Goodness-of-Fit Analysis of Radial Velocities Surveys

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ABSTRACT

Using eigenmode expansion of the Mark III and SFI surveys of cosmological radial velocities a goodness-of-fit analysis is applied on a mode-by-mode basis. This differential analysis complements the Bayesian maximum likelihood analysis that finds the most probable model given the data. Analyzing the surveys with their corresponding most likely models from the CMB-like family of models, as well as with the currently popular Λ-CDM model, reveals a systematic inconsistency of the data with these 'best' models. There is a systematic trend of the cumulative χ^2 to increase with the mode number (where the modes are sorted by decreasing order of the eigenvalues). This corresponds to a decrease of the χ^2 with the variance associated with a mode, and hence with its effective scale. It follows that the differential analysis finds that on small (large) scales the global analysis of all the modes 'puts' less (more) power than actually required by the data. This indicates that either the theoretical model (i.e. power spectrum) or the error model (or both) have an excess of power on large scales. Alternatively it may indicate that the velocity data suffers from still uncorrected systematic errors.

Subject headings: Cosmology: observations — cosmology: theory — dark matter — galaxies: distances and redshifts — large-scale structure of universe — methods: statistical

1. Introduction

Surveys of radial velocities of galaxies have played a major role in our study of the large scale structure. The analysis of such surveys has been conducted in two main directions, the mapping of the local cosmography and the estimation of the cosmological parameters (cf. Dekel 1994 for a review). The Bayesian framework provides one with very elegant and powerful tools for conducting both the mapping and parameter estimation, where the recovery of the large scale structure is done by means of the Wiener filter and the parameters are estimated by maximum likelihood (MaxLike) analysis (Zaroubi *et al.* 1995, ZHFL). In the case where the deviations from a homogeneous and isotropic universe constitute a Gaussian random field the Wiener filter and the MaxLike are the optimal tools for performing such an analysis (ZHFL). Indeed, the MARK III catalog of radial velocities (Willick *et al.* 1995, 1996, 1997a) have been recently analyzed by Wiener filtering (Zaroubi, Hoffman and Dekel 1999) and by MaxLike (Zaroubi *et al.* 1997). The SFI survey of da Costa *et al.* (1996) has been studied by MaxLike analysis by Freudling *et al.* (1999) and by Wiener filtering (Hoffman and Zaroubi, unpublished). Both surveys seem to yield similar results.

In the Bayesian MaxLike analysis one calculates the posterior probability of a model to be correct given the data (ZHFL, Vogeley and Szalay 1996). Thus the model that maximizes the likelihood function, over a given parameter (or model) space, is the most likely model in that space. The MaxLike analysis cannot guarantee, however, that the most probable model is indeed consistent with the data. It provides a relative measure for models to be correct, but it lacks an absolute measure. It is common to adopt an independent measure for the goodness-of-fit, which is often given by the requirement that the reduced χ^2 is close to unity. Often, when the most likely model (given the data) passes also the goodness-of-fit test one assumes that the 'correct' model has been nailed down. Here, the

χ^2 test is expanded and a much more critical test is suggested and then applied to the Mark III and SFI surveys.

The χ^2 'goodness-of-fit' is based on the assumptions that all the random variables that affect the observables are normally distributed. In the cosmological context this applies to both the underlying dynamical (e.g. density and velocity) field and the statistical errors. Thus for a survey containing N data observables (e.g. radial velocities) the χ^2 of the system of N degrees of freedom (DOF) is calculated given the model that maximizes the likelihood function. The goodness-of-fit is measured by how close is the χ^2/DOF to unity. This provides a global measure for the consistency of the data with the model, as it includes all the observables. A situation might occur of some 'conspiracy' where different pars of the data set deviate from the predictions of the model, but when combined together they 'conspire' to yield a reasonable χ^2. A much stronger test on the model is to decompose the data into statistical independent eigenmodes and observe the χ^2 behavior of the independent modes. Eigenmode analysis, also known as principal component analysis (PCA) and the Karhunen-Loeve transform, is not a new tool in the field. It has been applied to studies of redshift surveys (Vogeley and Szalay 1996),the cosmic microwave background (Bunn 1997, Bond 1995) and more recently radial velocities surveys (Hoffman, 1999). The later study is extended here to perform the 'goodness-of-fit' test on a mode-by-mode basis. The basic formalism is presented in § 2, and its application to the Mark III and SFI surveys is given in § 3. Our results are discussed and the conclusions are summarized in § 4.

