Constructive solution of a certain class of Riemann-Hilbert problems on multiply connected circular regions

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Abstract

A class of Riemann-Hilbert(RH) problems on multiply connected circular regions is treated iteratively by a method of successive conjugation (SC), which solves the problem on a single boundary circle each time, running through all circles in each iterative step. It is shown that the method converges. Convergence is linear with a rate which can be calculated with an eigenvalue problem. The RH problems considered have negative index. In order to fulfill the solvability conditions, free parameters are introduced in form of polynomials. The proof of convergence of the SC method shows at the same time that the RH problems with this specification of the free parameters have unique solutions. RH problems of this kind are of practical use in conformal mapping problems of multiply connected regions to circular regions. They play a role in the mapping of nearly circular regions (Lavrentev's principle). There is also a close relationship to Koebe's iterative method for the calculation of conformal mappings. The properties of RH problems are different for unbounded and bounded regions. The SC iteration must be adapted accordingly. The performance of the SC iteration is illustrated with several examples.

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1 Introduction

We consider boundary value problems for analytic functions of the following kind: Let G be a region in the complex plane, let A be a complex and ψ a real function on the boundary ∂G. It is required to find a function Ψ analytic in G, continuous in the closure $\bar{G}$, and satisfying

$$\mathrm{Re}(\bar{A}\Psi) = \psi \tag{1}$$

on the boundary ∂G. Such problems are called *Riemann-Hilbert problems* (RH-problems for short). For simply connected regions, RH-problems have been investigated by Muskhelishvili [6] and for multiply connected regions by Vekua [8]. By a conformal mapping, G can be transformed to a region H whose boundary consists only of circles C_μ. Such a region is called a *circular region.* The circles C_μ are parametized by

$$\zeta_\mu(t) = z_\mu + R_\mu e^{\pm it}. \tag{2}$$

The orientation of the circles, determined by the sign in the exponential in (2), is so that H is always to the left of C_μ. For an unbounded H all circles are negatively oriented. When H is bounded, then there is a positively oriented outer circle which we denote by C_0.

A RH-problem can be transformed into a *canonical form* ([8],p.241). The boundary condition on each C_μ reduces to

$$\mathrm{Re}(e^{\mp il_\mu t}\Psi_{|\mu}) = \psi_\mu \tag{3}$$

with real functions ψ_μ, integers l_μ and the opposite sign as in (2). Here and in the following we denote for analytic functions F in H the values $F(\zeta_\mu(t))$ on the boundary circle C_μ by $F_{|\mu}$. We consider only the case where all $l_\mu = l$ are equal to an integer $l \geq 0$.

Some problems from fluid dynamics or electrostatics lead to boundary problems for harmonic functions which can be reduced to RH-problems of form (3) with $l = 0$. We are particularly interested in the case $l = 1$ which occurs frequently in conformal mapping problems. Such is the case for the conformal mapping to a near circular region (section 6). Attempts to transfer the author's conformal mapping method from singly connected regions [9], and doubly connected regions [10] to multiply connected regions lead to RH-problems of this kind. This will be discussed in a forthcoming paper.

These examples give the motivation for the study of problems (3) with constant $l_\mu = l$ for all μ with a non-negative integer l. Such problems can be called *general conjugation* since the case $l = 0$ describes the problem of finding for an harmonic function with boundary values ψ_μ the conjugate harmonic function. We discuss these problems in section 2.

For $l \geq 0$ the RH-problems have negative index ([8], p.238). A solution exists if and only if the functions ψ_μ on the right hand sides satisfy certain constraints. In order to make the problem solvable one must indroduce free parameters. We introduce these free parameters in the form of polynomials of degree l. This appears to be the most natural way. The so-called accessory parameters occur in the problem of conformal mapping of multiply connected regions as linear polynomials (see section 6).

Since the problem (3) can readily be solved on each circle C_μ separately by conjugation, it is tempting to proceed iteratively, by solving the problem on one circle after another. We describe this method of *successive conjugation* in section 3. This method has been applied before (e.g. by Halsey [2]), but apparently there are no proofs of convergence. We prove convergence for $l \geq 1$ in section 4. As a consequence of convergence the existence and uniqueness of solutions of (3) is established for $l \geq 1$. This is quite analogous to the well-known result for $l = 0$ which is Vekua's "problem D" ([8],p. 261).

Convergence is linear. The rate is determined from an operator eigenvalue problem in section 5. This rate determines also the rate of convergence of some conformal mapping methods. In section 7 we show that the method of successive conjugation is related to Koebe's iterative method [4].

We consider mainly the case of an unbounded region H which is simpler since all boundary circles have the same orientation. But the method can be applied also for bounded regions. The necessary modifications are explained in section 8.

Finally, we exemplify in section 9 how the method of successive conjugation behaves in practice.

Sorokin [7] gives a solution of RH-problems on multiply connected circular regions in closed form. The formulas involve infinite series of functions which are constructed by reflection at the boundary circles. It is not quite clear how these formulas can be used for computational purposes.

2 General conjugation on unbounded regions

Let H be an unbounded circular region of connectivity m. The boundary ∂H consists of m circles C_μ with centers z_μ and radii R_μ. The circles C_μ are parametrized by the functions

$$\zeta_\mu(t) = z_\mu + R_\mu e^{-it}. \tag{4}$$

Let l be a non negative integer, and let ψ_μ be real functions. The problem of general conjugation requires to determine an analytic function Ψ in H with $\Psi(\infty) = 0$, real numbers $a_{\mu 0}$ and complex numbers $a_{\mu 1}, \ldots, a_{\mu l}$ such that

$$\operatorname{Re}\left(e^{ilt}\Psi_{|\mu} + a_{\mu l}e^{ilt} + \cdots + a_{\mu 1}e^{it} + a_{\mu 0}\right) = \psi_\mu \tag{5}$$

for $\mu = 1, \ldots, m$. The normalization $\Psi(\infty) = 0$ is not an essential restriction. One can apply instead the more general condition that Ψ behaves near infinity like a given polynomial $p(z)$

$$\Psi(z) = p(z) + O(1/z). \tag{6}$$

When on the right hand side of (5) the functions ψ_μ are replaced by $\psi_\mu - \mathrm{Re}(e^{ilt} p_{|\mu})$ this situation is reduced to the general conjugation problem (5) for the function $\tilde{\Psi} := \Psi - p$ with normalization $\tilde{\Psi}(\infty) = 0$. This covers the condition $\Psi(z) = z + O(1/z)$ which is usually applied for conformal mapping functions.

One can conclude from the results of Vekua [8], that the RH-problem

$$\mathrm{Re}(e^{ilt} \Psi_{|\mu}) = \tilde{\psi}_\mu \tag{7}$$

has a solution if and only if the right hand sides satisfy $(2l+1)m$ constraints of the kind

$$\sum_\mu \int_{C_\mu} \tilde{\psi}_\mu \phi_{k\mu} dt = 0 \tag{8}$$

with real functions $\phi_{k\mu}, k = 1, \ldots, (2l+1)m$, on C_μ. The functions ϕ_k on ∂H defined by $\phi_k = \phi_{k\mu}$ on C_μ are linearly independent. The solution Ψ of (7) is unique.

The $\phi_{k\mu}$ are derived from solutions of a homogeneous RH-problem with analytic data. Therefore, the $\phi_{k\mu}$ are analytic functions.

The Lebesgue space $L^2(C)$ on $C = \partial H$ is the product of the Lebesgue spaces on the boundary components C_μ. The same is true for Sobolev spaces $W(C)$ and Hoelder spaces $C^\alpha(C)$. The analytic functions Ψ in H with boundary values in the Lebesgue space $L^2(C)$, form a linear space which we denote by AL^2. One can identify a function $\Psi \in AL^2$ with its boundary values. In this way AL^2 can be considered as a closed subspace of $L^2(C)$. This defines the norm on AL^2. In a similar way one can define spaces AW and AC^α of analytic functions with boundary values in W or C^α. The mapping

$$AL^2 \ni \Psi \to \mathrm{Re}(e^{ilt} \Psi_{|\mu}) \in L^2(C_\mu) \tag{9}$$

is continuous for each μ. Combining the m components gives a continuous mapping from AL^2 to $L^2(C)$. We have seen before that this mapping is one-to-one. The solvability conditions (8) imply that the range is closed. Therefore, the inverse of the mapping (9) is a continuous operator on a subspace of $L^2(C)$ with codimension $(2l+1)m$. The same is true for Sobolev and Hoelder spaces.

