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Analytical solution for the evolution of a binary with stable mass transfer from a giant

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ABSTRACT

We derive a simple analytical solution for the evolution of a close binary with nuclear time-scale driven mass transfer from a giant. This solution is based on the well-known fact that the luminosity and the radius of a giant scale to a good approximation as simple power laws of the mass $M_{\rm c}$ of the degenerate helium core. Comparison with results of numerical calculations by Webbink, Rappaport & Savonije (1983) show the analytical solution and the power law approximation to be quite accurate. The analytical solution presented does also allow (in parametrized form) for non-conservative mass transfer. Furthermore it is shown that the near constancy of the mass transfer rate over most of the mass transfer phase seen in the results by Webbink, Rappaport & Savonije is not a generic feature of this type of evolution but rather a consequence of a particular choice of parameters. The analytical solution also demonstrates that the level of mass transfer is largely set by the core mass of the giant at the *onset* of mass transfer. Finally we show that the model is selfconsistent and discuss its applicability to low-mass X-ray binaries.

Key words: binaries: close – stars: evolution – stars: pulsars: general – X-rays: stars

1 INTRODUCTION

Stable mass transfer on to a (compact) accretor which is driven by the nuclear evolution of a giant is a process of considerable astrophysical interest. This type of evolution was first encountered in numerical simulations by (among others) Kippenhahn, Kohl & Weigert (1967) and then studied in some detail by Refsdal & Weigert (1969, 1971) in the context of the formation of low-mass white dwarfs in binaries. Later Webbink, Rappaport & Savonije (1983, hereafter WRS) and, independently, Taam (1983) have proposed that some of the most luminous among the galactic low-mass X-ray binaries could be powered by nuclear time-scale driven mass transfer from a giant. Finally with the detection of the millisecond pulsar binary PSR 1953+29 by Boriakoff, Buccheri & Fauci (1983) it became immediately clear that this system (and many other similar ones which have been found since then) must be the end product of an evolution of the type described by WRS and Taam (1983), see e.g. Joss & Rappaport (1983), Paczynski (1983), Savonije (1983), and Rappaport et al. (1995, hereafter RPJDH) for a more recent reference.

Whereas Taam (1983) performed full stellar structure calculations WRS took a simpler approach by making use of the well-known fact that giants obey a core mass-luminosity and a core mass-radius relation. WRS derived these relations from full stellar equilibrium models of giants and approximated them as power series of the mass of the degenerate helium core M_c. By using the core mass-luminosity and core mass-radius relation calculating the evolution of a binary with nuclear time-scale driven mass transfer from a giant reduces to preforming a simple integral in time which, however, in the approach taken by WRS has still to be performed numerically. The purpose of this paper is first to show that the evolution of such a binary can be described by an analytical solution if the core mass-luminosity and core mass-radius relation are simple power laws of M_c, and second to show that the results of full stellar structure calculations of giants given by WRS can indeed be approximated with little loss of accuracy by simple power laws. Moreover, the analytical solutions which we present are more general than the numerical ones of WRS in the sense that we allow for non-conservative mass transfer and a non-zero mass radius exponent of the donor star (see below section 3).

The advantages of having an analytical solution as compared to a numerical one are immediatedly clear: First, if sufficiently simple (as in our case), such solutions provide us with direct physical insight into how this type of evolution works, i.e. which parameters determine the characteristc properties and which ones are of minor importance. Second, whereas a numerical solution has to be computed separately for each set of input parameters of interest, an analytical solution provides us with the whole manifold of solutions, i.e. we get the explicit dependence of the solution on the input parameters. Among other things, this allows us to examine whether certain properties found in numerical solutions are generic or a simple consequence of a particular choice of parameters. A case in point is the well-known property of the solutions presented by WRS, namely that the mass transfer rate stays almost constant over most of the mass transfer phase.

Another area of application in which simple analytical solutions are superior to numerical ones is population synthesis. This is because population synthesis calculations usually involve integrals over the complete manifold of solutions of the particular evolutionary phase under consideration. Using numerical solutions for that task requires computation of a sufficiently dense grid of evolutions over a multi-dimensional parameter space and such computations can easily become prohibitively expensive (see e.g. Kolb 1993 for a detailed discussion in the context of cataclysmic variables). Using analytical solutions instead is comparatively easy and cheap.

This paper is organized as follows: In section 2 we present the basic model assumptions in the framework of which we shall work, in section 3 we give an analytical solution for the nuclear evolution of a giant, and in section 4 we derive expressions for the mass radius exponent of the donor star's critical Roche radius. Finally, in section 5 we present the analytical solution for nuclear time-scale driven mass transfer from a giant. In order to compare our results with those of WRS we derive in section 6 the parameters of the power law approximations of the core mass-luminosity and core mass-radius relations using the data from WRS and RPJDH. In section 7 we then compare the results obtained from the analytical solutions with the corresponding numerical ones of WRS. In section 8 we examine whether the model is selfconsistent and whether it is applicable to low-mass X-ray binaries. A discussion based on the analytical solution of the key features of nuclear time-scale driven mass transfer from a giant follows in section 9, and our main conclusions are summarized in the final section 10.

2 BASIC MODEL ASSUMPTIONS

In the following we shall list the basic model assumptions in the framework of which we shall work. These assumptions are:

(i) The donor star (of mass M_2) is on the first giant branch and has a degenerate helium core of mass M_c. Nuclear luminosity comes from hydrogen shell burning exclusively.

(ii) M_2 is small enough that the binary is adiabatically and thermally stable against mass transfer. For this to hold we must have (e.g. Ritter 1988)

$$\zeta_{ad} - \zeta_{R,2} > 0 \tag{1}$$

and

$$\zeta_e - \zeta_{R,2} > 0, \tag{2}$$

where

$$\zeta_{ad} = \left(\frac{\partial \ln R_2}{\partial \ln M_2}\right)_{ad} \tag{3}$$

is the adiabatic mass radius exponent of the donor star,

$$\zeta_e = \left(\frac{\partial \ln R_2}{\partial \ln M_2}\right)_{e} \tag{4}$$

the corresponding thermal equilibrium mass radius exponent, and

$$\zeta_{R,2} = \left(\frac{\partial \ln R_{R,2}}{\partial \ln M_2}\right)_{*} \tag{5}$$

the mass radius exponent of its critical Roche radius (see e.g. Ritter 1996 for details). In (5) the subscript $*$ indicates that for computing this quantity one has to specify how, as a consequence of mass transfer, mass and orbital angular momentum are lost from and redistributed within the binary system (see below section 4).

For stars on the first giant branch and with small relative core mass M_c/M $\zeta_{ad} \approx -1/3$. ζ_{ad} increases with relative core mass and becomes positive if $M_c/M \gtrsim 0.2$ (Hjellming & Webbink 1987). Therefore, as we shall see, adiabatic stability of mass transfer is less critical than thermal stability.