2. Eigenmode Analysis of Radial Velocities

Consider a data base of radial velocities $\{u_i\}_{i=1,\dots,N}$, where

$$u_i = \mathbf{v}(\mathbf{r}_i) \cdot \hat{\mathbf{r}}_i + \epsilon_i, \tag{1}$$

$\mathbf{v}$ is the three dimensional velocity, $\mathbf{r}_i$ is the position of the i-th data point and ϵ_i is the statistical error associated with the i-th radial velocity. The assumption made here is of a cosmological model that well describes the data, that systematic errors have been properly dealt with and that the statistical errors are well understood. The data auto-covariance matrix is then written as:

$$R_{ij} \equiv \left\langle u_i u_j \right\rangle = \hat{\mathbf{r}}_j \left\langle \mathbf{v}(\mathbf{r}_i) \mathbf{v}(\mathbf{r}_j) \right\rangle \hat{\mathbf{r}}_j + \sigma_{ij}^2. \tag{2}$$

(Here $\left\langle \ldots \right\rangle$ denotes an ensemble average.) The last term is the error covariance matrix. The velocity covariance tensor that enters this equation was derived by Górski (1988, see also Zaroubi, Hoffman and Dekel 1999) and it depends on the power spectrum and cosmological parameters.

The eigenmodes of the data covariance matrix provides a natural representation of the data:

$$R\eta^{(i)} = \lambda_i \eta^{(i)} \tag{3}$$

The set of N eigenmodes $\{\eta^{(i)}\}$ constitutes an orthonormal basis and the eigenvalues λ_i are arranged in decreasing order (in absolute values). A new representation of the data is given by:

$$\tilde{a}_i = \eta_j^{(i)} \, u_j \tag{4}$$

This provides a statistical orthogonal representation, namely:

$$\langle \tilde{a}_i \tilde{a}_j \rangle = \lambda_i \delta_{ij} \tag{5}$$

The normalized transformed variables are defined by:

$$a_i = \frac{\tilde{a}_i}{\sqrt{\lambda_i}} \tag{6}$$

Eq. 5 is written now as:

$$\langle a_i a_j \rangle = \delta_{ij} \tag{7}$$

Note that as the modes are statistically independent one can measure the χ^2 of a given mode, $\chi^2_i = a^2_i$, and the cumulative reduced χ^2 is given by:,

$$\chi^2_M = \frac{1}{M}\sum_{i=1}^{M} a^2_i \tag{8}$$

For normally distributed errors and a Gaussian random velocity field the a_i's are normally distributed with zero mean and a variance of unity.

In addition the probability of finding such χ^2_M is calculated as well. The probability is defined by

$$\begin{aligned} P\left(\chi^2_M\right) &= P_{\chi^2}(M\chi^2_M, M) \qquad \text{for } P_{\chi^2}(M\chi^2_M, M) < 0.5 \\ &= 1 - P_{\chi^2}(M\chi^2_M, M) \qquad \text{otherwise,} \end{aligned} \tag{9}$$

where $P_{\chi^2}(x, M)$ is the probability that a random variable drawn from a χ^2 distribution with M degrees of freedom is less than a given value x.