When we apply the conditions (8) to the functions

$$\tilde{\psi}_\mu := \psi_\mu - \mathrm{Re}\left(a_{\mu l}e^{ilt} + \cdots + a_{\mu 1}e^{it} + a_{\mu 0}\right) \tag{10}$$

we get a system of $(2l+1)m$ real linear equations for the $(2l+1)m$ real parameters $\mathrm{Re}\, a_{\mu k},\ k = 0, \ldots, l,$ and $\mathrm{Im}\, a_{\mu k},\ k = 1, \ldots, l.$ There is a unique solution of (5) for all functions ψ_μ if and only if the matrix of this system of equations is nonsingular.

3 Successive Conjugation

We treat the problem (5) by a procedure which we call *successive conjugation.*

An analytic function Ψ in H which vanishes at infinity can be written in a unique way as a sum

$$\Psi(z) = \sum_{\mu=1}^{m} h_\mu(z) \tag{11}$$

where each function h_μ is analytic outside C_μ and is represented by a Laurent series around z_μ

$$h_\mu(z) = \sum_{n=1}^{\infty} b_{\mu,n}(z - z_\mu)^{-n}. \tag{12}$$

We insert (11) into (5) and write the equation in the form

$$\mathrm{Re}\left(e^{ilt}h_{\mu|\mu} + a_{\mu l}e^{ilt} + \cdots + a_{\mu 1}e^{it} + a_{\mu 0}\right) \tag{13}$$

$$= \psi_\mu - \mathrm{Re}\left(\sum_{\nu \neq \mu} e^{ilt}h_{\nu|\mu}\right) =: \psi_\mu^*.$$

The left hand side of (13) contains only quantities on the circle C_μ.

Conjugation can be performed very efficiently in Fourier space. This carries over to the general conjugation. When the function ψ_μ^* on the right hand side of (13) is known, the equation can be solved simply by comparison of Fourier coefficients.

The left hand side of (13) must be compared with the Fourier series

$$\psi_\mu^*(t) = \sum_{n=-\infty}^{\infty} A_{\mu,n}e^{int}. \tag{14}$$

Since ψ_μ^* is a real function the coefficients satisfy $A_{\mu,-n} = \bar{A}_{\mu,n}$. In particular $A_{\mu,0}$ is real.

By comparing the coefficients of the left side of (5) and the right side of (14) we get the unique solution

$$a_{\mu 0} = A_{\mu,0}, \quad a_{\mu j} = 2A_{\mu,j} \quad \text{for} \quad j = 1, \dots, l, \tag{15}$$

$$b_{\mu,n} = 2R_\mu^n A_{\mu,n+l} \quad \text{for} \quad n = 1, 2, \dots \ . \tag{16}$$

This suggests an iterative procedure of successive conjugation:

Start with a set of functions $h_\mu^{(0)}$. When after the k-th iterative step functions $h_\mu^{(k)}$ are available, insert these functions on the right hand side of (13) and determine new functions $h_\mu^{(k+1)}$ from the equations

$$\begin{aligned} &\mathrm{Re}\left(e^{ilt} h_{\mu|\mu}^{(k+1)} + a_{\mu l} e^{ilt} + \cdots + a_{\mu 1} e^{it} + a_{\mu 0}\right) \\ &= \psi_\mu - \mathrm{Re}\left(\sum_{\nu \neq \mu} e^{ilt} h_{\nu|\mu}^{(k)}\right). \end{aligned} \tag{17}$$

This is analogous to the Jacobi method (or *Gesamtschrittverfahren*) for the solution of systems of linear equations. One can also use a sort of Gauss-Seidel method (or *Einzelschrittverfahren*) when one calculates successively $h_\mu^{(k+1)}$ from (17) but uses on the right hand side instead of $h_{\nu|\mu}^{(k)}$ the functions $h_{\nu|\mu}^{(k+1)}$ as far as these functions have already been determined. This gives the equation on C_μ

$$\begin{aligned} &\mathrm{Re}\left(e^{ilt} h_{\mu|\mu}^{(k+1)} + a_{\mu l} e^{ilt} + \cdots + a_{\mu 1} e^{it} + a_{\mu 0}\right) \\ &= \psi_\mu - \mathrm{Re}\left(\sum_{\nu < \mu} e^{ilt} h_{\nu|\mu}^{(k+1)}\right) - \mathrm{Re}\left(\sum_{\nu > \mu} e^{ilt} h_{\nu|\mu}^{(k)}\right). \end{aligned} \tag{18}$$

The cost of the computation is mainly determined by the evaluation of the values $h_{\nu|\mu}$ of the functions h_ν on the other circles C_μ. The new functions are then very easily calculated by Fourier transform using (16). The coefficients $a_{\mu 0}, \dots, a_{\mu l}$ in (17) and (18) depend on $(k+1)$. But they are not needed during the iteration. Therefore, it is sufficient to calculate these parameters at the end, when the iteration has converged. We omit the superscript $(k+1)$.

In view of equation (13) the function $h_{\mu|\mu}$ can be constructed from the right side ψ_μ^* by the operator K of conjugation on the unit circle. Since the functions $h_{\nu|\mu}$ are analytic, the function ψ_μ^* is as smooth as ψ_μ. Therefore, for any smoothness property which is preserved by the operator K, the function $h_{\mu|\mu}$ is as smooth as ψ_μ. This can in particular be applied to Sobolev and Hoelder spaces. Whenever all ψ_μ are in W or C^α then the solution Ψ of (5) is in AW or AC^α and depends in a continuous way on the right hand side in the appropriate norm.

4 Convergence

The iteration (17) converges when the differences $\tilde{h}_\mu^{(k)} := h_\mu^{(k)} - h_\mu^{(k-1)}$, $\mu = 1, \dots, m$, converge to 0 as $k \to \infty$. The difference functions solve also the equations (17) but with $\psi_\mu = 0$. Therefore, it is sufficient to study the iteration

$$\mathrm{Re}\left(e^{ilt} h_{\mu|\mu}^{(k+1)} + a_{\mu l} e^{ilt} + \dots + a_{\mu 1} e^{it} + a_{\mu 0}\right) = -\mathrm{Re}\left(\sum_{\nu \neq \mu} e^{ilt} h_{\nu|\mu}^{(k)}\right) \tag{19}$$

for the homogeneous problem (5). We have to show that the sequence $h_\mu^{(k)}$ converges to 0 as $k \to \infty$ for any choice of the initial functions $h_\mu^{(0)}$.

The reflection of a point z at a circle C_μ is described by

$$z^{\sharp\mu} := z_\mu + \frac{R_\mu^2}{\bar{z} - \bar{z}_\mu}. \tag{20}$$

For any set B in the complex plane we denote by $B^{\sharp\mu}$ the reflected set $\{z^{\sharp\mu} : z \in B\}$. Let f be a function defined and analytic in a region G. Then the reflected function defined by

$$f^{\sharp\mu}(z) := \overline{f(z^{\sharp\mu})} \tag{21}$$

is analytic in the reflected region $G^{\sharp\mu}$. The function

$$F_\mu(z) := \frac{R_\mu}{z - z_\mu} \tag{22}$$

is analytic in the complex plane with the exception of the simple pole at z_μ. It has the boundary values

$$F_{\mu|\mu} = e^{it}. \tag{23}$$

The reflected and the reciprocal functions coincide:

$$F_\mu^{\sharp\mu} = \frac{1}{F_\mu}. \tag{24}$$

For $\nu \neq \mu$ the reflected circle $C_\nu^{\sharp\mu}$ has radius

$$R_{\nu\mu} = \frac{R_\mu^2}{|z_\nu - z_\mu|^2 - R_\nu^2}\, R_\nu. \tag{25}$$

Since C_ν is outside C_μ the factor in front of R_ν is less than 1. The reflected circle $C_\nu^{\sharp\mu}$ is smaller than C_ν. The maximal distance of $C_\nu^{\sharp\mu}$ from z_μ is

$$\max_{\zeta \in C_\nu^{\sharp\mu}} |\zeta - z_\mu| = \frac{R_\mu^2}{|z_\nu - z_\mu| - R_\nu}. \tag{26}$$

Now we derive an explicit representation of the solution $h_\mu^{(k+1)}$ of (19) in terms of the reflected functions $h_\nu^{(k)\sharp\mu}$.

Lemma 1 *For all z outside C_μ there holds*

$$h_\mu^{(k+1)}(z) = -F_\mu^{-2l}(z) \sum_{\nu\neq\mu} h_\nu^{(k)\sharp\mu}(z) + Q_\mu^{(k+1)}(z) \tag{27}$$

with a polynomial $Q_\mu^{(k+1)}$ of degree $\leq 2l$.