Assuming that the donor obeys a core mass-luminosity relation, i.e. that $L = L(M_c)$ and $\partial L/\partial M = 0$, where L is the (nuclear) luminosity, ζ_e follows from the Stefan–Boltzmann law:

$$\zeta_e = \frac{1}{2}\left(\frac{\partial \ln L}{\partial \ln M}\right)_e - 2\left(\frac{\partial \ln T_{eff}}{\partial \ln M}\right)_e = -2\left(\frac{\partial \ln T_{eff}}{\partial \ln M}\right)_e \tag{6}$$

It is well known that a giant, as long as it has a sufficiently deep outer convective envelope, in the HRD stays close to and evolves along the Hayashi line to higher luminosity. This property allows us to get an approximation for ζ_e. Approximating the Hayashi line in the HRD in the form (e.g. Kippenhahn & Weigert 1990)

$$\log L = a \log T_{eff} + b \log M + c \tag{7}$$

this together with (6) yields

$$\zeta_e \approx -2b/a\,. \tag{8}$$

Numerical calculations typically yield $-0.3 \lesssim \zeta_e \lesssim -0.2$. Thus, except for small relative core mass, $\zeta_e < \zeta_{ad}$, i.e. the criterion for thermal stability (2) against mass transfer is stronger than the one for adiabatic stability (1).

(iii) Mass transfer is driven only by the donor's nuclear evolution, i.e. we assume that there is no systemic angular momentum loss in the absence of mass transfer. Consequential angular momentum loss (see e.g. King & Kolb 1995 for a discussion) is, however, not excluded a priori.

(iv) Mass transfer is so slow that the donor star remains close to thermal equilibrium. Accordingly we assume that its radius R_2 and luminosity L_2 are given by the corresponding thermal equlilibrium values, i.e. $R_2 = R_{2,\mathrm{e}}$ and $L_2 = L_{2,\mathrm{e}}$, and hence $\zeta_{\mathrm{eff}} = (d\ln R_2/d\ln M_2) = \zeta_{\mathrm{e}}$. This condition is well fulfilled because mass transfer occurs on the nuclear time scale τ_{nuc} which is much longer than the Kelvin-Helmholtz time $\tau_{\mathrm{KH}} = G{M_2}^2/R_2L_2$. We shall later (section 8) verify the validity of this assumption.

With the above assumptions we can now write the mass transfer rate, i.e. the donor's mass loss rate as (e.g. Ritter 1996)

$$-\dot{M}_2 = \frac{M_2}{\zeta_{\mathrm{e}} - \zeta_{\mathrm{R},2}} \left(\frac{\partial \ln R_2}{\partial t}\right)_{\mathrm{nuc}}, \tag{9}$$

where $(\partial \ln R_2/\partial t)_{\mathrm{nuc}}$ is the inverse time-scale of the donor's expansion due to nuclear evolution. Next we shall work out in more detail $(\partial \ln R_2/\partial t)_{\mathrm{nuc}}$ and $\zeta_{\mathrm{R},2}$.

3 NUCLEAR EVOLUTION OF A GIANT

Here we are going to exploit the fact that to a very good approximation the luminosity L of a star on the first giant branch scales as a simple power law of the mass M_{c} of the degenerate helium core but does not depend on its total mass (Refsdal & Weigert 1970). Accordingly we make the following ansatz for L:

$$L(M_{\mathrm{c}}) = L_0 \left(\frac{M_{\mathrm{c}}}{\mathrm{M}_\odot}\right)^{\lambda} \tag{10}$$

It then follows from Eqs. (6) and (7) that the radius of such a star cannot at the same time depend only on the core mass M_{c} but must also depend on its total mass M. Because along the Hayashi line the effective temperature remains almost constant it follows from (10) and the Stefan–Boltzmann law that the appropriate form of the 'core mass-radius relation' is

$$R(M_{\mathrm{c}}, M) = R_0 \left(\frac{M_{\mathrm{c}}}{\mathrm{M}_\odot}\right)^{\rho} \left(\frac{M}{\mathrm{M}_\odot}\right)^{\zeta_{\mathrm{e}}}. \tag{11}$$

Here, L_0, λ, R_0 and ρ are parameters which have to be determined from full numerical calculations. We shall give examples (taken form WRS and RPJDH) in section 6. Using homology techniques Refsdal & Weigert (1970) could derive an expression for λ, the numerical value of which depends on the conditions in the hydrogen shell source. A typical value for Pop. I giants is $\lambda \approx 7 - 8$.

Hydrogen burning in the shell source around the degenerate core adds to the core mass at a rate $\dot{M}_{\mathrm{c}}$. Thereby the nuclear luminosity generated is

$$L = L_{\mathrm{nuc}} = XQ\dot{M}_{\mathrm{c}}\,, \tag{12}$$

where X is the hydrogen mass fraction in the star's envelope and $Q \approx 6\;10^{18}$ erg g^{-1} the net energy yield of hydrogen burning per unit mass. In writing (12) we have assumed that the star is in thermal equilibrium, i.e. that $L = L_{\mathrm{nuc}}$ or, for practical purposes, that $|L_g| \ll L_{\mathrm{nuc}}$, where L_g is the gravo-thermal luminosity.

(10) and (12) together yield a simple differential equation for $M_{\mathrm{c}}(t)$ with the solution

$$M_{\mathrm{c}}(t) = M_{\mathrm{c}}(t=0)\left(1 - \frac{t}{t_\infty}\right)^{\frac{1}{1-\lambda}}, \tag{13}$$

where

$$t_\infty = \frac{XQ\mathrm{M}_\odot}{(\lambda-1)L_0}\left[\frac{M_\mathrm{c}(t=0)}{\mathrm{M}_\odot}\right]^{1-\lambda} \tag{14}$$

is the time over which, formally, the core mass reaches infinity. It is also the characteristic time for nuclear evolution on the first giant branch. (13) combined with respectively (10) or (11) yields

$$L(t) = L_0\left[\frac{M_\mathrm{c}(t=0)}{\mathrm{M}_\odot}\right]^{\lambda}\left(1-\frac{t}{t_\infty}\right)^{\frac{\lambda}{1-\lambda}} \tag{15}$$

and

$$R(t) = R_0\left(\frac{M}{\mathrm{M}_\odot}\right)^{\zeta_\mathrm{e}}\left[\frac{M_\mathrm{c}(t=0)}{\mathrm{M}_\odot}\right]^{\rho}\left(1-\frac{t}{t_\infty}\right)^{\frac{\rho}{1-\lambda}}. \tag{16}$$

Here $M_\mathrm{c}(t=0)$ is the core mass at the beginning of the evolution on the first giant branch. This quantity has also to be taken from full stellar evolution calculations or to be treated as a free parameter.

From (16) we finally obtain

$$\left(\frac{\partial \ln R}{\partial t}\right)_\mathrm{nuc} = \frac{\rho L_0}{XQ\mathrm{M}_\odot}\left(\frac{M_\mathrm{c}}{\mathrm{M}_\odot}\right)^{\lambda-1} = \frac{\rho}{\lambda-1}\,\frac{1}{t_\infty-t}, \tag{17}$$

the expression to be inserted in (9).

