Fast Conformal Mapping of Multiply Connected Regions

Rudolf Wegmann
Max-Planck-Institut für Astrophysik
D-85748 Garching Germany
e-mail:ruw@mpa-garching.mpg.de

Abstract

An iterative method is presented which constructs for an unbounded region G with m holes and sufficiently smooth boundary a circular region H and a conformal mapping Φ from H to G. With the usual normalization both H and Φ are uniquely determined by G. With a few modifications the method can also be applied for a bounded region G with m holes. The canonical region H is then the unit disc with m circular holes. The method requires in each iterative step the solution of a Riemann-Hilbert (RH) problem. This problem has a unique solution. It determines also the centers and the radii of the boundary circles of H. Numerically, the RH problem can be treated efficiently by the method of successive conjugation using the Fast Fourier Transform (FFT). The iteration for the solution of the RH problem converges linearly. The conformal mapping method converges quadratically. The results of some test calculations exemplify the performance of the method.

Key words: Numerical conformal mapping, multiply connected regions, convergence

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1 Introduction

An unbounded region G with m holes is conformally equivalent to an unbounded region H which is bounded by m circles. Such an H is called a *circular region*. When the conformal mapping Φ from H to G is normalized by the condition near infinity

$$\Phi(z) = z + O(1/z) \tag{1}$$

then the circular region H as well as the conformal mapping are uniquely determined by the region G.

Because the circular region is not known in advance it seems to be more natural to determine the inverse mapping Φ^{-1}. Then one can work on the fixed region G and the circular region H is simply the image of G under the mapping Φ^{-1}. All methods described in Gaier's book [3] are of this kind. Most methods calculate the mapping to slit regions (see [1, 3, 4]). Only the elegant method of Koebe [7] calculates the mapping to a circular region. Koebe obtains the mapping of the multiply connected region as a combination of mappings of simply connected regions to the exterior of circles.

Only very recently, methods have been developed which calculate directly the mapping from H to G. DeLillo et al. [2] use a Fornberg-type method. Klonowska & Prosnak [5] developed in 1996 a projection method. They show how the circle parameters can be calculated in a very natural way.

In this paper we carry over the fast methods for simply connected ([11, 12]) and doubly connected ([13, 8]) to multiply connected regions. We deal with bounded and unbounded regions separately since both need a somewhat different treatment.

The plan of the paper is as follows. In section 2 we fix the notation and quote some results about conformal mapping. In section 3 we give first some heuristic motivation and then write in compact form the formulas for the iterative step needed for unbounded and for bounded regions. The main computational task with this approach is an RH problem of a special form which can be interpreted as a generalization of conjugation. This problem has been treated in our previous paper [16], where we proved in particular that this problem has always a unique solution, and that this solution can be calculated iteratively by a method of *successive conjugation*. In section 4 we prove local quadratic convergence of the iterative method for regions G with three times Lipschitz continuously differentiable boundary curves. In section 5 we finally demonstrate the performance of the method by several test calculations.

2 Notation and auxilary results

We fix the notation and quote some general results. We consider regions of the following kind:

The boundary of an unbounded region G of connectivity $m \geq 1$ consists of m curves $\Gamma_\mu, \mu = 1, \ldots, m$. The boundary curves are parametrized by complex 2π-periodic functions $\eta_\mu(s)$. The orientation is clockwise, so that the region is to the left of the curves.

The boundary of a bounded region G of connectivity $m+1 \geq 1$ consists of $m+1$ curves $\Gamma_\mu, \mu = 0, 1, \ldots, m$. The outer boundary curve Γ_0 is oriented counterclockwise, the inner boundaries $\Gamma_\mu, \mu = 1, \ldots, m$, are oriented clockwise, so that also in this case the region is to the left of the curves.

The parameter functions η_μ are assumed to be differentiable with Hoelder continuous derivatives $\dot{\eta}_\mu(s) \neq 0$. The derivatives can be represented in the form

$$\dot{\eta}_\mu(s) = r_\mu(s) \exp(i\theta_\mu(s)) \tag{2}$$

with Hoelder continuous real functions $r_\mu > 0$ and θ_μ.