Proof: We replace in (19) e^{it} by F_μ using (23), and distribute the polynomial part on both sides

$$\mathrm{Re}\left(F_\mu^l h_\mu^{(k+1)} + b_l F_\mu^l + \cdots + b_1 F_\mu + b_0\right)_{|\mu} \tag{28}$$

$$= -\mathrm{Re}\left(\sum_{\nu\neq\mu} F_\mu^l h_\nu^{(k)} - c_l F_\mu^l - \cdots - c_1 F_\mu\right)_{|\mu}.$$

The first function in the bracket on the right hand side

$$g_1(z) := \sum_{\nu\neq\mu} F_\mu^l h_\nu^{(k)} \tag{29}$$

has a pole of order l at z_μ. When the $c_1, \ldots, c_l$ are equal to the coefficients of the principal part of the Laurent series

$$g_1(z) = c_l(z - z_\mu)^{-l} + \cdots + c_1(z - z_\mu)^{-1} + \cdots \tag{30}$$

then the function in brackets on the right side of (28) is analytic inside C_μ. On the left side of (28) the real part is taken from a function which is analytic outside C_μ. Equation (28) says that the real parts of two functions, one analytic outside C_μ the other analytic inside C_μ, coincide on C_μ. This implies that up to an imaginary constant $i\gamma$ either of these functions is the mirror image of the other:

$$F_\mu^l h_\mu^{(k+1)} + b_l F_\mu^l + \cdots + b_1 F_\mu + b_0 \tag{31}$$

$$= -\sum_{\nu\neq\mu} (F_\mu^{\sharp\mu})^l h_\nu^{(k)\sharp\mu} - \bar{c}_l (F_\mu^{\sharp\mu})^l - \cdots - \bar{c}_1 F_\mu^{\sharp\mu} + i\gamma.$$

We solve this for $h_\mu^{(k+1)}$ using (24). On the right side a polynomial of degree $\leq 2l$ in F_μ^{-1} occurs which gives the polynomial $Q_\mu^{(k+1)}$ represented in powers of $z - z_\mu$. QED

In view of this lemma the function $h_\mu^{(k+1)}$ can be calculated in a simple way: Reflect all $h_\nu^{(k)}$ with $\nu \neq \mu$ at C_μ, multiply by $-F_\mu^{-2l}$ and add up. The resulting function has a pole of order $2l$ at ∞ which must be compensated by the polynomial $Q_\mu^{(k+1)}$ of degree $2l$.

A set or a function can be reflected successively several times, first at the circle C_{ν_1}, then at the circle C_{ν_2}, and so on until the last reflection at C_{ν_k}. We denote the result by the superscript $\sharp\nu_1 \ldots \nu_k$. We abbreviate the k-tuple $\nu_1, \ldots, \nu_k$ by (ν).

By repeated application of Lemma 1 one can obtain an explicit representation of the functions $h_\mu^{(k)}$ in terms of the initial functions $h_\nu^{(0)}$.

Lemma 2 *The function $h_{\nu_k}^{(k)}$ is analytic outside the set*

$$\mathbf{C}_{\nu_k}^{(k)} := \bigcup C_{\nu_0}^{\sharp\nu_1 \ldots \nu_k}. \tag{32}$$

The union is over all indices $\nu_0, \nu_1, \ldots, \nu_{k-1}$, such that no two consecutive indices are the same and $\nu_{k-1} \neq \nu_k$. In the region outside $\mathbf{C}_{\nu_k}^{(k)}$ the function is represented by

$$h_{\nu_k}^{(k)} = (-1)^k \sum (F_{\nu_k} F_{\nu_{k-1}}^{\sharp\nu_k} \cdots F_{\nu_1}^{\sharp\nu_2 \ldots \nu_k})^{-2l} h_{\nu_0}^{(0)\sharp\nu_1 \ldots \nu_k} \tag{33}$$

$$+ \sum_{r=1}^{k-1} (-1)^{k-r} \sum (F_{\nu_k} \cdots F_{\nu_{r+1}}^{\sharp\nu_{r+2} \cdots \nu_k})^{-2l} Q_{\nu_r}^{(r)\sharp\nu_{r+1} \ldots \nu_k} + Q_{\nu_k}^{(k)}$$

with polynomials $Q_\nu^{(r)}$ of degree $\leq 2l$. The sum in the first line is over all indices $\nu_0, \nu_1, \ldots, \nu_{k-1}$, such that no two consecutive indices are the same and $\nu_{k-1} \neq \nu_k$. The inner sum in the second line is over all indices $\nu_r, \nu_{r+1}, \ldots, \nu_{k-1}$ such that no two consecutive indices are the same and $\nu_{k-1} \neq \nu_k$.

Proof by induction with respect to k.

For $k = 1$ the formula (33) reduces to (27) proved in Lemma 1. Since each $h_\nu^{(0)}$ is analytic outside C_ν, the reflected function $h_\nu^{(0)\sharp\mu}$ is analytic outside $C_\nu^{\sharp\mu}$. Therefore, $h_\mu^{(1)}$ represented by (27) is analytic outside

$$\mathbf{C}_\mu^{(1)} = \bigcup_{\nu \neq \mu} C_\nu^{\sharp\mu}.$$

Now we assume that Lemma 2 is true for some $k \geq 1$. Using (27) we calculate

$$h_\mu^{(k+1)} = -F_\mu^{-2l} \sum_{\nu_k \neq \mu} h_{\nu_k}^{(k)\sharp\mu} + Q_\mu^{(k+1)} \tag{34}$$

$$= -F_\mu^{-2l} \sum_{\nu_k \neq \mu} (-1)^k \sum (F_{\nu_k}^{\sharp\mu} F_{\nu_{k-1}}^{\sharp\nu_k\mu} \cdots F_{\nu_1}^{\sharp\nu_2\ldots\nu_k\mu})^{-2l} h_{\nu_0}^{(0)\sharp\nu_1\ldots\nu_k\mu}$$

$$-F_\mu^{-2l} \sum_{\nu_k \neq \mu} \sum_{r=1}^{k-1} (-1)^{k-r} \sum (F_{\nu_k}^{\sharp\mu} \cdots F_{\nu_{r+1}}^{\sharp\nu_{r+2}\ldots\nu_k\mu})^{-2l} Q_{\nu_r}^{(r)\sharp\nu_{r+1}\ldots\nu_k\mu}$$

$$-F_\mu^{-2l} \sum_{\nu_k \neq \mu} Q_{\nu_k}^{(k)\sharp\mu} + Q_\mu^{(k+1)}$$

$$= (-1)^{k+1} \sum (F_\mu F_{\nu_k}^{\sharp\mu} F_{\nu_{k-1}}^{\sharp\nu_k\mu} \cdots F_{\nu_1}^{\sharp\nu_2\ldots\nu_k\mu})^{-2l} h_{\nu_0}^{(0)\sharp\nu_1\ldots\nu_k\mu}$$

$$+\sum_{r=1}^{k-1} (-1)^{k+1-r} \sum (F_\mu F_{\nu_k}^{\sharp\mu} \cdots F_{\nu_{r+1}}^{\sharp\nu_{r+2}\ldots\nu_k\mu})^{-2l} Q_{\nu_r}^{(r)\sharp\nu_{r+1}\ldots\nu_k\mu}$$

$$-\sum_{\nu_k \neq \mu} F_\mu^{-2l} Q_{\nu_k}^{(k)\sharp\mu} + Q_\mu^{(k+1)}.$$

On the right side the first sum is over all indices $\nu_0, \nu_1, \ldots, \nu_k$ such that no two consecutive indices are the same and $\nu_k \neq \mu$. The inner sum in the second term is over all indices $\nu_r, \nu_{r+1}, \ldots, \nu_k$ such that no two consecutive indices are the same and $\nu_k \neq \mu$. The next to last term in (34) can be included as the term $r = k$ in the preceding sum over r. We write ν_{k+1} instead of μ and get the formula (34) for $k+1$.