4 COMPUTING $\zeta_{\mathrm{R},2}$

Here we follow standard arguments which have been given many times in the literature (e.g. Soberman, Phinney & van den Heuvel 1997 and references therein). Therefore, we shall give here only the definitions we use and the final results. In the context of this paper, we assume that mass transfer occurs from the secondary and that the primary (of mass M_1) accretes (on average) a fraction η of the transferred mass, i.e.

$$dM_1 = -\eta dM_2 \tag{18}$$

and that the mass $d(M_1+dM_2) = (1-\eta)dM_2$ leaves the binary system carrying away the orbital angular momentum

$$dJ = \nu J\frac{d(M_1+M_2)}{M_1+M_2}, \tag{19}$$

where, for the moment at least, η and ν are free parameters,

$$J = \left(\frac{G{M_1}^2{M_2}^2a}{M_1+M_2}\right)^{\frac{1}{2}} \tag{20}$$

is the orbital angular momentum, and a the orbital separation. Writing for the secondary's critical Roche radius

$$R_{\mathrm{R},2} = af_2(q), \tag{21}$$

where

$$q = \frac{M_1}{M_2} \tag{22}$$

is the mass ratio, one derives

$$\zeta_{\mathrm{R},2} = (1-\eta)\frac{2\nu+1}{1+q} + \frac{2\eta}{q} - 2 - \beta_2\left(1+\frac{\eta}{q}\right), \tag{23}$$

with

$$\beta_2 = \frac{d\ln f_2}{d\ln q}\,. \tag{24}$$

As a further simplification we use for f_2 in (21) and (24) the approximation (Paczynski 1971)

$$f_2(q) = \left(\frac{8}{81}\right)^{1/3}(1+q)^{-1/3}, \quad q \gtrsim 1.25 \tag{25}$$

which yields

$$\beta_2(q) = -\frac{q}{3(1+q)}\,. \tag{26}$$

Inserting (26) in (23) finally yields

$$\zeta_{\mathrm{R},2} = \frac{(1-\eta)(2\nu+1)}{1+q} + \frac{2\nu}{q} - 2 + \frac{\eta+q}{3(1+q)}\,. \tag{27}$$

In the following we shall also give the results for two specific choices of the parameters η and ν:

4.1 Conservative mass transfer

Here $\eta = 1$ and $\nu = 0$. From (23) and (26) we then have

$$\zeta_{\mathrm{R},2} = \frac{6-5q}{3q}\,. \tag{28}$$

4.2 Isotropic stellar wind from the primary

Here we are considering the case that all the transferred mass is ejected from the primary in an isotropic stellar wind carrying with it the specific orbital angular momentum of the primary. Accordingly $\eta = 0$ and $\nu = 1/q$. Inserted in (23) together with (26) this yields

$$\zeta_{\mathrm{R},2} = \frac{6-5q^2-3q}{3q(1+q)}\,. \tag{29}$$

5 EVOLUTION WITH MASS TRANSFER FROM A GIANT

Now that we have determined $(\partial \ln R/\partial t)_{\mathrm{nuc}}$ and $\zeta_{\mathrm{R},2}$ in equation (9) we can solve that equation. First we are going to give a rather general solution before discussing special cases of interest.

5.1 General analytical solution

General solution in this context means that we allow for a constant value $\eta \geq 0$, either a constant value of $\nu \geq 0$ or $\nu = 1/q$, and a constant value of ζ_{e} subject to the condition (2). Inserting (17) and (23) with (26) in (9) yields an ordinary differential equation for $M_2(t)$, or rather, as we shall see, for $t(M_2)$ which is solved by separating the variables M_2 and t:

$$\int_{M_{2,\mathrm{i}}}^{M_2} \frac{\zeta_{\mathrm{e}}-\zeta_{\mathrm{R},2}}{m_2}\,dm_2 = \int_0^t \left(\frac{\partial \ln R}{\partial \tau}\right)_{\mathrm{nuc}} d\tau \tag{30}$$

The solution of (30) can be written as follows:

$$t(M_2) = t_\infty \left[1 - \left(\frac{M_1}{M_{1,\mathrm{i}}}\right)^{-p_1} \left(\frac{M_2}{M_{2,\mathrm{i}}}\right)^{-p_2} \left(\frac{M_1+M_2}{M_{1,\mathrm{i}}+M_{2,\mathrm{i}}}\right)^{-p_3} e^{-p_4 \frac{M_{2,\mathrm{i}}-M_2}{M_1}}\right], \tag{31}$$

where the exponents p_1, p_2, p_3 and p_4 are given in Table 1 for various cases of interest. With (31) the mass transfer rate can be written as

$$-\dot{M}_2 = \frac{1}{\zeta_\mathrm{e} - \zeta_{\mathrm{R},2}} \frac{\rho L_0}{XQ} \left(\frac{M_{\mathrm{c,i}}}{\mathrm{M}_\odot}\right)^{\lambda-1} \left(\frac{M_2}{\mathrm{M}_\odot}\right) \left(1 - \frac{t}{t_\infty}\right)^{-1}. \tag{32}$$

Inserting (31) in (15) and (16) finally also yields

$$L_2 = L_0 \left(\frac{M_{\mathrm{c,i}}}{\mathrm{M}_\odot}\right)^{\lambda} \left[\left(\frac{M_1}{M_{1,\mathrm{i}}}\right)^{p_1} \left(\frac{M_2}{M_{2,\mathrm{i}}}\right)^{p_2} \left(\frac{M_1+M_2}{M_{1,\mathrm{i}}+M_{2,\mathrm{i}}}\right)^{p_3} e^{p_4 \frac{M_{2,\mathrm{i}}-M_2}{M_1}}\right]^{\frac{\lambda}{\lambda-1}} \tag{33}$$

and

$$R_2 = R_0 \left(\frac{M_{\mathrm{c,i}}}{\mathrm{M}_\odot}\right)^{\rho} \left(\frac{M_2}{\mathrm{M}_\odot}\right)^{\zeta_\mathrm{e}} \left[\left(\frac{M_1}{M_{1,\mathrm{i}}}\right)^{p_1} \left(\frac{M_2}{M_{2,\mathrm{i}}}\right)^{p_2} \left(\frac{M_1+M_2}{M_{1,\mathrm{i}}+M_{2,\mathrm{i}}}\right)^{p_3} e^{p_4 \frac{M_{2,\mathrm{i}}-M_2}{M_1}}\right]^{\frac{\rho}{\lambda-1}}. \tag{34}$$

Here $M_{1,\mathrm{i}}$, $M_{2,\mathrm{i}}$ and $M_{\mathrm{c,i}}$ are respectively the initial values, i.e. taken at $t = 0$, of M_1, M_2, and M_c. Note that from (18) we have

$$M_1 = M_{1,\mathrm{i}} + \eta\,(M_{2,\mathrm{i}} - M_2)\ . \tag{35}$$

As can be seen the equations (32) (together with (31)), (33) and (34) do not depend explicitly on time t. The solutions are completely fixed by the parameters of the core mass-luminosity relation (10), i.e. L_0 and λ, the 'core mass-radius relation' (11), i.e. R_0, ρ and ζ_e, the initial values and the current values of both masses and the secondary's core mass. Kowing $\dot{M}_2$, R_2 and L_2 as explicit functions of the initial values and current values of M_1, M_2 and M_c other quantities of interest such as the orbital period P, the orbital separation a, the size of the accretor's Roche radius $R_{\mathrm{R},1}$ and with it the size of the disc around the accretor and the secondary's effective temperature $T_{\mathrm{eff},2}$ follow directly. What this solution does not provide explicitly is the total duration of the mass transfer phase., i.e. the time t_f when the secondary's envelope is exhausted or, which is equivalent, the final core mass $M_{\mathrm{c,f}}$. A very good estimate of the latter together with t_f can be obtained by inserting (31) with $M_2 = M_{2,\mathrm{f}} \approx M_{\mathrm{c,f}}$ in (13). Note that the resulting equation for $M_{\mathrm{c,f}}$ cannot in general be solved in closed form. Reinserting the solution of this equation in (31) finally yields also t_f.

5.2 Special cases

Table 1 lists the powers p_1, p_2, p_3 and p_4 which appear in equations (31) – (34) for a number of special cases of interest.

