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Time-dependent quasi-spherical accretion

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ABSTRACT
Differentially rotating, 'advection-dominated' accretion flows are considered in which the heat generated by viscous dissipation is retained in the fluid. The equations of time-dependent quasi-spherical accretion are solved in a simplified one-dimensional model that neglects the latitudinal dependence of the flow. A self-similar solution is presented that has finite size, mass, angular momentum and energy. This may be expected to be an attractor for the initial-value problem in which a cool and narrow ring of fluid orbiting around a central mass heats up, spreads radially and is accreted. The solution provides some insight into the dynamics of quasi-spherical accretion and avoids many of the strictures of the steady self-similar solution of Narayan & Yi. Special attention is given to the astrophysically important case in which the adiabatic exponent $\gamma = 5/3$; even in this case, the flow is found to be differentially rotating and bound to the central object, and accretion can occur without the need for powerful outflows.

Key words: accretion, accretion discs – hydrodynamics.

1 INTRODUCTION

Accretion flows in which a significant fraction of the heat generated by viscous dissipation is retained in the fluid, rather than radiated away, have been the subject of considerable attention in recent years (e.g. Narayan & Yi 1994, 1995a,b; Abramowicz et al. 1995). These 'advection-dominated' flows occupy an intermediate position between the spherically symmetric accretion flow of a non-rotating fluid (Bondi 1952) and the cool, thin disc of classical accretion-disc theory (e.g. Pringle 1981). Although their self-consistency may be contingent on certain unresolved issues of plasma physics, they have been considered relevant to a variety of astrophysical objects, in particular to explain the low luminosity of the Galactic Centre (e.g. Narayan et al. 1998). A simple algebraic model describing the fluid-dynamical aspects of such a flow, which was assumed to be steady, axisymmetric and radially self-similar, was presented by Narayan & Yi (1994, hereafter NY). They adopted a set of approximate spherically (or vertically) averaged equations which were largely justified by subsequent calculations that included the latitudinal dependence (Narayan & Yi 1995a; see also Gilham 1981).

Two properties of this solution have been emphasized and identified as potential difficulties with its application. First, in the astrophysically important case in which the adiabatic exponent $\gamma = 5/3$, the solution of NY coincides with that of Bondi (1952). This is spherically symmetric and non-rotating, has no viscous torque or dissipation, and has been considered to depart too far from the concept of an accretion disc. In particular, it is not clear how such a solution could connect to the initial or external conditions if the supplied matter had any angular momentum. To circumvent this difficulty, some authors have investigated the possibility that the effective value of γ might be less than 5/3 if the magnetic field contributed a significant fraction of the total pressure (Narayan & Yi 1995b; Quataert & Narayan 1998).

Secondly, the Bernoulli parameter of the flow is positive. This has been interpreted as meaning that the fluid is unbound and would generate a powerful outflow if the constraints of the model were somehow relaxed. Xu & Chen (1997) therefore generalized the work of Narayan & Yi (1995a) by constructing self-similar solutions that involve inflow at equatorial latitudes and outflow at polar latitudes. However, to achieve this, the scalings of physical quantities with radius were adjusted arbitrarily from their natural values so that only an asymptotically small fraction of the mass reaches the central object, and in this sense the flow is hardly an accretion flow. The inflow is forced to turn into an outflow because the central object by assumption accretes essentially no mass. These solutions are relatively unconstrained because they are unbounded in size and energy and are not required to satisfy any radial boundary conditions. It is therefore uncertain whether they could be realized as intermediate asymptotic forms in practice.

In a recent paper, Blandford & Begelman (1999) have sought to resolve both these difficulties by elaborating on the basic model, adding terms that parametrize the removal of mass, angular momentum and energy by an outflow, which is considered necessary 'to allow accretion to proceed'. In their solution the flow is rotating even though $\gamma = 5/3$, and the Bernoulli parameter can be negative. However, this was again achieved by adjusting the scalings so that essentially no mass reaches the central object.