The functions $h_{\nu_k}^{(k)}$ are analytic outside all the circles $C_{\nu_0}^{\sharp\nu_1\ldots\nu_k}$. The reflected functions $h_{\nu_k}^{(k)\sharp\mu}$ are analytic outside the reflected circles $C_{\nu_0}^{\sharp\nu_1\ldots\nu_k\mu}$. QED

Lemma 3 *The function*

$$\Psi^{(k)} := \sum_{\mu=1}^{m} h_\mu^{(k)} \tag{35}$$

is analytic outside the union

$$\mathbf{C}^{(k)} := \bigcup_{\mu=1}^{m} \mathbf{C}_\mu^{(k)} \tag{36}$$

of circles $\mathbf{C}_\mu^{(k)}$ defined in (32). It can be represented by a sum of Cauchy integrals

$$\Psi^{(k)}(z) = (-1)^k \sum \frac{1}{2\pi i} \int_{C_{\nu_0}^{\sharp(\nu)}} \frac{(F_{\nu_k} F_{\nu_{k-1}}^{\sharp\nu_k} \cdots F_{\nu_1}^{\sharp\nu_2\ldots\nu_k})^{-2l} h_{\nu_0}^{(0)\sharp(\nu)}}{\zeta - z} d\zeta. \tag{37}$$

The sum is over all indices $\nu_0, \nu_1, \ldots, \nu_k$, such that no two consecutive indices are the same. The orientation of the circles $C_{\nu_0}^{\sharp(\nu)}$ is negative.

Proof: The function $\Psi^{(k)}$ is analytic outside the circles $C_{\nu_0}^{\sharp(\nu)}$ by lemma 2 and vanishes at infinity. Therefore, it can be represented by a Cauchy-integral over the boundary circles. We insert for $h_\mu^{(k)}$ the representation (33) from Lemma 2. The functions F_ν^{-2l} and the polynomials $Q_\nu^{(r)}$ are analytic everywhere in the plane but have poles of order $2l$ at ∞. By reflection at C_μ the point ∞ is mapped to z_μ. Therefore, the poles of $Q_{\nu_r}^{(r)\sharp\nu_{r+1}\ldots\nu_k}$ and of $(F_{\nu_r}^{\sharp\nu_{r+1}\ldots\nu_k})^{-2l}$ are at $z_{\nu_{r+1}}^{\sharp\nu_{r+2}\ldots\nu_k}$ which for $r \geq 1$ is outside all the circles $C_{\nu_0}^{\sharp(\nu)}$. Hence all these functions are analytic inside all these circles . This implies that the last two terms in (33) do not contribute to the Cauchy-integral for z outside these circles. The only remaining term is the first sum in (33) which gives (37). QED

By estimating the Cauchy-integral in (37) we can prove convergence. The number

$$q_H := \max_{\mu \neq \nu} \frac{|R_\mu + R_\nu|}{|z_\mu - z_\nu|} \tag{38}$$

is less than 1. The circles C_μ can be enlarged at most by a factor q_H^{-1} so that their interior regions remain disjoint. Let the functions $h_\mu^{(0)}$ be bounded with the upper bound

$$||h_\mu^{(0)}|| := \max_{\zeta \in C_\mu} |h_\mu^{(0)}(\zeta)|, \qquad ||h^{(0)}|| := \max_{\mu = 1,\ldots,m} ||h_\mu^{(0)}|| \tag{39}$$

Theorem 1 *There exists a constant M_H which depends only on H such that the estimate*

$$|\Psi^{(k)}(z)| \leq M_H ||h^{(0)}|| q_H^{4k} \tag{40}$$

holds for all $z \in H$ and for all $l \geq 1$.

Proof: The distance

$$\Delta := \min_{\nu \neq \mu} \min_{\zeta_1 \in C_\mu, \zeta_2 \in C_\nu^{\sharp\mu}} |\zeta_1 - \zeta_2| \tag{41}$$

is positive. We estimate the integrals in (37) for $k \geq 1$ by

$$\left| \frac{1}{2\pi i} \int_{C_{\nu_0}^{\sharp(\nu)}} \frac{(F_{\nu_k} F_{\nu_{k-1}}^{\sharp\nu_k} \cdots F_{\nu_1}^{\sharp\nu_2\ldots\nu_k})^{-2l} h_{\nu_0}^{(0)\sharp\nu_1\ldots\nu_k}}{\zeta - z} d\zeta \right| \leq \frac{1}{\Delta} R_{\nu_0}^{(\nu)} I_{\nu_0}^{(\nu)}$$

where $R^{(\nu)}_{\nu_0}$ is the radius of the circle $C^{\sharp(\nu)}_{\nu_0}$ and

$$I^{(\nu)}_{\nu_0} = \max \left| (F_{\nu_k} F^{\sharp\nu_k}_{\nu_{k-1}} \cdots F^{\sharp\nu_2 \ldots \nu_k}_{\nu_1})^{-2l} h^{(0)\sharp\nu_1 \ldots \nu_k}_{\nu_0} \right| \tag{42}$$

is the maximum of the numerator in the integrand on the circle of integration. Adding up and using Schwarz's inequality yields

$$|\Psi^{(k)}(z)| \le \frac{1}{\Delta} \sum R^{(\nu)}_{\nu_0} I^{(\nu)}_{\nu_0} \le \frac{1}{\Delta} \left(\sum (R^{(\nu)}_{\nu_0})^2 \sum (I^{(\nu)}_{\nu_0})^2 \right)^{\frac{1}{2}}. \tag{43}$$

We denote for any circle C by $D(C)$ the disc bounded by C, and by $\alpha(C)$ the area of this disc.

Let $\tilde{C}_\mu$ be the circle around z_μ with radius $q_H^{-1} R_\mu$. Then the inclusions

$$D(C_\mu) \subset D(\tilde{C}_\mu) \tag{44}$$

are preserved by every reflection $\sharp(\nu)$ provided $\nu_1 \ne \mu$. Since the doubly connected regions

$$D(\tilde{C}^{\sharp(\nu)}_{\nu_0}) \setminus D(C^{\sharp(\nu)}_{\nu_0})$$

all have modulus q_H^{-1} the areas are in the relation

$$\alpha(C^{\sharp(\nu)}_{\nu_0}) \le q_H^2 \alpha(\tilde{C}^{\sharp(\nu)}_{\nu_0})$$

([3], p. 503). This gives the estimate

$$\sum (R^{(\nu)}_{\nu_0})^2 = \frac{1}{\pi} \sum \alpha(C^{\sharp(\nu)}_{\nu_0}) \le \frac{1}{\pi} q_H^2 \sum \alpha(\tilde{C}^{\sharp(\nu)}_{\nu_0}). \tag{45}$$

With an estimate from the convergence proof of Koebe's method ([3],p.505) we get for the right side of (45) the bound

$$\sum \alpha(\tilde{C}^{\sharp(\nu)}_{\nu_0}) \le q_H^{4k} \sum_\mu \alpha(\tilde{C}_\mu) = q_H^{4k-2} \sum_\mu \alpha(C_\mu) \tag{46}$$

and finally

$$\sum (R^{(\nu)}_{\nu_0})^2 \le q_H^{4k} \sum_\mu R_\mu^2. \tag{47}$$

Now we estimate the integrand using

$$I^{(\nu)}_{\nu_0} \le \max |F^{-2l}_{\nu_k}| \max \left| (F^{\sharp\nu_k}_{\nu_{k-1}} \ldots F^{\sharp\nu_2 \ldots \nu_k}_{\nu_1})^{-2l} h^{(0)\sharp\nu_1 \ldots \nu_k}_{\nu_0} \right|. \tag{48}$$

The maximum is over $C^{\sharp(\nu)}_{\nu_0}$. By reflection, a function is replaced by its complex conjugate (see equation (21)). Therefore, we can omit in the second factor of (48) the reflection $\sharp\nu_k$ and take the maximum on $C_{\nu_0}^{\sharp\nu_1\ldots\nu_{k-1}}$. The first factor of (48) can easily be estimated for $l \geq 1$ using (26)

$$I_{\nu_0}^{(\nu)} \leq \left(\frac{R_{\nu_k}}{|z_{\nu_0}^{\nu_1\ldots\nu_{k-1}} - z_{\nu_k}| - R_{\nu_0}^{\nu_1\ldots\nu_{k-1}}} \right)^2 I_{\nu_0}^{\nu_1\ldots\nu_{k-1}} \tag{49}$$

with the center $z_{\nu_0}^{\nu_1\ldots\nu_{k-1}}$ and the radius $R_{\nu_0}^{\nu_1\ldots\nu_{k-1}}$ of the circle $C_{\nu_0}^{\sharp\nu_1\ldots\nu_{k-1}}$. On the other hand it follows from (25) that

$$R_{\nu_0}^{(\nu)} = \frac{R_{\nu_k}^2}{|z_{\nu_0}^{\nu_1\ldots\nu_{k-1}} - z_{\nu_k}|^2 - (R_{\nu_0}^{\nu_1\ldots\nu_{k-1}})^2} R_{\nu_0}^{\nu_1\ldots\nu_{k-1}}. \tag{50}$$