Case 1 is the most general case for constant values of η, ν and ζ_e. In case 2 the special choice $\nu = 1/q$ with $\eta > 0$ is made, whereas case 3 corresponds to case 2 in the limit $\eta = 0$. Cases 4 and 5 correspond to conservative mass transfer with respectively $\zeta_\mathrm{e} = \mathrm{const.} \neq 0$ and $\zeta_\mathrm{e} = 0$. Case 6 finally corresponds to case 3 with $\zeta_\mathrm{e} = 0$.

Case 3 (or 6) corresponds to the situation in which the accretor ejects all the transferred mass with the specific orbital angular momentum of the accretor's orbit. This approximates what apparently must happen in systems in which the mass transfer rate to a neutron star exceeds the Eddington accretion rate by many orders of magnitude (see e.g. King & Ritter 1999 for a discussion).

We shall later mostly make use of case 5 because this is the approximation within which

Table 1. Parameters and coefficients entering the analytical solution

case	η	ν	ζ_e	p_1	p_2	p_3	p_4
1	≥ 0, const.	const.	const.	$-\frac{2(\lambda-1)}{\rho}$	$-\frac{(3\zeta_e+5)(\lambda-1)}{3\rho}$	$\frac{2(3\nu+1)(\lambda-1)}{3\rho}$	0
2	> 0, const.	$1/q$	const.	$-\frac{2(\lambda-1)}{\eta\rho}$	$-\frac{(3\zeta_e+5)(\lambda-1)}{3\rho}$	$-\frac{4(\lambda-1)}{3\rho}$	0
3	0	$1/q$	const.	0	$-\frac{(3\zeta_e+5)(\lambda-1)}{3\rho}$	$-\frac{4(\lambda-1)}{3\rho}$	$-\frac{2(\lambda-1)}{\rho}$
4	1	—-	const.	$-\frac{2(\lambda-1)}{\rho}$	$-\frac{(3\zeta_e+5)(\lambda-1)}{3\rho}$	0	0
5	1	—-	0	$-\frac{2(\lambda-1)}{\rho}$	$-\frac{5(\lambda-1)}{3\rho}$	0	0
6	0	$1/q$	0	0	$-\frac{5(\lambda-1)}{3\rho}$	$-\frac{4(\lambda-1)}{3\rho}$	$-\frac{2(\lambda-1)}{\rho}$

WRS have performed their numerical calculations and with which we wish to compare the results of our analytical solution. Before we can do that we have to provide the parameters describing the core mass-luminosity and core mass-radius relation.

6 APPROXIMATIONS FOR THE CORE MASS-LUMINOSITY AND CORE MASS RADIUS RELATION

In general the values of L_0, λ, R_0, ρ and ζ_e in equations (10) and (11) have to be determined from full numerical calculations of stars in thermal equilibrium on the first giant branch. Here we restrict ourselves to deriving approximate values of these parameters by using results given by WRS (their equations 4 – 6). Because WRS have assumed $\zeta_e = 0$ their relation (5) is a true core mass-radius relation in the sense that the radius of the star does not depend on the total mass M. Their results for Pop. I and Pop. II giants are shown graphically respectively in Fig. 1 and Fig. 2. It is immediately apparent that in the range $0.15\mathrm{M}_\odot \lesssim M_c \lesssim 0.4\mathrm{M}_\odot$ the relations $L(M_c)$ and $R(M_c)$ are very nearly simple power laws in M_c. Based on the results of Refsdal & Weigert (1970) this is of course not surprising. Because our analytical solutions of section 5 are based on the power law approximations (10) and (11) we fit the results of WRS shown in Figs. 1 and 2 accordingly. The resulting parameters, together with the chemical composition of the underlying numerical models (characterized in the usual way by the hydrogen mass fraction X and the the metallicity Z) are summarized in Table 2. The corresponding relations are shown as thin full (L) or dashed (R) lines in Figs. 1 and 2.

For comparison we provide in Table 2 also approximate parameters of a power law fit in the range $0.2\mathrm{M}_\odot \lesssim M_c \lesssim 0.5\mathrm{M}_\odot$ to the core mass-radius relation derived by RPJDH (their equation 5).

7 NUMERICAL RESULTS VERSUS THE ANALYTICAL MODEL

Here we wish to compare the results obtained from the analytical approximation, i.e. from equations (31) – (34), with the numerical results obtained by WRS. For this we have to insert the parameters listed in Table 2 into the expressions for p_1, p_2, p_3, and p_4 corresponding to case 5 in Table 1 and in equations (31) – (34). Rather than listing all the resulting equations for both Pop. I and Pop. II composition, we restrict ourselves to giving only the corresponding expressions for the mass transfer rate. With $Q = 6\ 10^{18}$ erg g^{-1} this yields

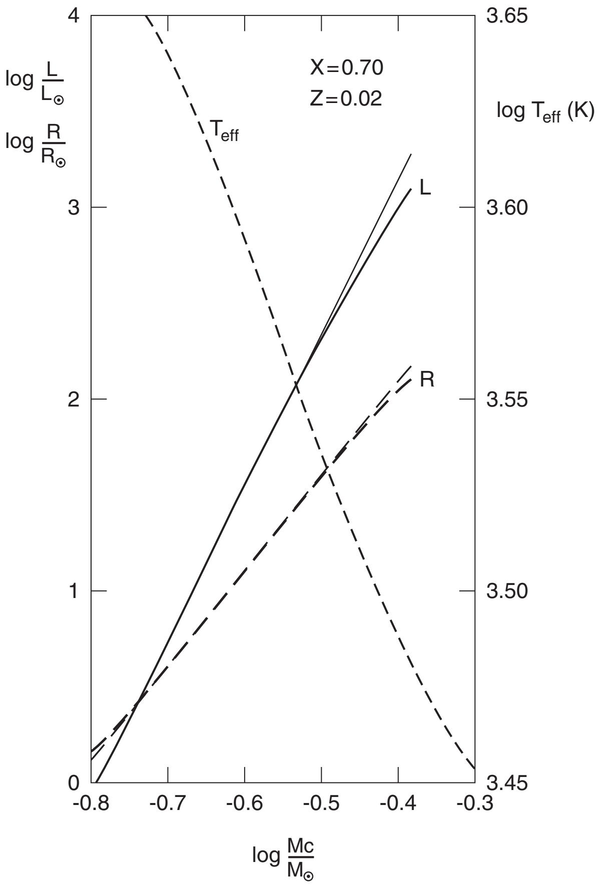

Figure 1. Core mass-luminosity relation (full line), core mass-radius relation (long-dashed line) and core mass-effective temperature relation (short-dashed line) for Pop. I giants according to numerical calculations by WRS. Our simple power law fits are shown respectively as thin full and thin long-dashed lines. The corresponding fit parameters are listed in Table 2.