The purpose of this letter is to investigate the fluid dynamics of quasi-spherical accretion from an alternative viewpoint which avoids many of the strictures of steady self-similar solutions. In the theory of thin discs, a consideration of the initial-value problem, in which a cool and narrow ring of viscous fluid orbiting around a central mass spreads radially and is accreted, provides an excellent demonstration of the dynamics of disc accretion and is in some ways more informative than steady solutions. In that problem the surface density satisfies a diffusion equation which, depending on the properties of the viscosity, is either linear, in which case the Green function can be obtained analytically (Lüst 1952; Lynden-Bell & Pringle 1974), or non-linear, in which case similarity methods can be used to obtain an exact special solution of the equation (Pringle 1974; Lin & Bodenheimer 1982; Lin & Pringle 1987; Lyubarskii & Shakura 1987). This self-similar solution accurately describes the behaviour of all solutions of the initial-value problem long after the inner radius of the fluid first reaches the central object. In the case of quasi-spherical accretion, the governing equations are non-linear and similarity methods can again be used to provide an exact solution which may similarly be expected to be an attractor for the initial-value problem.

2 BASIC EQUATIONS

A simplified one-dimensional model for non-relativistic quasi-spherical accretion is adopted in which physical quantities depend only on the spherical radius r and on time t. The model is described by the equations

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u), \tag{1}$$

$$\rho\left(\frac{\mathrm{D}u}{\mathrm{D}t} - r\Omega^2\right) = -\rho\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{\partial p}{\partial r}, \tag{2}$$

$$\rho\frac{\mathrm{D}\ell}{\mathrm{D}t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\mu r^4\frac{\partial\Omega}{\partial r}\right) \tag{3}$$

and

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = -\frac{p}{r^2}\frac{\partial}{\partial r}(r^2 u) + \mu r^2\left(\frac{\partial\Omega}{\partial r}\right)^2. \tag{4}$$

Here ρ is the density, u the radial velocity, Ω the angular velocity, $\Phi = -GM/r$ the gravitational potential (where M is the mass of the central object), p the pressure, $\ell = r^2\Omega$ the specific angular momentum, μ the (dynamic) viscosity (which, for simplicity, is assumed to act only on the differential rotation) and e the specific internal energy. The Lagrangian time derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r}. \tag{5}$$

It is further assumed that the fluid is an ideal gas with

$$p = (\gamma - 1)\rho e, \qquad 1 < \gamma \leq 5/3, \tag{6}$$

and that the viscosity is given, as in NY, by an α-model,

$$\mu = \alpha p/\Omega_\mathrm{K}, \tag{7}$$

where $\Omega_\mathrm{K} = (GM/r^3)^{1/2}$ is the Keplerian angular velocity. The parameters γ and α are assumed to be constant.

These equations may be obtained from the full axisymmetric equations in spherical polar coordinates (r, θ, ϕ) by considering the equatorial plane $\theta = \pi/2$ and neglecting terms associated with the θ-velocity and any θ-dependence. The variables may be considered as approximate spherically averaged quantities. Although approximate, the equations have two important properties: they reduce exactly to those of spherical accretion in the absence of rotation, and they possess exact conservation laws for mass, angular momentum and energy, which are of the form

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u) = 0, \tag{8}$$

$$\frac{\partial}{\partial t}(\rho\ell) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho\ell r^2 u - \mu r^4\frac{\partial\Omega}{\partial r}\right) = 0 \tag{9}$$

and

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{1}{r^2}\frac{\partial F_\varepsilon}{\partial r} = 0, \tag{10}$$

where

$$\varepsilon = \tfrac{1}{2}u^2 + \tfrac{1}{2}(r\Omega)^2 + e + \Phi \tag{11}$$

is the specific total energy and

$$F_\varepsilon = \rho\varepsilon r^2 u + p r^2 u - \mu r^4\Omega\frac{\partial\Omega}{\partial r} \tag{12}$$

the total energy flux. These equations are appropriate to quasi-spherical accretion, rather than disc accretion, because the dissipated energy is retained in the fluid and may be advected with the flow.