From (47) the estimate

$$R_{\nu_0}^{(\nu)} \leq M_1 q_H^{2k} \tag{51}$$

follows with $M_1^2 := \sum R_\mu^2$. Since the discs $D(C_{\nu_k})$ and $D(C_{\nu_0}^{\sharp\nu_1\ldots\nu_{k-1}})$ are disjoint, the denominator in (49) is $\geq m_1 := \min R_\mu$. The factors in (49) and (50) are almost equal. The ratio is

$$\frac{|z_{\nu_0}^{\nu_1\ldots\nu_{k-1}} - z_{\nu_k}|^2 - (R_{\nu_0}^{\nu_1\ldots\nu_{k-1}})^2}{(|z_{\nu_0}^{\nu_1\ldots\nu_{k-1}} - z_{\nu_k}| - R_{\nu_0}^{\nu_1\ldots\nu_{k-1}})^2} \tag{52}$$

$$= 1 + \frac{2R_{\nu_0}^{\nu_1\ldots\nu_{k-1}}}{|z_{\nu_0}^{\nu_1\ldots\nu_{k-1}} - z_{\nu_k}| - R_{\nu_0}^{\nu_1\ldots\nu_{k-1}}} \leq 1 + \frac{2M_1}{m_1} q_H^{2k} =: p_k.$$

The product

$$P := \prod_{k=1}^{\infty} p_k \tag{53}$$

converges. Combining (49) and (50) gives the estimate

$$I_{\nu_0}^{(\nu)} \leq P M_2 R_{\nu_0}^{(\nu)} \tag{54}$$

with

$$M_2 := \max_\mu \frac{||h_\mu^{(0)}||}{R_\mu} \leq \frac{||h^{(0)}||}{m_1}.$$

With (47),(54) and (43) we get finally

$$|\Psi^{(k)}(z)| \leq \frac{M_1^2 P}{m_1 \Delta} q_H^{4k} ||h^{(0)}||. \tag{55}$$

The factor $M_H := M_1^2 P / m_1 \Delta$ in (55) is combined of the quantities M_1, P, m_1 and Δ which are all defined in terms of the geometry of H. QED

When the functions $h_\mu^{(k)}$ are determined by the Einzelschritt iteration (18), then the same line of reasoning can be applied. The main difference is that in the k-th iteration sets and functions occur which are reflected at least k times but possibly more often.

In fact, $h_\mu^{(k)}$ is analytic outside the union of circles $C_{\nu_0}^{\sharp(\kappa)}$ with certain combinations κ of at least k indices. The corresponding discs $D(C_{\nu_0}^{\sharp(\kappa)})$ are all mutually disjoint but contained in the corresponding discs for the Gesamtschrittverfahren

$$\bigcup D(C_{\nu_0}^{\sharp(\kappa)}) \subset \bigcup D(C_{\nu_0}^{\sharp\nu_1 \dots \nu_k}). \tag{56}$$

This implies that

$$\sum (R_{\nu_0}^{(\kappa)})^2 \leq \sum (R_{\nu_0}^{(\nu)})^2 \tag{57}$$

where the sums are taken over all circles on the left and right sides of (56). The sum on the right side occurs in the Gesamtschritt method and is estimated by (47). The function $\Psi^{(k)}$ formed from these $h_\mu^{(k)}$ can be represented by a sum of Cauchy integrals as in (37). The indices of the factors F_μ correspond to the indices in (κ). Therefore, the integrand can be estimated by the radius as in formula (54). This proves (55) also for the Einzelschritt-method.

Convergence is linear. It becomes apparent from the proof that convergence is faster for larger l. Also convergence for the Einzelschritt-method is faster than for the Gesamtschritt-method.

For $l = 0$ the solvability of (5) is well established (see e.g. [8], p.265). For $l \geq 1$ the solvability follows from the convergence theorem 1.

Corollary 1 *For every integer $l \geq 0$ and for any functions ψ_μ in L^2, the Riemann-Hilbert problem (5) has a unique solution with an analytic function Ψ in H normalized by $\Psi(\infty) = 0$, and with complex numbers $a_{\mu 1}, \dots, a_{\mu_l}$ and real numbers $a_{\mu 0}$.*

5 Convergence factors

We combine the functions $h_{\mu|\mu} \in L^2(C_\mu)$ to an element

$$\mathbf{X} := (h_{1|1}, \dots, h_{m|m}) \in L^2(C) = L^2(C_1) \oplus \dots \oplus L^2(C_m). \tag{58}$$

Iteration by successive conjugation is described by (27). The transition from one iterate to the next is effected by an operator, which is represented by an integral with Cauchy kernel. The singularity is avoided by the condition

$\nu \neq \mu$. Therefore, the operator is compact. In view of the reflection $\sharp\mu$, the operator is antilinear. Hence, the iteration can be written in the form

$$\mathbf{X}^{(k+1)} = \mathbf{M}\overline{\mathbf{X}^{(k)}} \tag{59}$$

with a compact linear operator $\mathbf{M}$ in $L^2(C)$. The iteration converges linearly with a rate which is asymptotically equal to the spectral radius $r(\mathbf{M})$. Since $\mathbf{M}$ is compact $r(\mathbf{M}) = |\lambda_0|$ with an eigenvalue λ_0 of $\mathbf{M}$ of maximum modulus.

Let $\mathbf{X}$ be an eigenfunction corresponding to the eigenvalue λ. It follows from the eigenvalue equation $\mathbf{M}\overline{\mathbf{X}} = \lambda\mathbf{X}$ that

$$\mathbf{M}\overline{\mathbf{M}}\mathbf{X} = \mathbf{M}\bar{\lambda}\overline{\mathbf{X}} = \bar{\lambda}\mathbf{M}\overline{\mathbf{X}} = |\lambda|^2\mathbf{X}. \tag{60}$$

Therefore, the squares of the moduli of the eigenvalues of $\mathbf{M}$ can conveniently be calculated as eigenvalues of the operator $\mathbf{M}\overline{\mathbf{M}}$. The operator $\overline{\mathbf{M}}$ is also an integral operator whose kernel is the conjugate complex of the kernel of $\mathbf{M}$. It must not be confused with the adjoint operator. There is no transposition!

For numerical purposes one can use the isometry of $L^2(C_\mu)$ with l^2, when each function $f_{|\mu}$ is represented by the series of its Fourier coefficients. In view of the representation (27) one has to develop the functions h_ν, which are given as Fourier-series of $h_{\nu|\nu}$ on C_ν into Fourier-series of $h_{\nu|\mu}$ on C_μ. This can be done explicitly using the binomial series for the powers $(z - z_\nu)^{-n}$. From the Fourier series of $h_{\nu|\mu}$ the first $2l + 1$ terms have to be discarded. This is the effect of the Q_μ in (27). The rest gives already the Fourier-coefficient up to sign, complex conjugation and a power of R_μ. In the space of Fourier coefficients the operator $\mathbf{M}$ can be represented as an $m \times m$ block matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{O} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1m} \\ \mathbf{M}_{21} & \mathbf{O} & \cdots & \mathbf{M}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{M}_{m1} & \mathbf{M}_{m2} & \cdots & \mathbf{O} \end{pmatrix} \tag{61}$$

with zero diagonal blocks. The elements of the nondiagonal blocks $\mathbf{M}_{\mu\nu}$ are

$$d_{sr} = -\binom{-r}{s+2l} R_\nu^r R_\mu^{s+2l} (z_\mu - z_\nu)^{-r-s-2l} \tag{62}$$

for $s, r = 1, 2, \ldots$. The matrix applied to the conjugate complex Fourier coefficients of $h_{\nu|\nu}$ gives the Fourier coefficients of the new functions $h_{\mu|\mu}$. For numerical purposes the infinite matrices $\mathbf{M}_{\mu\nu}$ are truncated. One has to multiply the matrix $\mathbf{M}$ by its complex conjugate (not transposed!) and determine the eigenvalues of the product. These turn out to be real and positive. The square root of the maximum eigenvalue gives the convergence rate of the Gesamtschrittverfahren.