The boundary C of an unbounded circular region H consists of m circles C_μ with centers z_μ and radii R_μ. The circles are parametrized by the functions

$$\zeta_\mu(t) = z_\mu + R_\mu e^{-it}. \tag{3}$$

The inner boundaries of a bounded circular region are parametrized as in (3) but the outer circle has positive orientation:

$$\zeta_0(t) = z_0 + R_0 e^{it}. \tag{4}$$

The following theorem is well known:

Theorem 1 *a) For an unbounded region G with m holes there exists an unbounded circular region H and a conformal mapping $\Phi : H \to G$. Both, H and Φ, are uniquely determined by the region G when the normalization condition (1) near ∞ is imposed.*

b) For a bounded region G with m holes there exists a bounded circular region H and a conformal mapping $\Phi : H \to G$. Both, H and Φ, are uniquely determined by the region G when the normalization conditions are imposed:

1. *The outer boundary C_0 of H is the unit circle.*

2. *Φ interpolates at three fixed points of the outer boundary,i.e.,*

$$\Phi(\zeta_0(t_j)) = \eta_0(s_j) \tag{5}$$

for prescribed parameter values $0 \leq t_1 < t_2 < t_3 < 2\pi$ and $s_1 < s_2 < s_3 < s_1 + 2\pi$.

Proof: It is well known that a circular region H and a conformal mapping $\Phi : H \to G$ exist ([7, 1, 3, 4]). Without further specification, H and Φ are unique only up to a conformal mapping Ψ from one circular region H_1 to another circular region H_2. As shown by Koebe [6], such a mapping Ψ must be a Moebius transformation $\Psi(z) = (a + bz)/(c + dz)$. Since for unbounded regions infinity must be mapped to infinity, only translations $\Psi(z) = a + bz$ are admitted. The special form (1) of the pole in ∞ fixes this translation.

For bounded regions the outer boundary is required to be the unit circle. This determines the mapping up to a conformal mapping of the unit disc onto itself. This is uniquely determined when three points on the unit circle are fixed. Therefore, the conditions of theorem 1 determine the parameters a, b, c, d of the Moebius transformation and so ensure uniqueness. QED

The conformal mapping Φ is completely described by its boundary values. Here and in the following we denote for analytic functions F in H the values $F(\zeta_\mu(t))$ on the boundary circle C_μ by $F_{|\mu}$. These are 2π-periodic functions of t. With the aforementioned parametrizations the boundary correspondence

$$\Phi_{|\mu} = \eta_\mu(S_\mu(t)), \qquad \mu = 1, \ldots, m \tag{6}$$

is described by the *parameter mapping functions* $S_\mu(t)$. These functions have the property that $S_\mu(t) - t$ is 2π-periodic. For bounded regions (6) holds also for $\mu = 0$.

A real or complex 2π-periodic function $f(t)$ is in L^2 when $|f|^2$ is integrable over the interval $[0, 2\pi]$. The norm $||f||_2$ is the square-root of the integral. The function f is in the Sobolev space W when it is completely continuous and the derivative f' is in L^2. W is a Banach space with the norm $||f||_W := ||f||_2 + ||f'||_2$.

A function F which is analytic at ∞ has a Laurent series representation $F(z) = b_0 + b_1/z + b_2/z^2 \cdots$. The residue at infinity is defined as (note the minus sign!)

$$\mathrm{Res}_\infty F := -b_1. \tag{7}$$

The functions g, analytic outside the unit disc with L^2 boundary values and vanishing in ∞, form a linear space A, which can be interpreted as a closed linear subspace of L^2.

A function Ψ analytic in the unbounded circular region H, vanishing at ∞ and with L^2 boundary values can be represented by Cauchy's integral formula. This gives the representation

$$\Psi(z) = \sum_{\mu=1}^{m} g_\mu \left(\frac{z - z_\mu}{R_\mu} \right) \tag{8}$$

with functions $g_\mu \in A$ and the circle parameters z_μ and R_μ. This representation has derivatives of all orders with respect to all parameters g_μ, z_μ, and R_μ. The boundary values are

$$\Psi_{|\mu} = g_\mu(e^{-it}) + \sum_{\nu \neq \mu} g_\nu \left(\frac{\zeta_\mu - z_\nu}{R_\nu} \right). \tag{9}$$