3 SELF-SIMILAR SOLUTION

3.1 Analysis

NY solved essentially equations (1)–(4) for the case of a steady, radially self-similar flow. Such a solution is unbounded in size, mass, angular momentum and energy, and must therefore be interpreted as an approximation to a finite steady solution within a region far removed from its boundaries. In contrast, let us seek a time-dependent solution of finite size, for which the integrals representing the total mass, angular momentum and energy,

$$M = 4\pi\int_0^\infty \rho\, r^2\,\mathrm{d}r, \tag{13}$$

$$J = 4\pi\int_0^\infty \rho\ell\, r^2\,\mathrm{d}r, \tag{14}$$

$$E = 4\pi\int_0^\infty \rho\varepsilon\, r^2\,\mathrm{d}r, \tag{15}$$

are all convergent. The central object is considered to be arbitrarily small and to act as a sink for mass and (negative) energy but not as a source for angular momentum. Therefore J is strictly conserved, but M and E are not.

The time-dependent self-similar solution is found by a well-known procedure (e.g. Barenblatt 1979). In the standard terminology the solution is of type I, which means that the similarity variable can be deduced simply from dimensional considerations. In this case the dimensional constants present are the quantity GM and the total angular momentum J. One therefore identifies the similarity variable

$$\xi = r(GMt^2)^{-1/3} \tag{16}$$

and seeks a solution of the form

$$\rho(r,t) = \rho_*(\xi)\, J(GM)^{-5/3}t^{-7/3}, \tag{17}$$
$$u(r,t) = u_*(\xi)\,(GM)^{1/3}t^{-1/3}, \tag{18}$$
$$\Omega(r,t) = \Omega_*(\xi)\, t^{-1}, \tag{19}$$
$$e(r,t) = e_*(\xi)\,(GM)^{2/3}t^{-2/3}. \tag{20}$$

In such a solution each physical quantity retains a similar spatial form as the flow evolves, but the characteristic length-scale of the flow increases proportionally to $t^{2/3}$. Note that the total mass M is proportional to $t^{-1/3}$ and the total energy E (which is in fact negative) proportional to t^{-1}.

Substitution into equations (1)–(4) yields the dimensionless equations

$$(u_* - \tfrac{2}{3}\xi)\rho_*' - \tfrac{7}{3}\rho_* = -\rho_*\xi^{-2}(\xi^2 u_*)', \tag{21}$$

$$\rho_*\left[(u_* - \tfrac{2}{3}\xi)u_*' - \tfrac{1}{3}u_* - \xi\Omega_*^2 + \xi^{-2}\right] = -(\gamma-1)(\rho_* e_*)', \tag{22}$$

$$\rho_*\left[(u_* - \tfrac{2}{3}\xi)(\xi^2\Omega_*)' + \tfrac{1}{3}\xi^2\Omega_*\right] = (\gamma-1)\alpha\xi^{-2}(\rho_* e_*\xi^{11/2}\Omega_*')' \tag{23}$$

and

$$(u_* - \tfrac{2}{3}\xi)e_*' - \tfrac{2}{3}e_* = -(\gamma-1)e_*\xi^{-2}(\xi^2 u_*)' + (\gamma-1)\alpha e_*\xi^{7/2}(\Omega_*')^2, \tag{24}$$

where a prime denotes differentiation with respect to ξ. The constraint

$$4\pi\int_0^\infty \rho_*\xi^2\Omega_*\,\xi^2\,\mathrm{d}\xi = 1 \tag{25}$$

provides a normalization condition for the density.

This is a fifth-order system of non-linear ordinary differential equations. Critical points occur wherever

$$u_* - \tfrac{2}{3}\xi = 0 \quad \text{or} \quad \pm\left[\gamma(\gamma-1)e_*\right]^{1/2}, \tag{26}$$

i.e. wherever the radial velocity measured with respect to a self-similarly expanding coordinate system is equal to zero or to the local sound speed.