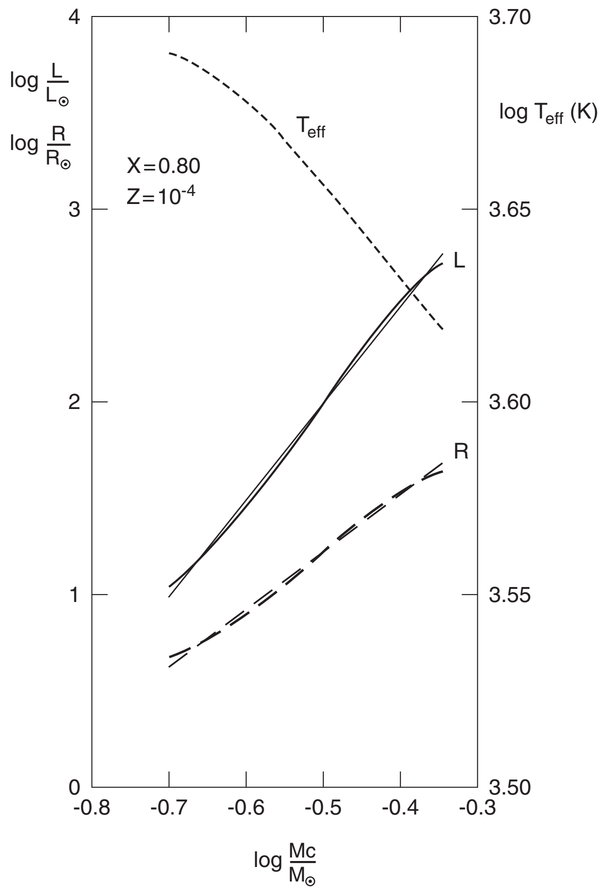

Figure 2. The same as Fig. 1, but for Pop. II giants.

Table 2. Approximate parameters of the core mass-luminosity and core mass-radius relations derived from numerical results

Pop.	X	Z	$\log \frac{L_0}{L_\odot}$	λ	$\log \frac{R_0}{R_\odot}$	ρ	ζ_e	source
I	0.70	0.02	6.3	8	4.1	5	0	WRS
II	0.80	0.0001	4.5	5	2.7	3	0	WRS
I	0.70	0.02	—	—	3.58	4.21	0	RPJDH
II	0.70	0.001	—	—	3.36	4.17	0	RPJDH

$$-\dot{M}_2 = 2.64\ 10^{-8}\,\mathrm{M_\odot yr^{-1}} \left(\frac{M_{\rm c,i}}{0.25\mathrm{M_\odot}}\right)^7 \frac{M_1}{5M_1-6M_2}\left(\frac{M_2}{\mathrm{M_\odot}}\right)\left(\frac{M_2}{M_{2,\rm i}}\right)^{-7/3}\left(\frac{M_1}{M_{1,\rm i}}\right)^{-14/5} \quad (36)$$

for Pop. I, and

$$-\dot{M}_2 = 1.40\ 10^{-8}\,\mathrm{M_\odot yr^{-1}} \left(\frac{M_{\rm c,i}}{0.25\mathrm{M_\odot}}\right)^4 \frac{M_1}{5M_1-6M_2}\left(\frac{M_2}{\mathrm{M_\odot}}\right)\left(\frac{M_2}{M_{2,\rm i}}\right)^{-20/9}\left(\frac{M_1}{M_{1,\rm i}}\right)^{-8/3} \quad (37)$$

for Pop. II chemical composition.

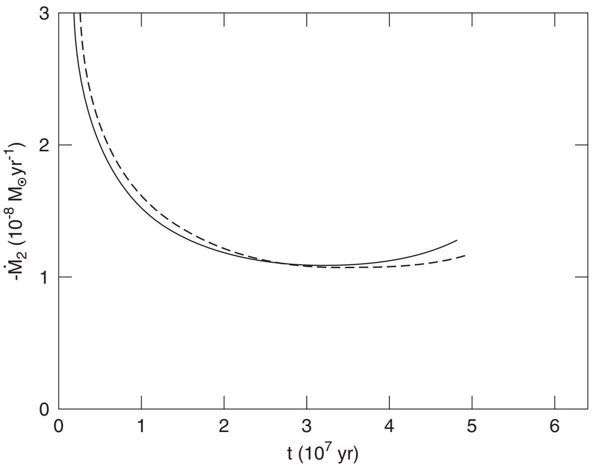

Figure 3. Comparison of the result obtained with the analytical solution (full line) with that of a matching numerical one by WRS (dashed line). Shown is the mass loss rate from a Pop. I giant as a function of the time for the parameters used by WRS in their figure 3, i.e. $X = 0.70$, $Z = 0.02$, $M_{1,\mathrm{i}} = 1.4\mathrm{M}_\odot$, $M_{2,\mathrm{i}} = 1\mathrm{M}_\odot$, and $M_{\mathrm{c,i}} = 0.26\mathrm{M}_\odot$.

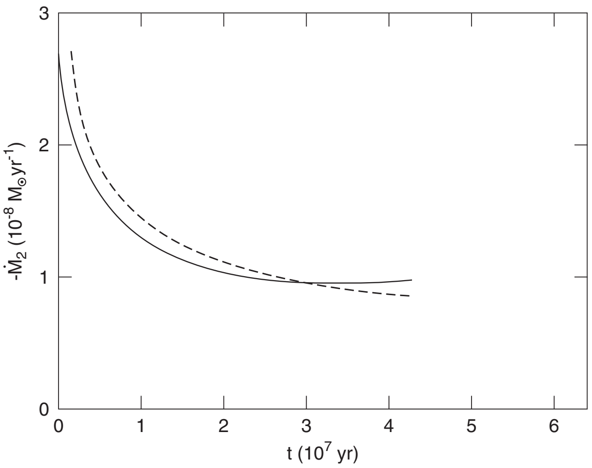

Figure 4. Comparison of the result obtained with the analytical solution (full line) with that of a matching numerical one by WRS (dashed line). Shown is the mass loss rate from a Pop. II giant as a function of the time for the parameters used by WRS in their figure 4, i.e. $X = 0.80$, $Z = 10^{-4}$, $M_{1,\mathrm{i}} = 1.4\mathrm{M}_\odot$, $M_{2,\mathrm{i}} = 0.9\mathrm{M}_\odot$, and $M_{\mathrm{c,i}} = 0.31\mathrm{M}_\odot$.

Results of detailed numerical computations by WRS are shown in their figs. 3 and 4. Comparing the results obtained from our analytical solution for the same parameters with those of WRS shows that the differences are small, too small in fact to be adequately shown on the scale of these figures. That the differences in $L(t)$ and $R(t)$ are small was to be expected based on the quality of the fits used (see Figs. 1 and 2). That the differences in $-\dot{M}_2$ are small too is shown in Figs. 3 and 4. In that context we should emphasize that our fits of the $L(M_\mathrm{c})$ and $R(M_\mathrm{c})$ relations of WRS with power laws are the simplest possible ones (with integer powers λ and ρ) and that no attempt has been made to adjust the power law parameters such as to minimize the differences between the numerical results of WRS and the analytical solution. With $-\dot{M}_2$ being in close agreement with the numerical results it follows that also $M_2(t)$ and $P(t)$ should agree very well, which is indeed the case. Finally also $M_\mathrm{c}(t)$ must be in close agreement with the numerical result of WRS, otherwise the temporal evolution of the other quantities would not match either.

In fact we can summarize the result of this comparison as follows: if the power law fits to the core mass-luminosity and core mass-radius relations obtained from full stellar structure calculations are adequate, the analytical model provides an adequate approximation to the results of numerical computations.