One can calculate the convergence rate of the Einzelschrittverfahren as the maximum modulus $|\lambda|$ of an eigenvalue of the real general eigenvalue problem

$$\lambda \left(\mathbf{I} - \begin{pmatrix} \mathbf{M}_{rL} & \mathbf{M}_{iL} \\ \mathbf{M}_{iL} & -\mathbf{M}_{rL} \end{pmatrix} \right) \begin{pmatrix} \mathbf{X}_r \\ \mathbf{X}_i \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{rU} & \mathbf{M}_{iU} \\ \mathbf{M}_{iU} & -\mathbf{M}_{rU} \end{pmatrix} \begin{pmatrix} \mathbf{X}_r \\ \mathbf{X}_i \end{pmatrix} \tag{63}$$

with the real and imaginary parts $\mathbf{X}_r$ and $\mathbf{X}_i$ of the vector $\mathbf{X}$ and with real block matrices which are constructed in the following way. Let $\mathbf{M}_L$ and $\mathbf{M}_U$ be the lower (upper) triangular part of the matrix $\mathbf{M}$. The matrices $\mathbf{M}_{rL}, \mathbf{M}_{iL}, \mathbf{M}_{rU}, \mathbf{M}_{iU}$ in (63) are the real and imaginary parts of these matrices $\mathbf{M}_L$ and $\mathbf{M}_U$. The convergence rates reported in section 9 are calculated in this way as matrix eigenvalues.

6 Conformal mapping to near circular regions

General conjugation with $l = 1$ can be used to construct approximate conformal mappings to near circular regions. This is an extension of Lavrentev's principle ([5], [7]).

To be specific, let H be an unbounded circular region whose boundary consists of circles C_μ with centers z_μ and radii R_μ, and let ψ_μ be twice continuously differentiable real functions. Then one can define for small real numbers τ, neighbouring regions G_τ with boundary curves parametrized by

$$\eta_\mu(s) = z_\mu + R_\mu e^{-is} + \tau \psi_\mu(s) e^{-is}. \tag{64}$$

On the other hand one can consider the RH-problem

$$\mathrm{Re}\left(e^{it}\Psi_{|\mu} + a_{\mu 1} e^{it} + a_{\mu 0}\right) = \psi_\mu \tag{65}$$

as a boundary problem for an analytic function Ψ in H with $\Psi(\infty) = 0$. According to Corollary 1 this problem has a unique solution Ψ. The complex numbers $a_{\mu 1}$ and real numbers $a_{\mu 0}$ are uniquely determined by (65).

Let H_τ be the circular region bounded by circles $C_{\tau\mu}$ with centers and radii

$$z_{\tau\mu} := z_\mu + \tau a_{\mu 1}, \qquad R_{\tau\mu} := R_\mu + \tau a_{\mu 0}. \tag{66}$$

The boundary values $\Psi_{|\mu}$ of Ψ on ∂H are in general not boundary values of an analytic function in H_τ. But they are close to the boundary values of the function

$$\Psi_\tau(z) = \sum_\mu \frac{1}{2\pi i} \int_{C_{\tau\mu}} \frac{\Psi_{|\mu}}{\zeta - z} d\zeta \tag{67}$$

which is analytic in H_τ. The difference between the boundary values $\Psi_{|\mu}$ of Ψ on H and the boundary values of Ψ_τ on ∂H_τ is uniformly of order $O(\tau)$.

One may expect, that G_τ is conformally equivalent to a circular region which is close to H, and that the conformal mapping is close to the identity. This is made precise by the following theorem.

Theorem 2 *For sufficiently small $|\tau|$ the function*

$$\Phi_\tau(z) := z + \tau\Psi_\tau(z) \tag{68}$$

maps H_τ conformally to a region $\tilde{G}_\tau$ which is close to G_τ in the sense that to each point $\zeta \in G_\tau$ there is a point $\tilde{\zeta} \in \tilde{G}_\tau$ with distance $|\zeta - \tilde{\zeta}| \leq M\tau^2$. When δ is a bound for $|\psi_\mu|, |\psi'_\mu|, |\psi''_\mu|$ then M depends only on H and on δ.

Proof: In view of (65) one can write

$$e^{it}\Psi_{|\mu} + a_{\mu 1}e^{it} + a_{\mu 0} = \psi_\mu + i\, R_\mu U_\mu \tag{69}$$

with real functions U_μ. Since the solution of the RH-problem (65) depends continuously on the right hand sides ψ_μ in the Sobolev norm, the maximum norm of U_μ can be estimated by $||U_\mu|| \leq M_1\delta$ with a constant M_1 which depends only on H. The remainder in

$$\begin{aligned} \eta_\mu(t - \tau U_\mu(t)) &= z_\mu + R_\mu e^{-it} + \tau(\psi_\mu(t) + i\, R_\mu U_\mu(t))e^{-it} + \cdots \\ &= z_\mu + \tau a_{\mu 1} + (R_\mu + \tau a_{\mu 0})e^{-it} + \tau\Psi_{|\mu} + \cdots \end{aligned} \tag{70}$$

is bounded by $\tau^2(R_\mu + 4\tau\delta)||U_\mu||^2$.

When we replace $\Psi_{|\mu}$ in (70) by the boundary values of Ψ_τ on ∂H_τ we introduce an error which can be estimated by $M_2\delta^2\tau^2$ with an M_2 depending only on H. Equation (70) then says, that the boundary curve of the image region $\tilde{G}_\tau$ approximates the reparametrized boundary curve η_μ of G_τ with an error less than $M\tau^2$. The constant M depends only on H and on the bound δ for the first two derivatives of ψ_μ.

The mapping Φ_τ is conformal for all sufficiently small $|\tau|$. QED.

7 Relation to Koebe's method

Koebe's iteration method [4] determines the conformal mapping F from an unbounded m-connected region G with boundary components Γ_μ to a circular region H. The mapping F is calculated as a composition of maps

$$F = \cdots \circ F^{k,\mu} \circ \cdots \circ F^{1,2} \circ F^{1,1} \tag{71}$$

where the analytic function $F^{k,\mu}$, normalized at ∞ by $F^{k,\mu}(z) = z + O(1/z)$, maps the exterior of the image of Γ_μ under the composition of all previous mappings, conformally to the exterior of a circle. The method converges (see [1],p.230,[3],p.499). The image regions become more and more circular. Therefore, the following results describe the asymptotic behaviour of Koebe's method.

We consider the case of a region G_τ which is close to a circular region H as described by formula (64) in section 6.

Lemma 4 *The mapping after the μ-th circle in the k-th iteration*

$$f^{k,\mu} := F^{k,\mu} \circ \cdots \circ F^{1,2} \circ F^{1,1} \tag{72}$$

is equal to

$$f^{k,\mu}(z) = z - \tau \sum_{\nu \le \mu} h_\nu^{(k)}(z) - \tau \sum_{\nu > \mu} h_\nu^{(k-1)}(z) + O(\tau^2) \tag{73}$$

with the functions $h_\nu^{(k)}$ obtained by the Einzelschritt iteration (18) with $l = 1$ starting from $h_\mu^{(0)} = 0$ for $\mu = 1, \ldots, m$.

Proof: 1) When we apply Theorem 2 with $m = 1$ for the case of the exterior of a single curve with parametrization

$$\eta_1(t) = z_1 + (R_1 + \tau\psi_1(s))e^{-is} \tag{74}$$

we obtain the normalized conformal mapping Φ to the exterior of this curve as $\Phi(z) = z + \tau\Psi(z) + O(\tau^2)$ with a function Ψ which is analytic for $|z - z_1| > R_1$, vanishes at ∞ and solves the RH-problem

$$\mathrm{Re}\left(e^{it}\Psi_{|1} + a_1 e^{it} + a_0\right) = \psi_1 \tag{75}$$

on the circle C_1. The inverse mapping is

$$\Phi^{(-1)}(z) = z - \tau\Psi(z) + O(\tau^2). \tag{76}$$

2) We prove (73) by induction with respect to the lexicographically ordered index pairs (k, μ).