3.2 Inner limit

When $\gamma < 5/3$, an appropriate asymptotic solution as $\xi \to 0$ is of the form

$$\rho_*(\xi) \sim \xi^{-3/2}(A_\rho + B_\rho\xi^\lambda + \cdots), \tag{27}$$
$$u_*(\xi) \sim \xi^{-1/2}(A_u + B_u\xi^\lambda + \cdots), \tag{28}$$
$$\Omega_*(\xi) \sim \xi^{-3/2}(A_\Omega + B_\Omega\xi^\lambda + \cdots), \tag{29}$$
$$e_*(\xi) \sim \xi^{-1}(A_e + B_e\xi^\lambda + \cdots), \tag{30}$$

in which

$$A_u = -\frac{(9\gamma-5)}{9(\gamma-1)}\frac{g}{\alpha}, \tag{31}$$

$$A_\Omega^2 = \frac{2(5-3\gamma)(9\gamma-5)g}{81(\gamma-1)^2\alpha^2}, \tag{32}$$

$$A_e = \frac{2(9\gamma-5)g}{27(\gamma-1)^2\alpha^2}, \tag{33}$$

where

$$g = \left[1 + \frac{162(\gamma-1)^2\alpha^2}{(9\gamma-5)^2}\right]^{1/2} - 1, \tag{34}$$

while A_ρ is determined subsequently from the density normalization. The parameter λ (such that $0 < \lambda \le 1$) and the vector $[B_\rho, B_u, B_\Omega, B_e]^{\mathrm{T}}$ satisfy a certain algebraic eigenvalue problem. At leading order, we have essentially the solution of NY. This is reasonable because, for small ξ, the flow has experienced many orbits and, except for the declining density, may be expected to approach a steady state.

When $\gamma = 5/3$, we have instead

$$\rho_*(\xi) \sim \xi^{-3/2}(\tilde{A}_\rho + \tilde{B}_\rho\xi + \cdots), \tag{35}$$
$$u_*(\xi) \sim \xi^{-1/2}(\tilde{A}_u + \tilde{B}_u\xi + \cdots), \tag{36}$$
$$\Omega_*(\xi) \sim \xi^{-1/2}(\tilde{B}_\Omega + \cdots), \tag{37}$$
$$e_*(\xi) \sim \xi^{-1}(\tilde{A}_e + \tilde{B}_e\xi + \cdots), \tag{38}$$

in which

$$\tilde{A}_u = -\frac{5\tilde{g}}{\alpha} \quad \text{and} \quad \tilde{A}_e = \frac{15\tilde{g}}{\alpha^2}, \tag{39}$$

where

$$\tilde{g} = \left(1 + \frac{2\alpha^2}{25}\right)^{1/2} - 1. \tag{40}$$

The limit $\gamma \to 5/3$ is a singular one because $\lambda \to 0$ so that $\tilde{A}_u \ne \lim A_u$ and $\tilde{A}_e \ne \lim A_e$. In other words, the NY solution is subject to fractional corrections which become of order unity at all radii as $\gamma \to 5/3$, and the Bondi solution is never attained. The further development of the inner solution consists in general of an irregular power series which is beyond the scope of this paper.

3.3 Outer limit

Any solution of finite size must possess a free surface at some point $\xi = \xi_\mathrm{s}$. At such a point the velocity must equal $u = (\mathrm{d}r/\mathrm{d}t)_\xi$, which implies $u_* = \frac{2}{3}\xi_\mathrm{s}$. Assuming that the entropy tends to a finite limit, the surface resembles that of a polytrope. In terms of $s = \xi_\mathrm{s} - \xi$, we have

$$\rho_*(\xi) \sim s^{1/(\gamma-1)}(C_\rho + D_\rho s + \cdots), \tag{41}$$
$$u_*(\xi) \sim \tfrac{2}{3}\xi_\mathrm{s} + D_u s + \cdots, \tag{42}$$
$$\Omega_*(\xi) \sim C_\Omega + D_\Omega s + \cdots, \tag{43}$$
$$e_*(\xi) \sim s(C_e + D_e s + \cdots). \tag{44}$$

All the coefficients in this expansion, except for the density normalization, follow algebraically from a knowledge of ξ_s. The expansion fails if $\gamma < 9/7$.