3. Differential χ^2 Analysis

Here the goodness-of-fit of the Mark III and SFI surveys is studied. The models studied here are the MaxLike solutions for these surveys, which are slightly different from one another. The most likely model given Mark III is a tilted-CDM (T-CDM) of $\Omega_0 = 1,\ h = 0.75$ and $n = 0.8$ where Ω_0 is the cosmological density parameter, h is Hubble's constant in units of 100 $km/s/Mpc$ and n is the power spectrum index (Zaroubi *et al.* 1977). The most likely model given SFI is an open CDM (OCDM) of $\Omega_0 = 0.79,\ h = 0.6$ and $n = 0.92$ (Fruedling *et al.* 1999). For both cases the MaxLike best model has a total $\chi^2_{M=N}$ very close to unity. Thus, from the point of view of the integral χ^2 the MaxLike solutions seem to be very consistent with the data. This is extended to perform a differential χ^2 analysis, namely to study the χ^2 behavior across the modes spectrum.

To study the robustness of this probe it is first applied to a linear mock catalog of Mark III, constructed from an unconstrained realization of the velocity field. This field is sampled at the location of the Mark III data points, to which normally distributed errors are added according to Mark III's error covariance matrix. The cumulative χ^2 of such a catalog should oscillate around unity, given that the model used to generate the catalog is known. Indeed, Fig. 1 confirms this expectation, and the probability of finding such a distribution is within the 90% confidence level. The non-trivial result of this test is that the very poor sampling of the long wavelength Fourier waves, i.e. cosmic variance, does not affect the goodness-of-fit test.

EDITOR: PLACE FIGURE 1 HERE.

The differential $\chi^2$and its associated probability of the Mark III and SFI surveys are presented in Fig. 2 and Fig. 3 (respectively). A clear trend is noticed, namely over almost the entire mode spectrum the cumulative χ^2 increases monotonically. When all modes are included the total χ^2/DOF is indeed close to 1, but if we had to take half the modes, starting from the top or the bottom, a very different χ^2 would have obtained.

EDITOR: PLACE FIGURE 2 HERE.

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The differential χ^2 analysis is repeated for the currently popular model of Λ-CDM($\Omega_0 = 0.4,\ h = 0.6$ and $n = 1$). Indeed, the same trend is found in this case as well but the total χ^2converges to a value outside the 90% confidence level.

EDITOR: PLACE FIGURE 4 HERE.

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The conclusions that follows is that for both data sets, Mark III and SFI, and for a variety of theoretical models the differential χ^2 increase monotonically with the mode number (with the exception of the first 10 modes of the Mark III). The theoretical expectation is that if indeed the data is consistent with the assumed model then χ^2_M will fluctuate around unity. The probability of observing such a trend given a model is very small across most of the mode number range.

4. Discussion

What have we learned from the differential χ^2 analysis? The analysis shows that even the most probable CDM-like model, the one that maximizes the likelihood function given the data, is not fully consistent with the data. The cumulative χ^2 has been calculated both downwards and upwards (namely starting from the modes with the largest and smallest eigenmode, respectively). Over more than 90% of the modes the cumulative χ^2 lies well outside the 90% confidence level, indicating a very small probability of measuring such data given the assumed model. Over most of the mode number range χ^2_M increases monotonically. It is this systematic behavior of the χ^2 which indicates a systematic inconsistency of the model with the data. The assumed model actually contains two ingredient, the theoretical power spectrum and the error model. However, the present analysis cannot indicate which one is to be 'blamed' for the systematic trend. It should be noted here that apart from the first few $(10 - 20)$ modes there is a clear correlation of the eigenvalues with its weighted mean distance (of data points of the given mode). Namely, the variance associated with a mode (i.e. its eigenvalue) increases with its mean distance (Zehavi, private communication, Silverman *et al.* in preparation). It follows that the χ^2 trend seen here is closely correlated with the distance and that the data 'asks' for less power on large scales than the model (power spectrum and noise) provides. A detailed study of the power spectrum and error model possible modifications is to be given elsewhere (Silverman *et al.* in preparation). (Note that these first $10 - 20$ modes are the ones dominated by the underlying velocity field and not the noise, Hoffman 1999.)