Since all $h_\mu^{(0)}$ are equal to 0 the statement for $(k, \mu) = (1, 1)$ is true in view of (76). We assume now that the statement is true for an index pair (k, μ). We consider only the case $\mu < m$. The mapping $F^{k,\mu+1}$ can be written in the form $F^{k,\mu+1}(z) = z - \tau l_{\mu+1}^{(k)}(z) + O(\tau^2)$. After inserting into (72) and using (73) we get

$$f^{k,\mu+1}(z) = z - \tau \sum_{\nu \le \mu} h_\nu^{(k)}(z) - \tau \sum_{\nu > \mu} h_\nu^{(k-1)}(z) - \tau l_{\mu+1}^{(k)}(z) + O(\tau^2). \tag{77}$$

With $h_{\mu+1}^{(k)} := h_{\mu+1}^{(k-1)} + l_{\mu+1}^{(k)}$ we get the representation (73) for the indices $(k, \mu+1)$. It follows from (75) and (76) that this function $h_{\mu+1}^{(k)}$ satisfies the equation (18) for the Einzelschritt method for $(k, \mu+1)$. QED

The $O(\tau^2)$-terms depend on H and on bounds for the first two derivatives of ψ_μ as in Theorem 2. But there may be different bounds for each index pair (k, μ). Therefore, one cannot pass to the limit $k \to \infty$. But it follows from Theorem 1 that the functions

$$\hat{f}^{k,\mu}(z) = z - \tau \sum_{\nu \leq \mu} h_\nu^{(k)}(z) - \tau \sum_{\nu > \mu} h_\nu^{(k-1)}(z) \tag{78}$$

converge to $z - \tau\Psi$ where Ψ solves the RH-problem (65). On the other hand it follows from Theorem 2 that $z - \tau\Psi(z)$ approximates $F(z)$ up to terms of order $O(\tau^2)$. This shows that (73) holds also in the limit $k \to \infty$. The Koebe iteration is in first order a successive conjugation with $l = 1$ in the Einzelschritt version. This implies that the Koebe method converges asymptotically as fast as the Einzelschritt iteration on the corresponding circular region. The convergence rate for the latter can be calculated by the methods described in section 5.

8 General conjugation on bounded regions

Now let H be a bounded circular region with boundary circles $C_\mu, \mu = 0, 1, \ldots, m$ parametrized by (2). We assume that the circle C_0 is the outer circle, and adopt the convention that in all formulas the upper sign is for $\mu = 0$ and the lower sign for $\mu = 1, \ldots, m$. Let l be a non negative integer and ψ_μ real functions for $\mu = 0, 1, \ldots, m$. Then determine an analytic function Ψ in H, real numbers $a_{\mu 0}$ and complex numbers $a_{\mu 1}, \ldots, a_{\mu l}$ such that

$$\mathrm{Re}\left(e^{\mp ilt}\Psi_{|\mu} + a_{\mu l}e^{\mp ilt} + \cdots + a_{\mu 1}e^{\mp it} + a_{\mu 0}\right) = \psi_\mu \tag{79}$$

for $\mu = 0, 1, \ldots, m$.

As shown by Vekua [8] the Riemann Hilbert-problem

$$\mathrm{Re}(e^{\mp ilt}\Psi_{|\mu}) = \tilde{\psi}_\mu \tag{80}$$

has a solution if and only if the right hand side satisfies $(2l+1)(m-1)$ constraints of the kind (8). Although there is one condition more in (79) than in (5) for the same m there are $2l+1$ less solvability conditions (8). On the other hand, there are $2l+1$ more free parameters

$a_{00}, \ldots, a_{0l}$ in (79). We will show how the parameters must be reduced (see (85)) and how additional constraints can be introduced (see (88)).

An analytic function Ψ in the bounded region H can be written in a unique way as a sum

$$\Psi(z) = \sum_{\mu=0}^{m} h_\mu(z) \tag{81}$$

where each function h_μ for $\mu = 1, \ldots, m$ is analytic outside C_μ and is represented by a Laurent series (12). The function h_0 is analytic inside C_0 and represented by a Taylor series

$$h_0(z) = \sum_{n=0}^{\infty} b_{0,n}(z - z_0)^n. \tag{82}$$

We insert (81) into (79) and get for $\mu = 1, \ldots, m$ again the equations (13). But for $\mu = 0$ we arrive at the condition

$$\begin{aligned} &\mathrm{Re}\left(e^{-ilt}h_{0|0} + a_{0l}e^{-ilt} + \cdots + a_{01}e^{-it} + a_{00}\right) \\ &= \psi_0 - \mathrm{Re}\left(\sum_{\nu=1}^{m} e^{-ilt}h_{\nu|0}\right) =: \psi_0^*. \end{aligned} \tag{83}$$

The left hand side of (83) contains only quantities on the circle C_0. It can be rewritten in terms of the Taylor coefficients of h_0

$$\begin{aligned} &\mathrm{Re}\left(e^{-ilt}h_{0|0} + a_{0l}e^{-ilt} + \cdots + a_{01}e^{-it} + a_{00}\right) \\ &= \mathrm{Re}\left(\sum_{n=0}^{\infty} b_{0,n}R_0^n e^{i(n-l)t} + a_{0l}e^{-ilt} + \cdots + a_{01}e^{-it} + a_{00}\right). \end{aligned} \tag{84}$$

By comparing the series (84) with the Fourier series (14) for ψ_0^* we get the higher coefficients:

$$b_{0,n} = 2R_0^{-n}A_{0,n-l} \quad \text{for} \quad n > 2l. \tag{85}$$

The lower order coefficients satisfy certain linear relations. Since the a_{0j} compete with the first Taylor coefficients, one can get a unique h_0 only when the a_{0j} are specified. Therefore, we put

$$a_{0l} = 0, \ldots, a_{01} = 0, a_{00} = 0. \tag{86}$$

In the conformal mapping problem, for instance, this is enforced by the condition that the outer circle remains fixed.

For the lower order coefficients we get an equation for the real part

$$\operatorname{Re}\left(\sum_{n=0}^{2l} b_{0,n} R_0^n e^{i(n-l)t}\right) = \sum_{n=-l}^{l} A_{0,n} e^{int}. \tag{87}$$

In order to determine the $b_{0,j}$ uniquely one must be able to calculate also the imaginary part of the Fourier polynomial in (87). This can be done in a very natural way when Ψ is restricted by $2l+1$ interpolatory conditions

$$\operatorname{Im}\left(e^{-ilt_j}\Psi(\zeta_0(t_j))\right) = \beta_j \tag{88}$$

at $2l+1$ different points $\zeta_0(t_j)$ on the outer circle with real numbers β_j. After insertion of the representation (81) an interpolation problem

$$\operatorname{Im}\left(\sum_{n=0}^{2l} b_{0,n} R_0^n e^{i(n-l)t_j}\right) = \beta_j - \operatorname{Im}\left(e^{-ilt_j}\sum_{\mu=1}^{m} h_\mu(\zeta_0(t_j))\right) \tag{89}$$

$$-\operatorname{Im}\left(\sum_{n=2l+1}^{\infty} b_{0,n} R_0^n e^{i(n-l)t_j}\right)$$

remains which must be satisfied at the $2l+1$ points t_j. Since the higher order coefficients $b_{0,n}$ are already determined by (85) the right hand side of (89) is known. The left hand side of (89) is a trigonometric polynomial of degree l. This polynomial is uniquely determined by the $2l+1$ interpolation conditions. This yields the imaginary part of the Fourier polynomial. Since the real part is given by (87), this yields the polynomial and the missing Taylor coefficients of h_0.

In the conformal mapping problem interpolatory conditions of the kind (88) are provided by the condition that the images of three outer boundary points are prescribed.

9 Examples

In order to give an impression how the method works for different regions, and for different l, and how the Einzel- and Gesamtschrittverfahren differ, we present the results of several test calculations.

Example 1: Three well separated circles with centers $z_1 = -1.5,\ z_2 = -1.5\,i,\ z_3 = 2+i$ and radii $R_1 = 1,\ R_2 = .6,\ R_3 = 1$.

Example 2: Four circles which are close together. The centers are $z_1 = -1.5, z_2 = i, z_3 = 2+i, z_4 = .8-i$ and the radii $R_1 = 1, R_2 = .6, R_3 = 1, R_4 = 1.2$.

Example 3: Three circles of very different size with centers $z_1 = 0, z_2 = -2.5 + 2.2\,i, z_3 = -2.2 + 2.4\,i$ and radii $R_1 = 3, R_2 = .2, R_3 = .1$.

Example 4: A small circle between two big circles. The centers are $z_1 = -2.1, z_2 = 2.1, z_3 = 0$ and the radii $R_1 = 1.9, R_2 = 1.9, R_3 = .1$.

Example 5: A bounded region whose boundary consists of circles with centers $z_0 = 0., z_1 = .4, z_2 = -.4 - .2\,i$ and radii $R_0 = 1.2, R_1 = .2, R_2 = .4$.