3.4 Numerical solution

If the value of ξ_s is guessed, the equations can be integrated inwards from a point very near the surface, using the above expansion. By adjusting ξ_s the solution that has the correct asymptotic behaviour as $\xi \to 0$ can be identified. By this method satisfactory solutions have been obtained without internal critical points.

Examples of these solutions are shown in Figs 1 and 2. There are several points of interest. For $\gamma < 5/3$, the solution matches only asymptotically on to that of NY, deviating

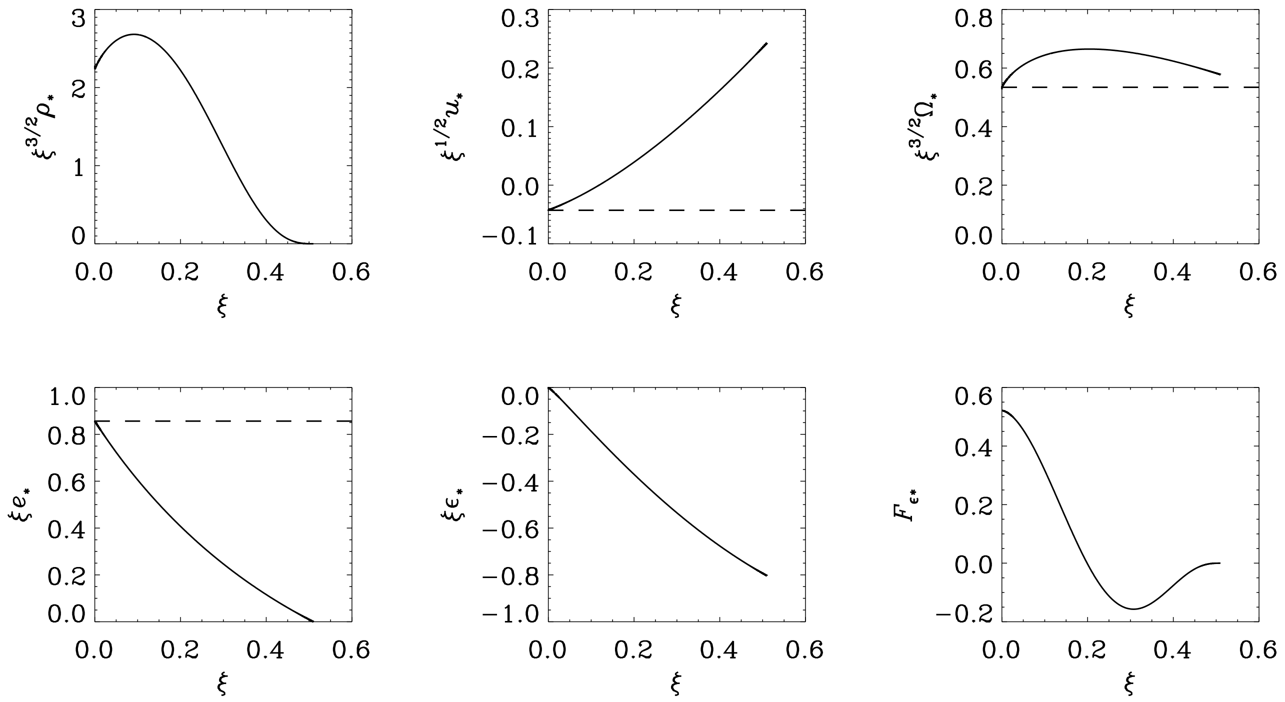

Figure 1. Time-dependent self-similar solution for $\gamma = 4/3$ and $\alpha = 0.1$. The surface occurs at $\xi_s \approx 0.5101$. The quantities ε_* and $F_{\varepsilon*}$, defined by $\varepsilon = \varepsilon_*(GM)^{2/3}t^{-2/3}$ and $F_\varepsilon = F_{\varepsilon*}Jt^{-2}$, are dimensionless versions of the specific total energy and the total energy flux. Note that $\xi^{3/2}\Omega_*$ is the ratio of the angular velocity to the Keplerian angular velocity, while ξe_* is proportional to the ratio of the temperature to the virial temperature. The dashed lines represent the steady solution of NY, which is the limiting form of the present solution as $\xi \to 0$.