8 SELFCONSISTENCY AND APPLICABILITY OF THE MODEL

8.1 Deviation from thermal equilibrium

The equations we have derived in section 5 are valid only to the extent that the donor star is not significantly driven out of thermal equilibrium as a consequence of mass loss. That is to say that the time-scale of mass loss

$$\tau_{\mathrm{M}_2} = -\frac{dt}{d\ln M_2} \tag{38}$$

must be significantly longer than the shortest time-scale on which the donor's convective envelope can react, i.e. longer than the Kelvin–Helmholtz time of the convective envelope (e.g. King et al. 1997a)

$$\tau_{\rm KH,ce} = \frac{3}{7}\,\frac{M_{\rm ce}}{M}\,\tau_{\rm KH} = \frac{3}{7}\,\frac{M-M_{\rm c}}{M}\,\frac{GM^2}{RL}\,. \tag{39}$$

Here $M_{\rm ce} = M - M_{\rm c}$ is the mass of the convective envelope and $\tau_{\rm KH}$ the Kelvin–Helmholtz time of the whole star. A measure for the star's deviation from thermal equilibrium is thus the the ratio $\tau_{\rm M_2}/\tau_{\rm KH,ce}$. Using equations (9) and (15) – (17) in (38) and (39) we can now write

$$\tau_{\rm M_2} = (\zeta_{\rm e} - \zeta_{\rm R,2})\,\frac{XQ{\rm M}_\odot}{\rho L_0}\left(\frac{M_{\rm c,i}}{{\rm M}_\odot}\right)^{1-\lambda}\left(1-\frac{t}{t_\infty}\right) \tag{40}$$

and

$$\tau_{\rm KH,ce} = \frac{3}{7}\,\frac{G{\rm M}_\odot{}^2}{R_0L_0}\,\frac{M_2-M_{\rm c}}{M_2}\left(\frac{M_2}{{\rm M}_\odot}\right)^{2-\zeta_{\rm e}}\left(\frac{M_{\rm c,i}}{{\rm M}_\odot}\right)^{-(\lambda+\rho)}\left(1-\frac{t}{t_\infty}\right)^{\frac{\lambda+\rho}{\lambda-1}}\,, \tag{41}$$

and, therefore,

$$\frac{\tau_{\rm M_2}}{\tau_{\rm KH,ce}} = (\zeta_{\rm e} - \zeta_{\rm R,2})\,\frac{7}{3}\,\frac{XQR_0}{\rho G{\rm M}_\odot}\,\frac{M_2}{M_2-M_{\rm c}}\left(\frac{M_2}{{\rm M}_\odot}\right)^{\zeta_{\rm e}-2}\left(\frac{M_{\rm c,i}}{{\rm M}_\odot}\right)^{1+\rho}\left(1-\frac{t}{t_\infty}\right)^{\frac{1+\rho}{1-\lambda}}\,. \tag{42}$$

Inserting now for illustrative puposes the parameters of Table 2 which correspond to the results of WRS into (42) yields

$$\frac{\tau_{\rm M_2}}{\tau_{\rm KH,ce}} = 3.16\;10^3\;(-\zeta_{\rm R,2})\,\frac{M_2}{M_2-M_{\rm c}}\left(\frac{M_2}{{\rm M}_\odot}\right)^{-2}\left(\frac{M_{\rm c,i}}{0.25\;{\rm M}_\odot}\right)^{6}\left(1-\frac{t}{t_\infty}\right)^{-6/7} \tag{43}$$

for Pop. I, and

$$\frac{\tau_{\rm M_2}}{\tau_{\rm KH,ce}} = 3.83\;10^3\;(-\zeta_{\rm R,2})\,\frac{M_2}{M_2-M_{\rm c}}\left(\frac{M_2}{{\rm M}_\odot}\right)^{-2}\left(\frac{M_{\rm c,i}}{0.25\;{\rm M}_\odot}\right)^{4}\left(1-\frac{t}{t_\infty}\right)^{-1} \tag{44}$$

for Pop. II chemical composition.

What (42) – (44) demonstrate is that for all practical purposes $\tau_{\rm M_2}/\tau_{\rm KH,ce} \gg 1$ unless the mass ratio is such that the binary system is close to thermal instability, i.e. that $\zeta_{\rm e}-\zeta_{\rm R,2} \ll 1$. Thus mass transfer driven by nuclear evolution of a star on the first giant branch is always sufficiently slow for the donor star to remain essentially in thermal equilibrium. To this extent, the analytical model presented in section 5 is selfconsistent.

8.2 Irradiation of the donor star

Since one of the main areas of application of this model is the evolution of luminous X-ray binaries (see e.g. Taam 1983 and WRS) and their transformation into millisecond pulsar binaries (see e.g. RPJDH and references therein) we have to examine whether this model remains applicable if irradiation of the cool donor star by X-rays from the vicinity of a very compact accretor is taken into account. As has first been shown by Podsiadlowski (1991) cool stars, when exposed to sufficiently strong external heating by X-rays, respond by expansion on the thermal time-scale of the convective envelope, thereby giving rise to very high mass transfer rates in a semi-detached binary. In the context of mass transfer from a giant to a compact accretor the stability of mass transfer with taking into account irradiation has been examined in some detail by King et al. (1997a). Their finding was that if the accretion rate on to the compact star equals the mass transfer rate at any time, the evolutionary model discussed here is not in general applicable, because mass transfer is very likely to be unstable. Rather than being continuous, mass transfer occurs in cycles which are characterized by short high states during which very high mass transfer rates are achieved and long low states during which the binary is detached. As has been shown by King et al. (1997a) this instability occurs mainly because for giants the ratio of mass loss time-scale to Kelvin–Helmholtz time (equation 42) is so large. It is therefore somewhat ironic that self-consistency of the model of nuclear time-scale driven stationary mass transfer from a giant is possible only at the price that such systems are in danger of being unstable to irradiation-driven mass transfer.

However, the above-mentioned condition, namely that the accretion rate equals the mass transfer rate at any time, is not likely to be fulfilled. Rather, as has been shown by King et al. (1997b), the accretion disc in such binary systems is likely to be unstable and that even for the rather optimistic assumptions made by King et al. (1997b) about the efficiency of X-ray irradiation of the accretion disc. Based on newer results of Dubus et al. (1998) one could conclude that the accretion discs in essentially all systems with a giant donor are unstable. If this is the case, then accretion on to the compact star is spasmodic because the disc undergoes a dwarf nova-like limit-cycle instability. Estimates of the properties of the outbursts (e.g. King & Ritter 1998) show that they are characterized by a very small duty cycle. This, in turn, is now of great importance for the stability of mass transfer from the giant in the presence of X-ray irradiation: King et al. (1997a) show that if accretion on to a neutron star or a black hole is spasmodic with a small duty cycle, then mass transfer from the giant is stable despite the donor being irradiated. If that is the case, the model considered by Taam (1993), WRS and us is applicable.

On the other hand, if the accretor is a white dwarf the situation is more complicated: If the core mass of the donor is $M_{\rm c} \lesssim 0.2{\rm M}_\odot$, irradiation cannot destabilize mass transfer and, therefore, the model is applicable. Yet if $M_{\rm c} \gtrsim 0.2{\rm M}_\odot$, mass transfer is unstable. Although spasmodic accretion has a stabilizing effect, for mass transfer to become stable in this case the required small duty cycle has to be smaller still by a factor of $\sim 10 \cdots 100$ than in the case of a black hole or a neutron star (see King et al. 1997a, fig. 4). Because the value of the duty cycle does not depend explicitely on the nature of the accretor but rather on the average mass transfer rate and on the size of the accretion disc, it appears unlikely that the duty cycle in a white dwarf system is systematically smaller than in a black hole or neutron star system. Therefore it is more likely that in a white dwarf system in which $M_{\rm c} \gtrsim 0.2{\rm M}_\odot$ mass transfer is unstable despite spasmodic accretion. In this case the model in question is not applicable. However, we should emphasize that detailed numerical calculations of mass transfer from a giant with taking into account the effects of irradiation of the donor star and of disc instabilites have yet to be done. Before results of such computations are available, a final decision about whether this model is applicable cannot be made.

