, \nu_0)}{dt_0\, d\Omega_0\, d\nu_0} = \frac{1}{\gamma^2(1-\beta\mu)}\frac{dN_0(\mu_0, \nu_0)}{dt_0\, d\Omega_0\, d\nu_0} \\ &= \frac{3\sigma_T c u}{64\tilde{\nu}\pi\gamma^2\beta}\left[3 + \frac{3}{\beta^2}\left(\frac{\mu-\beta}{1-\beta\mu}\right)^2\left(1 - \frac{1}{\gamma^2(1-\beta\mu)u}\right)^2 - \left(\frac{\mu-\beta}{1-\beta\mu}\right)^2 - \frac{1}{\beta^2}\left(1 - \frac{1}{\gamma^2(1-\beta\mu)u}\right)^2\right],\end{aligned} \tag{83}$$

where $u = \nu/\tilde{\nu}$.

The derivation given above is analogous to the one in (Sazonov & Sunyaev 1998). The difference is that that paper considered the scattering of a Planckian spectrum[2].

For a given μ, the photon frequency change, u, ranges in $[(1-\beta)/(1-\beta\mu), (1+\beta)/(1-\beta\mu)]$. Integrating equation (83) over the solid angle of emergence of the scattered photon yields the spectrum that forms as a result of scattering of monochromatic radiation on electrons moving at a speed of β. For a given u, the integration limits are

$$\begin{cases} -1 \le \mu \le (u-1+\beta)/(\beta u), & u < 1; \\ (u-1-\beta)/(\beta u) \le \mu \le 1, & u > 1. \end{cases} \tag{84}$$

The result is

$$\begin{aligned}\left(\frac{dN(\nu)}{dt\, d\nu}\right)_{+,-} &= \frac{3\sigma_T c}{32\tilde{\nu}u\gamma^2\beta^6}\left\{\mp(u-1)\left(\frac{u^2+6u+1}{\gamma^4} + 4u\right)\right. \\ &\left. + 2u(u+1)\left[2\beta\left(\frac{3}{\gamma^2} + \beta^4\right) + \frac{3-\beta^2}{\gamma^2}\left(\ln\frac{1-\beta}{1+\beta} \pm \ln u\right)\right]\right\},\end{aligned} \tag{85}$$

[2] A formula similar to equation (83) made it possible to find relativistic corrections to the amplitude of CMB distortions in the direction of clusters of galaxies. These corrections are of the order of $(kT_e/m_e c^2)^2$, $(V/c)^2$, and $(V_r/c) \times (kT_e/m_e c^2)$, where V and V_r are the cluster peculiar velocity and its component along the line of sight, respectively.

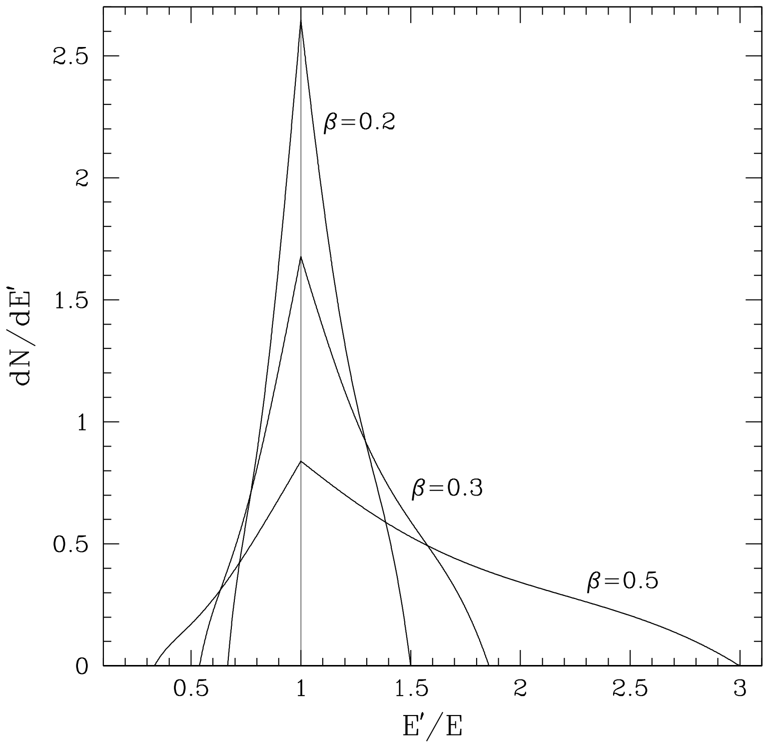

FIG. 17.— Spectra resulting from the single scattering of isotropic monochromatic radiation on an ensemble of electrons moving isotropically at a given speed, as calculated from formula (85) for different values of the electron velocity ($v = \beta c$). Compton recoil is ignored, which corresponds to the limit $h\nu \ll m_e v^2$.