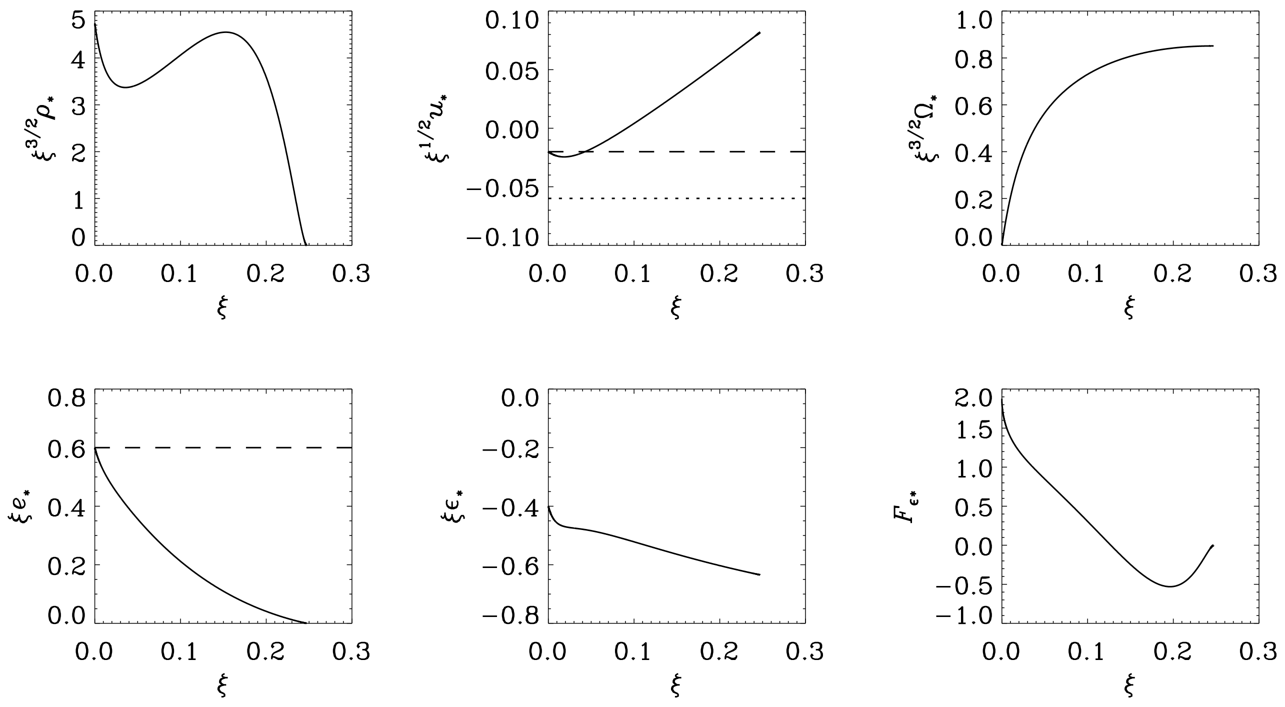

Figure 2. Time-dependent self-similar solution for $\gamma = 5/3$ and $\alpha = 0.1$. The surface occurs at $\xi_s \approx 0.2470$. The dashed lines represent the limiting form (35)–(38) of the solution. The dotted line represents the radial velocity according to NY.