The cosmological implications of the present findings are that either the error and/or the theoretical model need to be modified. The theoretical model assumed in the analysis of large scale radial velocity surveys is that the velocities are drawn from a Gaussian random field defined by a given power spectrum. The present study might indicate the inconsistency of the power spectrum with the data. A less likely possibility is that it indicates a departure from the Gaussian statistics. Alternatively, the present work might indicate a systematic error that has not been accounted for that causes this trend.

The conclusions reached here should not be taken as a contradiction of the results of Zaroubi *et al.* (1997) and Fruedling *et al.* (1999), but rather as extending and complementing them. The Bayesian MaxLike analysis can be performed only within the assumed parameter/model space. The differential χ^2 allows one to go beyond this and analyze the nature of the agreement, or the lack of it, between a given model and the data on a mode by mode basis.

The PCA transforms the data to a statistically independent representation and enables the study of the compatibility of the data with the model on a mode by mode basis. This differential analysis is in contrast to the more traditional approach where a data set is analyzed as a whole. The differential χ^2 analysis should be performed together with the Bayesian MaxLike analysis and complement it. This should be useful in fields where the MaxLike is the basic tool of analysis such as the mapping of the CMB angular fluctuations and the study of redshift surveys as well as all radial velocity surveys. The present analysis can prove to be very useful and powerful in those fields where systematic errors play a crucial roles, such as redshift and radial velocities surveys.

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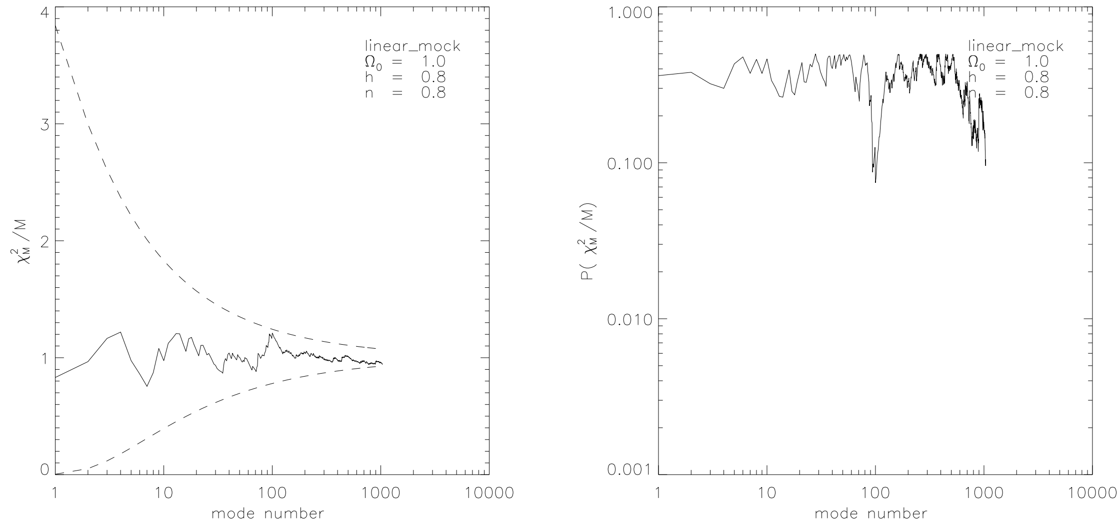

Fig. 1.— A linear mock catalog of radial : The cumulative χ^2 (left) and the probability of gettingg such a χ^2 (right) are plotted against the mode number. The modes are arranged by decreasing order.