where the subscript plus sign corresponds to the values $u > 1$, and the minus sign to $u < 1$. Equation (85) was earlier derived by Fargion et al. (1997). The derivation of these authors differs from the one above in the order of integrations: they first considered the scattering of a beam of photons by a beam of electrons with a given angle between the two, and then implemented the integration over this angle.

Let us now make a transition from a single electron to an ensemble of electrons with density N_e and introduce a quantity, related to the result of equation (85), that gives the probability of a scattering event calculated per unit dimensionless time $\tau = \sigma_T N_e ct$, defined as

$$P(\tilde{\nu} \to \nu, \beta) = \frac{1}{\sigma_T c} \frac{dN(\nu)}{dt\, d\nu}. \qquad (86)$$

Equation (86) describes the kernel of the integral kinetic equation for the problem of Comptonization on monoenergetic electrons. Let us mention its basic properties: (1) $P(\nu \to \tilde{\nu}, \beta) = (\tilde{\nu}/\nu)^2 P(\tilde{\nu} \to \nu, \beta)$, (2) $\int_{\nu_{min}}^{\nu_{max}} P\, d\nu = 1$, and (3) $\int_{\nu_{min}}^{\nu_{max}} (\tilde{\nu}/\nu - 1) P\, d\nu = 4(\gamma^2 - 1)/3$, where $\nu_{min} = (1-\beta)/(1+\beta)$, $\nu_{max} = (1+\beta)/(1-\beta)$ are the minimal and maximal possible values of the photon frequency after scattering. Property 1 ensures that the detailed balance principle is obeyed (see also eq. [21] for the case of Maxwellian electrons). In property 2, photon conservation manifests itself. Property 3 is a well-known relation (see, e.g., eq. [2.33] in the review by Pozdnyakov et al. 1983), which describes the rate at which the electrons transfer energy to the photons. There exists a more general expression than equation (85) (which was obtained in the Thomson limit) for the $P(\tilde{\nu} \to \nu, \beta)$ kernel (see eqs. [8.1.8] and [8.1.12] in Nagirner & Poutanen 1994, and references therein), which is valid for arbitrary photon and electron energies.

In Figure 17, we present examples of spectra described by equations (85) and (86). Their characteristic feature is the presence of a cusp at $\nu/\tilde{\nu} = 1$. Note that the right wing of the line contains more photons ($\approx 1/2 + 69\beta/140$) than the left one ($\approx 1/2 - 69\beta/140$), the asymmetry becoming more pronounced as the electrons get more relativistic. Such spectra will build up when isotropic monochromatic radiation is scattered off an optically thin cloud of electrons that are moving isotropically at the same speed. Note that equation (85) retains valid for ultrarelativistic ($\gamma \gg 1$) electrons (the case that interested Fargion et al. 1997).

In the nonrelativistic limit ($\beta \ll 1$), the frequency is changed upon scattering by a small amount: $|u-1|/(u+1) \ll 1$. This allows us, by expanding equation (85) in powers of β and

$$\xi = \frac{u-1}{u+1}, \qquad (87)$$

to derive the formula

$$P(\tilde{\nu} \to \nu, \beta) = \frac{1}{\tilde{\nu}\beta} \left[\left(\frac{11}{20} - \frac{73}{140}\beta^2 + O(\beta^4) \right) + \frac{|\xi|}{\beta} \left(-\frac{3}{4} + \frac{3}{4}\beta^2 + O(\beta^4) \right) \right.$$

$$+\left(\frac{\xi}{\beta}\right)^2\left(\frac{11}{20}\beta^2+O(\beta^4)\right)+\left(\frac{|\xi|}{\beta}\right)^3\left(\frac{1}{2}-\frac{7}{4}\beta^2+O(\beta^4)\right)+\left(\frac{|\xi|}{\beta}\right)^5\left(-\frac{3}{10}+\frac{17}{10}\beta^2+O(\beta^4)\right)$$
$$+\left(\frac{|\xi|}{\beta}\right)^7\left(-\frac{51}{70}\beta^2+O(\beta^4)\right)\Bigg](1+\xi), \quad (88)$$

which approximates well the scattered photon distribution. The series given in equation (88) ensures photon conservation to an accuracy of $\int P\,d\nu = 1+O(\beta^4)$, or, putting the same expression in explicit form, $\int P\,d\nu = 1+19\beta^4/300+\ldots$.