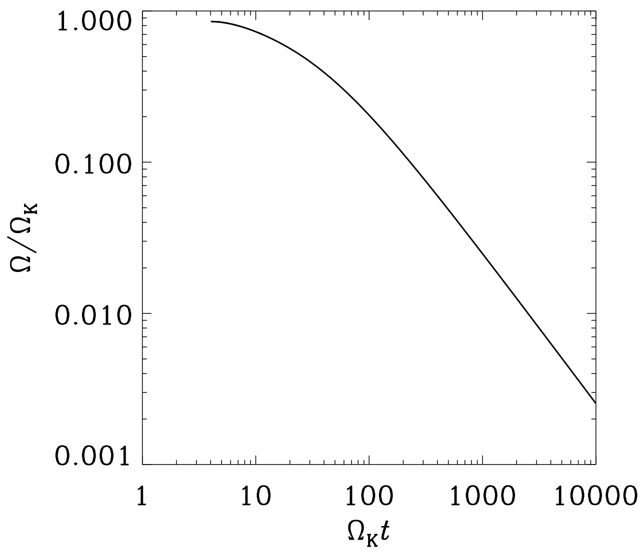

Figure 3. The variation with time of the ratio of the angular velocity to the Keplerian angular velocity at any fixed radius, for the same solution as in Fig. 2.

substantially from it at larger radii. In the case $\gamma = 5/3$ the flow is differentially rotating, although it is highly sub-Keplerian at small radii. For $\alpha \ll 1$, as in these examples, the radial velocity is everywhere highly subsonic with respect to a self-similarly expanding coordinate system. Although the temperature is comparable to the virial temperature over much of the flow, this does not drive a supersonic outflow at large radii. Finally, the flows are bound to the central object in the sense that the total energy is negative. As in a thin disc, there is a net outflow of energy from the centre of the disc because the central object is accreting bound matter.

4 DISCUSSION

In this paper the equations of time-dependent quasi-spherical accretion have been solved by semi-analytical similarity methods. The solution describes asymptotically the fate of an initially cool and narrow ring of viscous fluid orbiting around a central mass when it is unable to radiate efficiently. The flow is finite in size, mass, energy and angular momentum and avoids many of the strictures of the steady self-similar solution of NY.

In particular, a solution was found for the important case $\gamma = 5/3$ that has differential rotation and viscous dissipation, and resembles a disc at large radii. No barrier to accretion was found to exist in this case. At a fixed radius, the flow gradually becomes increasingly sub-Keplerian (Fig. 3). The limit $\gamma \to 5/3$ was shown to be singular so that the Bondi solution is not attained. This conclusion applies also to steady solutions that are not self-similar.

The flows are bound to the central object in the sense that the total energy is negative. Indeed, for $\gamma \geq 4/3$ the specific total energy is negative at all radii. It is suggested that previous interpretations based on the sign of the Bernoulli parameter may be insecure because Bernoulli's theorem does not hold for a viscous fluid (nor for a time-dependent one, as here). The meaning of the Bernoulli parameter is especially unclear in a flow that is driven specifically by viscous stresses.

The present model admits, at least in principle, the possibility of a supersonic outflow at large radii, but this was not found to be necessary. This suggests, contrary to Blandford & Begelman (1999), that quasi-spherical accretion can take place without the need for powerful outflows. However, because the model is topologically restricted, this conclusion must be regarded as tentative and one cannot exclude the possibility that an outflow would nevertheless occur from the hot inner part of the flow if the one-dimensional assumption were relaxed. This might be answered by restoring the latitudinal dependence in the equations and finding whether the analogous time-dependent self-similar solution includes a bipolar outflow. However, this would require the solution of a non-linear free-boundary problem of elliptic (or possibly mixed) type, which is technically demanding. It is perhaps more likely that the initial-value problem with latitudinal dependence could be solved using a time-dependent numerical method. Indeed, Igumenshchev, Chen & Abramowicz (1996) have already performed a calculation of this type, adopting $\gamma = 4/3$ and starting from a thick torus. They found rather complex behaviour, including convection, but no evidence for bipolar outflows.

The time-dependence of the flow is not essential to the conclusions of this paper. However, the present method of investigation allows a description of generic properties of solutions of the initial-value problem that is essentially free from infinities and contains no undetermined parameters, only the physical ones γ and α. Questions concerning the boundness of the flow and the necessity of outflows can be addressed with greater confidence within such a framework.

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