9 DISCUSSION

Having established that the analytical approximation is both accurate and selfconsistent, and that the model is probably also applicable to the most important case of interest, i.e. to low-mass X-ray binaries, we can now proceed to discuss the properties of these solutions. One of the remarkable properties of this type of evolution which has already been pointed out by WRS is that the mass transfer rate remains almost constant for most of the mass transfer phase. Here we shall examine whether this is a generic property of mass transfer from a giant or whether this is due to a particular choice of parameters. Other aspects which we wish examine are the dependence of the temporal evolution on the initial parameters, in particular on the initial core mass $M_{\rm c,i}$, and systematic differences between Pop. I and Pop. II systems.

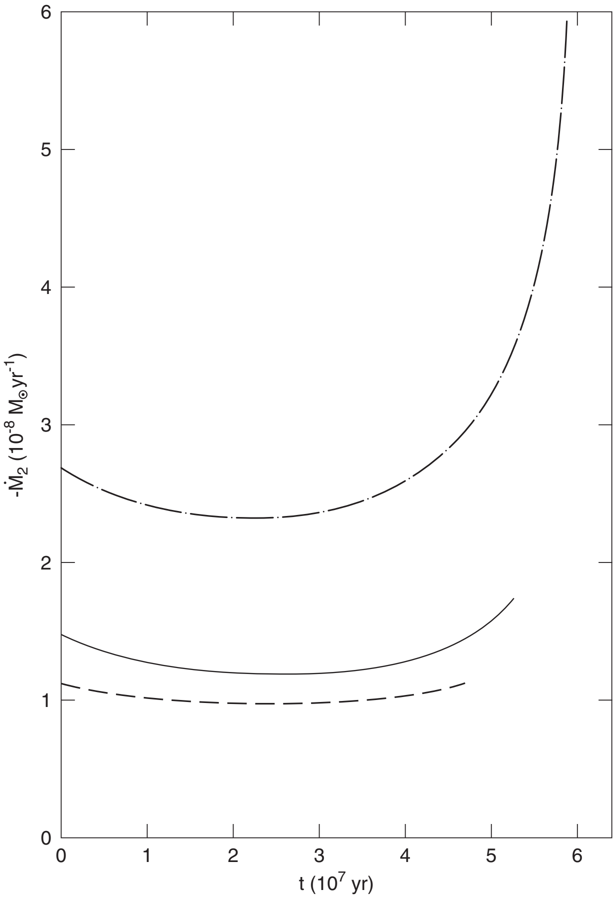

Figure 5. Mass transfer rate computed from the analytical solution for three different cases: The full and dashed lines show respectively the mass transfer rate as a function of time in an evolution of a Pop. I and a Pop. II system with same initial parameters as those used in Figs. 3 and 4, but now for non-conservative mass transfer characterized by $\eta = 0$ and $\nu = 1/q$. The dash-dotted line shows the mass transfer rate in a conservative evolution of a Pop. I system with the parameters $M_{1,\mathrm{i}} = 5\mathrm{M}_\odot$, $M_{2,\mathrm{i}} = 2\mathrm{M}_\odot$, and $M_{\mathrm{c,i}} = 0.26\mathrm{M}_\odot$.

9.1 The mass transfer rate as a function of time

One of the remarkable properties of the evolutions shown by WRS in their figs. 3 and 4 (see also our Figs. 3 and 4) is that the mass transfer rate remains constant within a small factor ($\lesssim 2$) in the case of conservative mass transfer, and, as Fig. 5 shows, within an even smaller factor ($\lesssim 1.5$) in the case that $\eta = 0$ and $\nu = 1/q$ throughout the evolution. This raises the question why this is so and whether this is a generic property of this type of evolution. For our discussion we combine (9) and (17) to write the mass transfer rate as

$$-\dot{M}_2 = \frac{M_2}{\zeta_{\mathrm{e}} - \zeta_{\mathrm{R},2}} \frac{\rho}{\lambda - 1} \frac{1}{t_\infty - t} \,. \tag{45}$$

From (45) it is immediately seen, that it contains two counteracting factors: first, with time $\zeta_{\mathrm{e}} - \zeta_{\mathrm{R},2}$ increases because q increases. This factor has thus the tendency to reduce $-\dot{M}_2$ with time. On the other hand, $t_\infty - t$ decreases with time and thus has the tendency to increase $-\dot{M}_2$. Therefore, there is a possibility that $-\dot{M}_2$ goes through a minimum during an evolution. Yet, an evolution does not necessarily include this minimum. Rather, whether or not $-\dot{M}_2(t)$ has a minimum depends on the initial mass ratio q_i and on the subsequent change of q with time. Starting from equations (31) and (32) and working out $\ddot{M}_2 = 0$ leads to a (rather complicated) equation for the mass ratio q_0 at which $-\dot{M}_2$ becomes extremal. From the above discussion we know already that this extremum is a minimum. Denoting the final mass ratio by $q_{\rm f} = M_{1,\rm f}/M_{2,\rm f} \approx M_{1,\rm f}/M_{\rm c,f}$, we can now distinguish three different cases: 1) if $q_{\rm crit} < q_{\rm i} < q_0 < q_{\rm f}$ then the mass transfer rate goes through a minimum, 2) if $q_{\rm crit} < q_0 < q_{\rm i} < q_{\rm f}$, the mass transfer rate will increase monotonically, and if 3) $q_{\rm crit} < q_{\rm i} < q_{\rm f} < q_0$, the mass transfer rate will decrease monotonically. Here $q_{\rm crit}$ is the critical mass ratio for both thermal and adiabatic stability against mass transfer. Its value is given by the larger of the positive solutions of the equations $\zeta_{\rm ad} - \zeta_{\rm R,2} = 0$ and $\zeta_{\rm e} - \zeta_{\rm R,2} = 0$ (see e.g. Ritter 1988).

For illustrative purposes we give the values of $q_{\rm crit}$ and q_0 for the cases we show in Figs. 3 – 5. For these cases we have assumed with WRS $\zeta_{\rm e} = 0$ and implicitly $\zeta_{\rm e} < \zeta_{\rm ad}$. From this and (28) it follows that $q_{\rm crit} = 1.2$ in the case of conservative mass transfer, and with (29) that $q_{\rm crit} = 0.836$ in the case where $\eta = 0$ and $\nu = 1/q$. The corresponding values of q_0 are in the case of conservative mass transfer $q_0 = 3.77$ for Pop. I giants (with parameters taken from Table 2) and $q_0 = 3.95$ for Pop. II giants. In fact for all combinations (λ, ρ) which describe the evolution of giants adequately we find $q_0 \lesssim 4$. In the case of non-conservative mass transfer with $\eta = 0$ and $\nu = 1/q$ we find $q_0 = 1.89$ for Pop. I giants and $q_0 = 1.97$ for Pop. II giants, again with parameters taken from Table 2, and an upper limit of $q_0 \lesssim 2$ for all acceptable combinations of (λ, ρ).