The spectrum that forms as a result of the single scattering of isotropic monochromatic radiation on a group of electrons with a given isotropic distribution of velocities, f_β, is described in general by the formula

$$P(\tilde{\nu}\to\nu)=\int P(\tilde{\nu}\to\nu,\beta)f_\beta\,\beta^2\,d\beta, \quad (89)$$

where $P(\tilde{\nu}\to\nu,\beta)$ is governed by equations (85) and (86). In the case of a thermal plasma, one must insert the relativistic Maxwellian distribution function into equation (89). The result can then be evaluated numerically.

In the present study, we are interested in the mildly relativistic case, $\eta = kT_e/m_ec^2 \lesssim 0.1$. In this limit, we can derive $P(\tilde{\nu}\to\nu)$ analytically by making use of the approximate equation (88). To this end, the f_β function must be expanded in terms of β. In order to obtain the result with the accuracy we need, it is enough to retain only relativistic correction terms of order η in this series, i.e., to substitute equation (59) of §3 for f_β.

The integral in equation (89) is to be performed in the range of values $\beta \geq \beta_m = |\xi| = |(u-1)/(u+1)|$. Upon elementary calculations, we arrive at equation (20) (with the transition from $\tilde{\nu}$, ν to ν, ν'), which was our aim.

This work was supported in part by the Russian Foundation for Basic Research through grants 97-02-16264, 00-15-96649, and 00-02-16681.