This representation has also derivatives of all orders. The first derivative is

$$D(\Psi_{|\mu}) = Dg_\mu(e^{-it}) + \sum_{\nu\neq\mu} Dg_\nu \left(\frac{\zeta_\mu - z_\nu}{R_\nu}\right) \tag{10}$$

$$+\sum_{\nu\neq\mu} g'_\nu \left(\frac{\zeta_\mu - z_\nu}{R_\nu}\right)\left(\frac{Dz_\mu + DR_\mu e^{-it} - Dz_\nu}{R_\nu} - \frac{\zeta_\mu - z_\nu}{R_\nu^2} DR_\nu\right).$$

When the derivative of g_μ on the unit circle exists, one can include the term $\nu = \mu$ in the last sum, since for this case the second factor vanishes. After rearranging the terms we get the equation

$$D(\Psi_{|\mu}) = (D\Psi)_{|\mu} + (\Psi')_{|\mu} D\zeta_\mu \tag{11}$$

with a function $D\Psi$ analytic in H. This equation, though intuitively clear, must be taken with some care. It holds true as it stands, when the function Ψ has derivatives also at the boundary. But when the boundary values of Ψ are only in L^2, one must use (10), where the "rough" part of $(\Psi')_{|\mu}$ does not occur. One might expect, that the derivative $(\Psi')_{|\mu}$ reduces the differentiability. But this is not the case. With the same argument one can conclude that $D(\Psi_{|\mu})$ has the same smoothness as $(D\Psi)_{|\mu}$. The term $(\Psi')_{|\mu}$ in (11) does not reduce the smoothness.

3 The iterative method

3.1 Motivation

The boundary correspondence (6) can be interpreted as an equation which determines the conformal mapping Φ, the centers z_μ and the radii R_μ of the circles, and the parameter mapping functions S_μ.

We solve the boundary correspondence equation (6) by a Newton-type iterative method. When only approximations $\Phi, z_\mu, R_\mu, S_\mu$ to the conformal mapping, the circles and the parameter mapping functions are available one can try to change these approximations so that the boundary correspondence equation (6) is satisfied at least in first order. Then the necessary changes $\delta\Phi, \delta z_\mu, \delta R_\mu, \delta S_\mu$ are connected by the linear equation

$$\Phi_{|\mu} + \delta\Phi_{|\mu} + \Phi'_{|\mu}\left(\delta z_\mu + \delta R_\mu e^{\pm it}\right) = \eta_\mu + \dot{\eta}_\mu \delta S_\mu \tag{12}$$

which is a linearized version of (6). The derivative of the left side of (6) has been calculated using (11). The functions η_μ and $\dot{\eta}_\mu$ are evaluated for the parameters $S_\mu(t)$. Here and in the following the upper sign is for $\mu = 0$ and the lower sign is for $\mu = 1, \dots, m$.

Since δS_μ are real functions, they can be eliminated from the equation (12) and a boundary value problem

$$\mathrm{Im}\frac{\Psi_{|\mu}}{\dot{\eta}_\mu} = \mathrm{Im}\frac{\eta_\mu}{\dot{\eta}_\mu} - \mathrm{Im}\frac{\Phi'_{|\mu}\,(\delta z_\mu + \delta R_\mu e^{\pm it})}{\dot{\eta}_\mu} =: \psi_\mu \tag{13}$$

for the analytic function $\Psi := \Phi + \delta\Phi$ remains. The function Ψ must satisfy the same normalization condition (1) or (5) as Φ. Equation (13) is a so-called Riemann-Hilbert (RH) problem . Such problems have been investigated on multiply connected regions by Vekua [10].

The properties of this problem depend on the winding-numbers of the derivatives $\dot{\eta}_\mu$. For unbounded regions all Γ_μ have negative orientation and each $\dot{\eta}_\mu$ has winding-number -1. It follows from the results of Vekua [10] that the problem (13) has a solution if and only if the right hand sides ψ_μ satisfy $3m$ constraints. The solution is unique. The $3m$ constraints give $3m$ linear equations for the $3m$ (real) parameters $\delta z_\mu, \delta R_\mu$ in the right hand side of (13).