In each example we take as right hand side $\psi_\mu = \exp(\cos(t))$ for all μ. In the figures 1 to 4 we show the geometry and the convergence measured by the maximum error

$$e_k := \max \left| \mathrm{Re}\left(\sum_{\nu=1}^{m} e^{ilt} h_{\nu|\mu}^{(k)} + a_{\mu l} e^{ilt} + \cdots + a_{\mu 1} e^{it} + a_{\mu 0} \right) - \psi_\mu \right| \tag{90}$$

between both sides of (13) after the k-th step. Each example is calculated with 40 grid points on each circle. For the bounded region the interpolation conditions are applied at the first $2l + 1$ of the grid points number $1, 9, 17, 25, 33$ on the outer circle.

For comparison the convergence rate q^k is added with the rate q calculated from a matrix eigenvalue problem according to section 5. These convergence rates are collected in table 1.

Table 1: Convergence rates q for the examples

Example	Gesamtschritt			—	Einzelschritt		
l=	0	1	2		0	1	2
1	.23	.040	.0076		.062	.0019	.000065
2	.63	.23	.085		.40	.055	.0073
3	.39	.074	.017		.16	.0063	.00029
4	.52	.17	.066		.28	.024	.0042

These examples show that successive conjugation works well, even for the case $l = 0$ where convergence could not be proved in section 4. It becomes apparent that convergence is faster for larger l. It is faster for the Einzelschritt than for the Gesamtschritt-method. The method works best for well separated circles.

References

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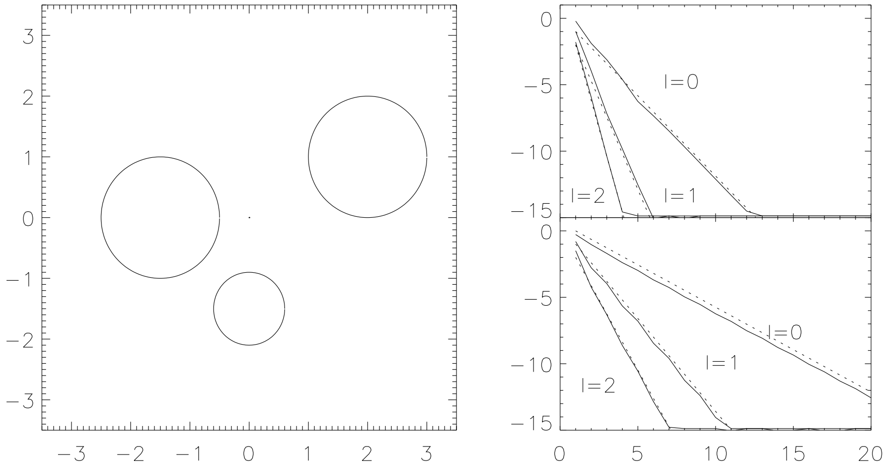

Figure 1: For the first example the geometry (left) and the logarithm of the error e_k after the k-th step for the Gesamtschritt (lower right) and the Einzelschritt method (upper right) for l=0,1,2. The dotted line is Cq^k with the convergence factor q from table 1

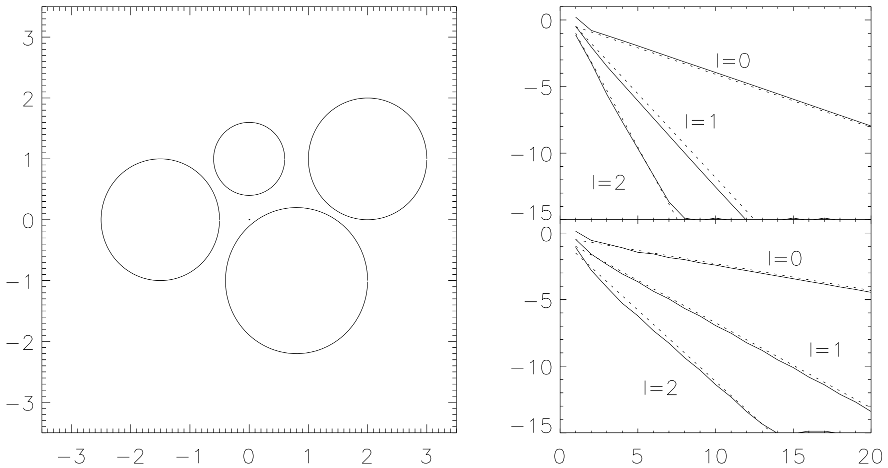

Figure 2: The same as Figure 1 for the second example

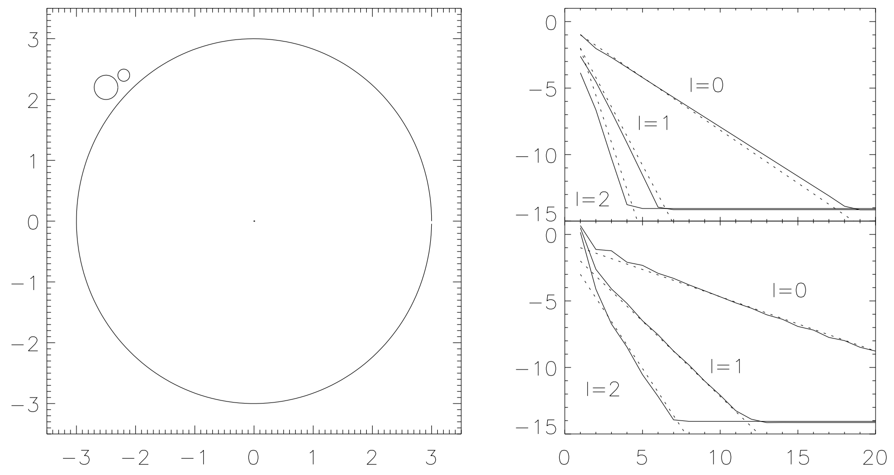

Figure 3: The same as Figure 1 for the third example

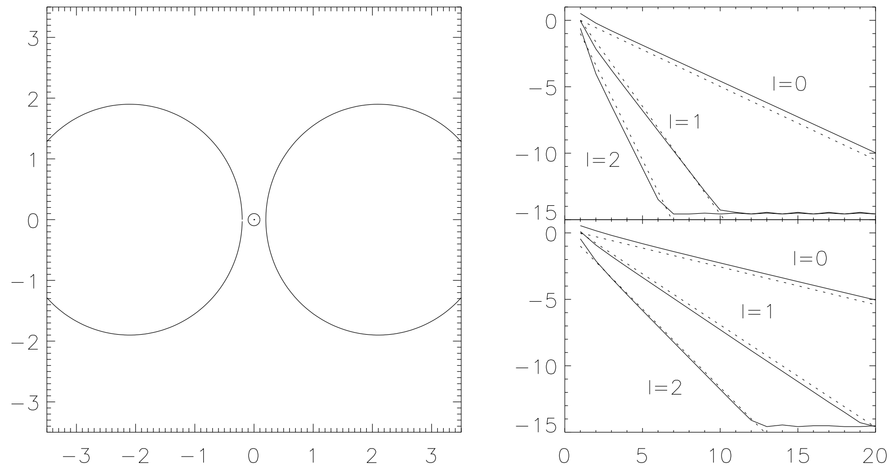

Figure 4: The same as Figure 1 for the fourth example

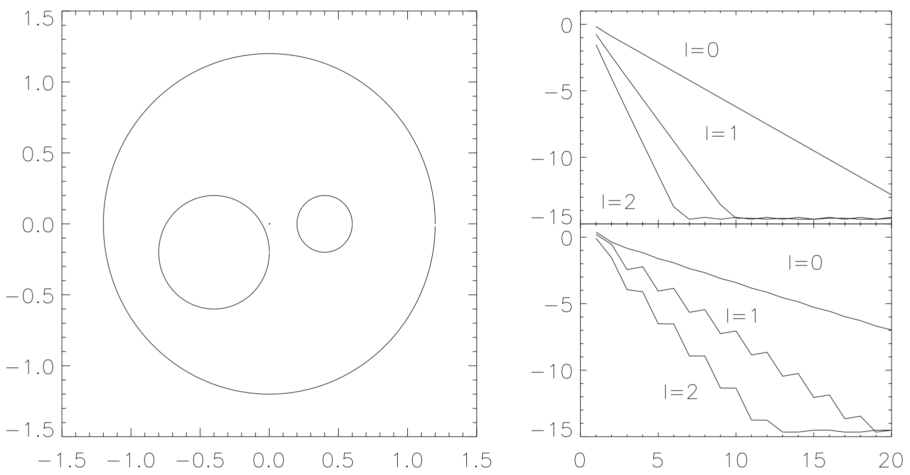

Figure 5: The same as Figure 1 for the fifth example

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