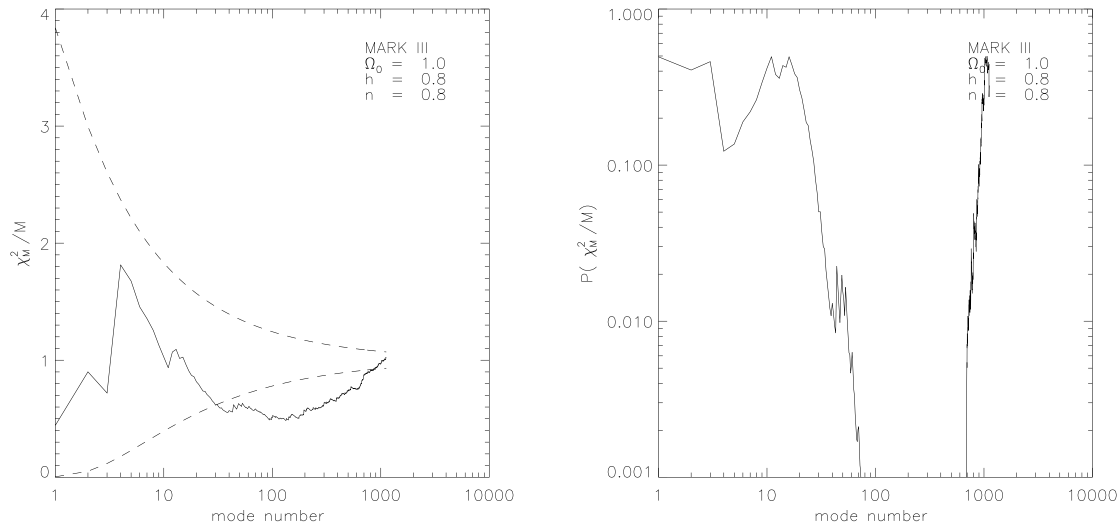

Fig. 2.— Same analysis as in 1 applied to the Mark III survey. The model is a tilted-CDM $\Omega_0 = 1,\ h = 0.75$ and $n = 0.8$ (Zaroubi *et al.* 1997)

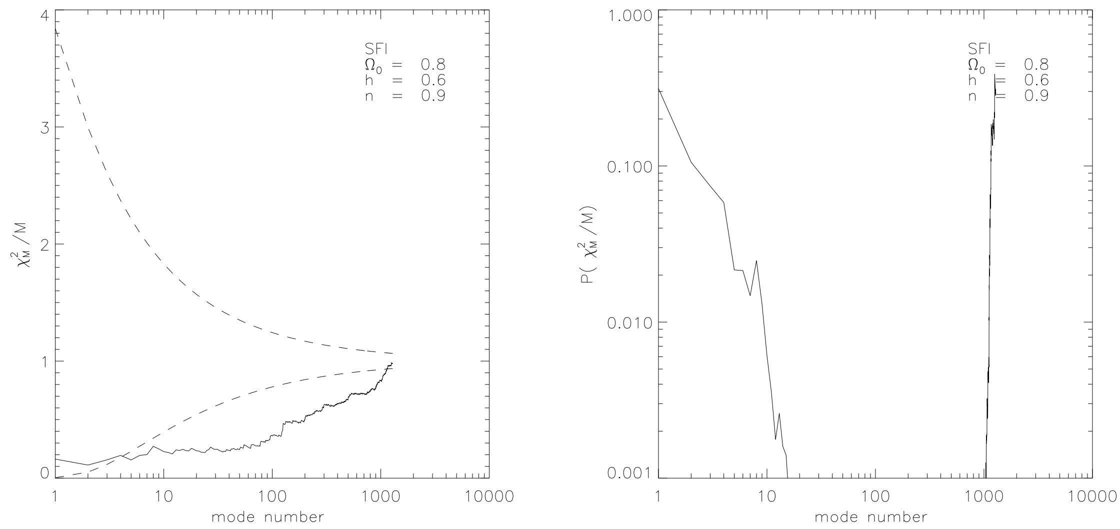

Fig. 3.— Same analysis as in 1 applied to the SFI survey. The model is an open CDM of $\Omega_0 = 0.79,\ h = 0.6$ and $n = 0.92$ (Fruedling *et al.* 1999).