For bounded regions Γ_0 has positive orientation and the winding-number of $\dot{\eta}_0$ is $+1$. In this case the RH problem has a solution if and only if the right hand sides ψ_μ satisfy $3m-3$ constraints. The solution is unique. The $3m-3$ constraints give $3m-3$ linear equations for the $3m$ parameters $\delta z_\mu, \delta R_\mu, \mu = 1, \dots, m$. (Observe that $\delta z_0 = 0$ and $\delta R_0 = 0$ since C_0 is supposed to be the unit circle in view of the first normalization condition of Theorem 1). The interpolatory conditions (5) provide 3 more real equations. Therefore, also in this case the number of equations equals the number of free parameters.

In the treatment of the RH problem on the multiply connected circular region H we follow as closely as possible the procedure on the disc and on the annulus ([11, 13, 8]).

First, an analytic function Y is determined such that $\pm ie^{\pm it}\exp(Y_{|\mu})$ has the same argument as $\dot{\eta}_\mu$ up to a constant λ_μ. With a positive proportionality factor e^{w_μ} the equation holds

$$\frac{1}{e^{i\lambda_\mu}\dot{\eta}_\mu} = \mp i\frac{e^{w_\mu}}{r_\mu}\frac{e^{\mp it}}{\exp(Y_{|\mu})}. \tag{14}$$

In order to simplify the problem we replace in (12) and (13) $\dot{\eta}_\mu$ by $e^{i\lambda_\mu}\dot{\eta}_\mu$. This has the advantage that a RH problem for the analytic function $\Xi := \Psi/\exp(Y)$ of the special form

$$\mathrm{Re}\left(e^{\mp it}\,\Xi_{|\mu}\right) = \mp\xi_\mu \tag{15}$$

remains. The factor is equal to $e^{\mp it}$ on each of the circles C_μ. We will see in section 4 that this simplification does not destroy the convergence properties.

The right hand side of (15)

$$\xi_\mu := \frac{r_\mu}{e^{w_\mu}} \mathrm{Im} \frac{\eta_\mu}{e^{i\lambda_\mu}\dot{\eta}_\mu} - \mathrm{Im}\left(\frac{r_\mu}{e^{w_\mu}} \frac{\Phi'_{|\mu}}{e^{i\lambda_\mu}\dot{\eta}_\mu}(\delta z_\mu + \delta R_\mu e^{\pm it})\right) \tag{16}$$

still contains the factors $\Phi'_{|\mu}$ in front of the shifts of the circle parameters. Using arguments from [8] one can get rid of this derivative in the following way.

Differentiation of the boundary correspondence equation (6) with respect to t gives

$$\Phi'_{|\mu}\zeta'_\mu = \dot{\eta}_\mu S'_\mu. \tag{17}$$

We replace $\dot{\eta}_\mu$ in (17) using (14) and obtain

$$\frac{\Phi'_{|\mu}}{\exp(Y_{|\mu})} = \frac{S'_\mu r_\mu}{e^{w_\mu} R_\mu e^{i\lambda_\mu}}. \tag{18}$$

This means that the analytic function $\Xi_0 := \Phi'/\exp(Y)$ is a solution of the homogeneous RH-problem $\mathrm{Im}(e^{i\lambda_\mu}\Xi_{0|\mu}) = 0$. The function Ξ_0 is required to be analytic also in ∞. This problem has a non trivial solution only if the numbers $e^{i\lambda_\mu}$ are all equal up to the sign ([10], p.257). The solution is then Ξ_0 = constant. By suitable normalization of Y one can enforce that this constant is positive. Then it follows that

$$B\Phi' = \exp(Y) \tag{19}$$

with a constant $B > 0$. Using this and (14) we get

$$\frac{r_\mu}{e^{w_\mu}} \frac{\Phi'_{|\mu}}{e^{i\lambda_\mu}\dot{\eta}_\mu} = \mp \frac{\Phi'_{|\mu}}{\exp(Y_{|\mu})} ie^{\mp it} = \mp \frac{ie^{\mp it}}{B}. \tag{20}$$

This means that the factor in front of the circle shift in (16) can be replaced by a simple $\mp ie^{\mp it}/B$. This holds exactly only at the solution. We apply this relation already during the iteration. This has the decisive advantage that the RH problem (15) has the special form discussed in [16]. We take from [16] that a unique solution exists. This solution can be calculated efficiently by the method of "successive conjugation". Therefore, the method which we are going to present in the following, is feasible.