REFERENCES

Aharonyan, F.A., & Atoyan, A.M. 1981, Ap&SS, 79, 321
Babuel-Peyrissac, J.P., & Rouvillois, G. 1970, J. Quant. Spectr. Rad. Transf., 10, 1277 (BR70)
Beresteskii, V.B., Lifshitz, E.M., & Pitaevskii, L.P. 1982, Quantum Electrodynamics: Landau and Lifshitz Course of Theoretical Physics (2nd ed., Oxford: Pergamon)
Birkinshaw, M. 1999, Phys. Rep., 310, 97
Blandford, R.D., & Scharlemann, E.T. 1975, Ap&SS, 36, 303
Challinor, A., & Lasenby, A. 1998, ApJ, 499, 1
Danese, L., & De Zotti, G. 1977, Nuovo Cimento, 7, 277
Dirac, P.A.M. 1925, MNRAS, 85, 825
Fargion, D., Konoplich, R.V., & Salis A. 1997, J. Physics. C., 74, 571
Fixsen, D.J., Cheng, E.S., Gales, J.M., Mather, J.C., Shafer, R.A., & Wright, E. 1996, ApJ, 473, 576
Ghierlinski, M., Zdziarski, A.A., Poutanen, J., Coppi, P.S., Ebisawa, K., & Johnson, W.N. 1999, MNRAS, 309, 496
Ghisellini, G., George, I.M., Fabian, A.C., & Done, C. 1991, MNRAS, 248, 14
Goldin, V.Y., Sunyaev, R.A., & Chetverushkin, B.N. 1975, Soviet Phys. JETP, 41, 18
Grebenev, S.A., & Sunyaev, R.A. 1987, Soviet Astron. Lett., 13, 438
Haardt, F. 1993, ApJ, 413, 680
Hummer, D.G., & Mihalas, D. 1967, ApJ, 150, L57
Illarionov, A., Kallman, T., McCray, R., & Ross, R. 1979, ApJ, 228, 279
Illarionov, A.F., & Kompaneets, D.A. 1977, Soviet Phys. JETP, 44, 930
Itoh, N., Kohyama, Y., & Nozawa, S. 1998, ApJ, 502, 7
Jauch, J.M., & Rohrlich, F. 1976, The theory of photons and electrons (2nd ed., New York: Springer)
Kershaw, D.S., Prasad, M.K., & Beason, J.D. 1986, J. Quant. Spectr. Rad. Transf., 36, 273
Kompaneets, A.S. 1957, Soviet Phys.–JETP, 4, 730
Landau, L.D., & Lifshitz, E.M. 1975, The Classical Theory of Fields: Landau and Lifshitz Course of Theoretical Physics (4th ed., Oxford: Pergamon)
Levich, E.V., & Sunyaev, R.A. 1971, Soviet Astron., 15, 363
Levich, E.V., Sunyaev, R.A., & Zeldovich, Y.B. 1972, A&A, 19, 135
Loeb, A., McKee, F., & Lahav, O. 1991, ApJ, 374, 44
Molnar, S.M., & Birkinshaw, M. 1999, ApJ, 523, 728
Nagirner, D.I., & Poutanen, J. 1994, in Astrophys. Space Phys. Rev., Vol.9, ed. R.A. Sunyaev (Chur: Harwood Academic Publishers), 1
Narayan, R., Mahadevan, R., & Quataert E., 1998, in The Theory of Black Hole Accretion Discs, ed. M.A. Abramowicz et al. (Cambridge: Cambridge University Press), 148
Peebles, P.J.E. 1993, Principles of physical cosmology (Princeton, NJ: Princeton Univ. Press)
Peyraud, J. 1968, J. de Phys., 29, 88
Pomraning, G.C. 1973, The Equations of Radiation Hydrodynamics (Oxford: Pergamon)
Poutanen, J., & Svensson, R. 1996, ApJ, 470, 249
Pozdnyakov, L.A., Sobol, I.M., & Sunyaev, R.A. 1979, A&A, 75, 214
Pozdnyakov, L.A., Sobol, I.M., & Sunyaev R.A. 1983, in Astrophys. & Space Phys. Rev. Vol.2, ed. R.A. Sunyaev (Chur: Harwood Academic Publishers), 189
Rephaeli, Y. 1995, ApJ, 445, 33
Ross, R.R., Weaver, R., & McCray, R. 1978, ApJ, 219, 292
Sazonov, S.Y., & Sunyaev, R.A. 1998, ApJ, 508, 1
Sazonov, S.Y., & Sunyaev, R.A. 2000, A&A, 354, L53
Shapiro, S.L., Lightman, A.P., & Eardley, D.M. 1976, ApJ, 204, 187
Shestakov, A.I., Kershaw, D.S., & Prasad, M.K. 1988, J. Quant. Spectr. Rad. Transf., 40, 577
Sunyaev, R.A. 1971, Soviet Astron., 15, 190
Sunyaev, R.A. 1980, Soviet Astron. Lett., 6, 213
Sunyaev, R.A., & Churazov, E.M. 1996, Astron. Lett., 22, 648
Sunyaev, R.A., & Titarchuk, L.G. 1980, A&A, 86,121
Sunyaev, R.A., & Truemper, J. 1979, Nature, 279, 506
Sunyaev, R.A., & Zeldovich, Y.B. 1970, Astrophys. Space Sci., 7, 20
Sunyaev, R.A., & Zeldovich, Y.B. 1980, ARA&A, 18, 537
Sunyaev, R.A., & Zeldovich, Y.B. 1972, Comments Astrophys. Space Phys., 4, 173
Tanaka, Y., & Shibazaki N. 1996, ARA&A, 34, 607
Titarchuk, L.G. 1994, ApJ, 434, 570
Vinogradov, A.V., & Pustovalov, V.V. 1972, Soviet Phys.–JETP, 35, 517
Weymann, R.J. 1970, ApJ, 160, 31
Zeldovich, Y.B., & Levich, E.V. 1970, Soviet Phys.–JETP, 11, 35
Zeldovich, Y.B., Levich, E.V., & Sunyaev, R.A. 1972, Soviet Phys.–JETP, 35, 733
Zeldovich, Y.B., & Sunyaev, R.A. 1969, Ap&SS, 4, 301
Zeldovich, Y.B., & Sunyaev, R.A. 1972, Soviet Phys.–JETP, 35, 81
Zeldovich, Y.B., & Sunyaev, R.A. 1976, Soviet Phys.–JETP, 41, 391
Zdziarski, A.A., Poutanen, J., Mikolajewska, J., Gierlinski, M., Ebisawa, K., & Johnson, W.N. 1998, MNRAS, 301, 435