From the foregoing it is now clear that the near constancy of the mass transfer rate in the evolutions considered by WRS is not a generic property. Rather it is the result of a particular choice of initial parameters. In particular, if $q_{\rm i}$ is close to (but larger than) $q_{\rm crit}$ the mass transfer rate initially drops very strongly. On the other hand, the larger $q_{\rm f}$ the more strongly the mass transfer rate increases beyond q_0. Because conservative mass transfer results in the largest possible values of $q_{\rm f}$ we expect the mass transfer rate to increase substantially beyond q_0 and that in particular if the initial mass of the primary is already much larger than that of the donor star. This is e.g. the case if the accretor is a black hole rather than a neutron star. To illustrate this we show in Fig. 5 the run of $-\dot M_2$ in a conservative mass transfer with $M_{1,\rm i} = 5{\rm M}_\odot$, $M_{2,\rm i} = 2{\rm M}_\odot$ and $M_{\rm c,i} = 0.26{\rm M}_\odot$. It is clearly seen how the mass transfer rate grows because of the high final mass ratio of $q_{\rm f} = 25.9$. Nevertheless, the mass transfer rate stays reasonably constant if $q_{\rm i}$ is not close to $q_{\rm crit}$ and the accretor is a neutron star, i.e. $M_{1,\rm i} \approx 1.4{\rm M}_\odot$. This is the case for the simulations shown by WRS.

On the other hand, in the case that $\eta = 0$ and $\nu = 1/q$ the variation of the mass transfer rate is systematically smaller, first because for the same $q_{\rm i}$ mass transfer is more stable than in the conservative case, and second because M_1 remains constant, thus resulting in systematically lower values of $q_{\rm f}$. This is illustrated in Fig. 5 where we show the run of $-\dot M_2$ for the same initial parameters as for the conservative evolutions shown in Figs. 3 and 4, but now with $\eta = 0$ and $\nu = 1/q$. As can be seen, the mass transfer varies much less than in the conservative case.

9.2 Dependence on the initial core mass

As can be seen from equation (32), apart from the variation of the mass transfer rate during a mass transfer phase discussed in section 9.1, the level of mass transfer is set by the initial core mass $M_{\rm c,i}$ and depends on a large power, i.e. $\lambda - 1$, of it. Using the parameters given in Table 2 which correspond to the results of WRS we have respectively $-\dot M_2 \propto {M_{\rm c,i}}^7$ and $-\dot M_2 \propto {M_{\rm c,i}}^4$ for Pop. I and Pop. II giants (see equations (36) and (37)). This strong dependence on $M_{\rm c,i}$ is a direct consequence of the strong dependence of the stellar luminosity on $M_{\rm c,i}$ (see equation (10)). Also the stellar radius shows a strong dependence on $M_{\rm c,i}$ as can be seen from equation (16). Therefore, the initial core mass is the single most important parameter characterizing the evolution of a binary with mass transfer from a giant.

9.3 Dependence on ζ_e

We have already discussed the possible influence of ζ_e on the value of q_{crit} in section 9.1. However, as Table 1 shows, the analytical solutions (31) – (34) depend on ζ_e also via the exponent p_2. Going from a solution with $\zeta_e = 0$ to one with $\zeta_e \neq 0$ changes p_2 by $\Delta p_2 = -\zeta_e(\lambda - 1)/\rho$. Thus, with the parameters from Table 2 which correspond to the results of WRS we have $\Delta p_2 = -7/5\,\zeta_e$ for Pop. I, and $\Delta p_2 = -4/3\,\zeta_e$ for Pop. II chemical composition. Taking for an estimate $\zeta_e \approx -0.3$ we find that for both cases, i.e. Pop. I and Pop. II, $\Delta p_2 \approx 0.4$. Therefore, apart from increasing q_{crit}, going from $\zeta_e = 0$ to $\zeta_e < 0$ leads to an increase of the mass transfer rate via the increased factor $1/(\zeta_e - \zeta_{R,2})$ and to a decrease by the factor $(M_2/M_{2,i})^{\Delta p_2}$. Sufficiently far from the stability limit, i.e. from q_{crit}, it is the latter factor which dominates. Because $\zeta_e \gtrsim -0.3$ this factor is never very small. For all practical purposes we find it to be between ~ 0.5 and ~ 0.7. Thus, qualitatively, the main effect of going from $\zeta_e = 0$ to $\zeta_e < 0$ is to increase q_{crit} and therefore to reduce the maximum allowed value for the initial secondary mass.

9.4 Pop. I versus Pop. II systems

Because for Pop. II giants L_0, R_0, λ and ρ are significantly smaller than for Pop. I giants (see Table 2), the evolution of a Pop. II system, all other parameters being the same, is significantly slower than that of a Pop. I system if $M_{c,i} \gtrsim 0.2\mathrm{M}_\odot$. From equation (32) it follows that the differential evolutionary speed scales as ${M_{c,i}}^{\lambda_{II}-\lambda_I}$, i.e. using the parameters of Table 2 which correspond to the results of WRS, as ${M_{c,i}}^{-3}$. For an initial core mass of $0.3\mathrm{M}_\odot$ and $0.4\mathrm{M}_\odot$ the evolution of a Pop. II system is thus slower by respectively a factor of about 3 and 8. Furthermore, because Pop. II giants are also smaller than Pop. I giants if $M_c \gtrsim 0.2\mathrm{M}_\odot$ the orbital period of a Pop. II binary is shorter too by a corresponding factor of about $(M_{c,i}/0.2\mathrm{M}_\odot)^{-3}$ than that of a comparable Pop. I system.

10 CONCLUSIONS

We have shown that the evolution of a binary with stable mass transfer from a giant donor can be described by a simple analytical solution (see section 5) if the core mass-luminosity and core mass-radius relation are simple power laws of the mass of the degenerate helium core M_c. Furthermore, we have seen that the results of full stellar structure calculations of giants given by WRS and others can indeed be approximated with little loss of accuracy by simple power laws (section 6) and that, therefore, the analytical solutions derived here are relevant for the cases studied by Taam (1983) and WRS and similar cases. Moreover, the analytical solutions which we have derived are more general than the numerical ones of WRS in the sense that we allow for non-conservative mass transfer and a non-zero mass radius exponent ζ_e.

A direct comparison of the results obtained from the analytical solution with those of numerical calculations by WRS (section 7) shows that both the analytical solution and the power law approximations are quite accurate (see Figs. 1 – 4).

Using the analytical solution we have also made an a posteriori check of whether one of the key assumptions made in this model is fulfilled, namely that the donor star remains near thermal equilibrium despite being subject to mass loss (section 8.1), and have found that this is indeed the case. In section 8.2 we put forward arguments, why, despite strong irradiation of the donor star from a compact accretor, the model is probably applicable to low-mass X-ray binaries.

Furthermore we have seen (section 9.1) that the near constancy of the mass transfer rate over most of the mass transfer phase seen in the results by WRS is not a generic feature of this type of evolution but rather a consequence of a particular choice of parameters. In particular, we found considerable variation of the mass transfer rate if the final mass ratio $q_{\rm f}$ gets large. Therefore, particularly in conservative evolutions in which the accretor is massive, e.g. a black hole, the mass transfer rate increases quite substantially towards the end of the mass transfer phase (see Fig. 5). On the other hand, in evolutions in which the mass of the accretor remains constant or nearly so, $q_{\rm f}$ gets much less extreme and as a consequence of that the mass transfer rate varies to a much lesser degree.

From the analytical solution (32) it is also directly seen that the level of mass transfer is largely set by $M_{\rm c,i}$, the core mass at the *onset* of mass transfer.

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