For an unbounded region one can impose the condition $Y(\infty) = 0$. Since $\Phi'(\infty) = 1$, this implies that $\exp(Y) = \Phi'$ and $B = 1$ in (19). For bounded regions there is no such normalization. One must admit values $B \neq 1$. The value can be calculated by integrating the equation $B|\Phi'_{|\mu}| = |\exp(Y_{|\mu})|$ over the circle C_μ. Since $\Phi_{|\mu}$ parametrizes Γ_μ the differential of arclength on Γ_μ is $R_\mu|\Phi'_{|\mu}|dt$. Therefore, the integral over $|\Phi'_{|\mu}|$ is simply the length $L(\Gamma_\mu)$ of the curve Γ_μ divided by R_μ. Using $|\exp(Y_{|\mu})| = \exp(w_\mu)$ we get finally

$$B = \frac{R_\mu \int_0^{2\pi} e^{w_\mu} dt}{L(\Gamma_\mu)} \tag{21}$$

independent of μ.

Now the RH problem is solved and the necessary changes can be calculated from (12). We give the complete formulas in compact form in the next subsections.

3.2 Unbounded regions

We assume that the functions S_μ are Hoelder continuous. Then the functions

$$v_\mu(t) := \theta_\mu(S_\mu(t)) + t + \frac{\pi}{2} \tag{22}$$

are Hoelder continuous and 2π-periodic. There are uniquely defined Hoelder continuous functions w_μ, real numbers λ_μ and an analytic function Y in H which has boundary values

$$Y_{|\mu} = w_\mu + iv_\mu + i\lambda_\mu \tag{23}$$

and vanishes at infinity. This is Vekua's "problem D" ([10],p. 261). The Y, w_μ and λ_μ satisfy the equation (14). With the w_μ and λ_μ the auxilary complex functions

$$F_\mu := \frac{r_\mu}{e^{i\lambda_\mu} e^{w_\mu}} \frac{\eta_\mu}{\dot{\eta}_\mu} = \exp(-w_\mu - i\theta_\mu - i\lambda_\mu)\eta_\mu \tag{24}$$

are calculated. The imaginary part of F_μ is the first term on the right hand side of (16). Then an analytic function Ξ in H, complex numbers a_μ, real numbers α_μ and real functions g_μ are determined such that

$$e^{it}\Xi_{|\mu} + a_\mu e^{it} + \alpha_\mu = \mathrm{Im} F_\mu + ig_\mu. \tag{25}$$

The function Ξ is required to behave near infinity like

$$\Xi(z) = z + O\left(\frac{1}{z}\right). \tag{26}$$

This problem is discussed in detail in [16]. We quote from [16] the result, that (25) has a unique solution Ξ, a_μ, and α_μ. The function Ξ is a solution of the Riemann-Hilbert-problem (15).

With the numbers a_μ, α_μ and the functions g_μ the necessary changes in S_μ, z_μ and R_μ are calculated by the formulas

$$\delta S_\mu = -\frac{e^{w_\mu}}{r_\mu}(g_\mu + \text{Re } F_\mu), \tag{27}$$

$$\delta z_\mu = a_\mu - \text{Res}_\infty Y, \qquad \delta R_\mu = \alpha_\mu.$$

These formulas give an approximate solution of the linearized boundary correspondence equation (12). They are explained by the heuristic arguments of subsection 3.1. They will be rigorously justified by the convergence proof of section 4. The convergence proof will also explain why the residue of Y in (27) is necessary.

Based on (27) one can devise an iterative method: When in the k-th step functions $S_\mu^{(k)}$ and circle parameters $z_\mu^{(k)}$ and $R_\mu^{(k)}$ are given, insert these values into the formulas (22) etc. and calculate a shift according to (27), then change the data

$$S_\mu^{(k+1)} = S_\mu^{(k)} + \delta S_\mu, \qquad z_\mu^{(k+1)} = z_\mu^{(k)} + \delta z_\mu, \tag{28}$$

$$R_\mu^{(k+1)} = R_\mu^{(k)} + \delta R_\mu$$

and perform the next step.