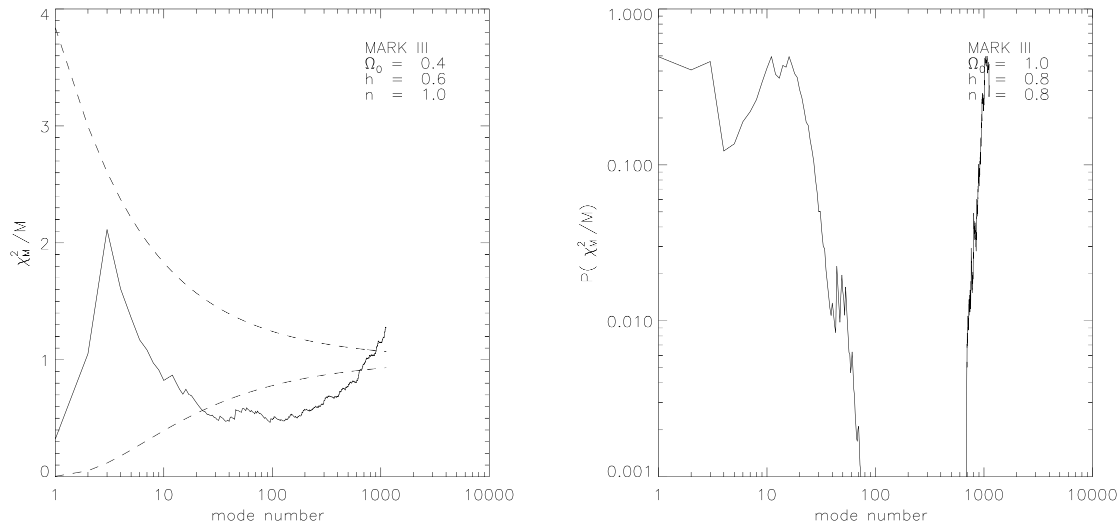

Fig. 4.— Same analysis as in 1 applied to the Mark III survey. The model is a Λ-CDM($\Omega_0 = 0.4,\ h = 0.6$ and $n = 1$)

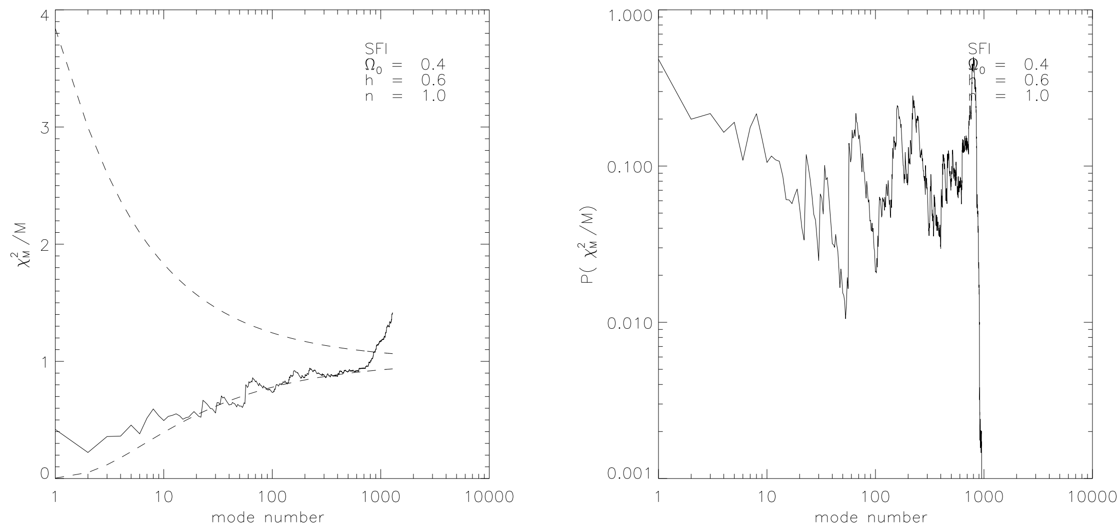

Fig. 5.— Same analysis as in 1 applied to the SFI survey. The model is a Λ-CDM($\Omega_0 = 0.4,\ h = 0.6$ and $n = 1$).