3.3 Bounded regions

Recall that there are $m+1$ boundary components of G and H, labeled by $\mu = 0, 1, \ldots, m$. The outer boundary corresponds to $\mu = 0$. It has a different orientation and must be treated separately. In the following formulas the upper sign is for $\mu = 0$, the lower sign for $\mu = 1, \ldots, m$.

In order to satisfy the first normalization condition of Theorem 1 we assume that $z_0 = 0$ and $R_0 = 1$, i.e., the outer boundary of H is already the unit circle. The interpolatory conditions (5) give restrictions

$$S_0(t_j) = s_j \quad \text{for} \quad j = 1, 2, 3 \tag{29}$$

on the function S_0.

When the functions S_μ are Hoelder continuous, then the functions

$$v_\mu(t) := \theta_\mu(S_\mu(t)) \mp t \mp \frac{\pi}{2} \tag{30}$$

are Hoelder continuous and 2π-periodic. There are Hoelder continuous functions w_μ, real numbers λ_μ, and an analytic function Y in H which has boundary values

$$Y_{|\mu} = w_\mu + iv_\mu + i\lambda_\mu. \tag{31}$$

The function Y is uniquely defined up to an additive complex constant. We single out a unique solution by the constraints

$$\lambda_0 = 0, \qquad w_0(0) = 0. \tag{32}$$

Equation (31) is Vekua's "problem D"([10],p. 261). With the w_μ and λ_μ so determined calculate the auxilary complex functions

$$F_\mu := \frac{r_\mu}{e^{i\lambda_\mu} e^{w_\mu}} \frac{\eta_\mu}{\dot{\eta}_\mu} = \exp(-w_\mu - i\theta_\mu - i\lambda_\mu)\eta_\mu. \tag{33}$$

Then determine an analytic function Ξ in H, complex numbers a_μ, real numbers α_μ and real functions g_μ such that for $\mu = 1, \ldots, m$

$$e^{it}\Xi_{|\mu} + a_\mu e^{it} + \alpha_\mu = \mathrm{Im}F_\mu + ig_\mu \tag{34}$$

and for $\mu = 0$

$$e^{-it}\Xi_{|0} = -\mathrm{Im}F_0 - ig_0. \tag{35}$$

The function Ξ is required to satisfy at the outer boundary the interpolation constraints

$$\mathrm{Im}(e^{-it_j}\Xi(e^{it_j})) = \frac{r_0(S_0(t_j))}{\exp(w_0(t_j))}(s_j - S_0(t_j)) + \mathrm{Re}F_0(t_j) \tag{36}$$

for $j = 1, 2, 3$. This problem is discussed in detail in [16]. The condition (35) is quite analogous to (34) with

$$a_0 = 0, \qquad \alpha_0 = 0. \tag{37}$$

It has been shown in [16] that the free parameters on the outer circle must be specified as in (37) and interpolatory conditions of the kind (36) must be added in order to make the solution of the RH problem unique.

Let L_0 be the length of the curve Γ_0 and put

$$B = \frac{\int_0^{2\pi} e^{w_0} dt}{L_0}. \tag{38}$$

This is the same as defined in (21) for $\mu = 0$. We have used $R_0 = 1$ from the normalisation of theorem 1.

With the numbers a_μ, α_μ and B and the functions g_μ so determined, the necessary changes in S, z,and R can be calculated by the formulas

$$\delta S_\mu = -\frac{e^{w_\mu}}{r_\mu}(g_\mu + \mathrm{Re}\ F_\mu), \qquad \delta z_\mu = B\, a_\mu, \qquad \delta R_\mu = B\, \alpha_\mu. \tag{39}$$

The definition (37) implies that the outer circle remains unchanged. These formulas give an approximate solution of the linearized boundary correspondence equation (12). They will be justified by the convergence proof of section 4.

4 Convergence

We follow the method of proof of [15] and prove that the iteration converges in the Sobolev norm whenever the boundary of the target region G is sufficiently smooth.

For the description of the functions S_μ it is convenient to introduce the set W^* of functions f of form $f(t) = g(t) + t$ with $g \in W$. This set inherits from W the metric, and expressions like $||f_1 - f_2||_W$ are well defined for $f_1, f_2 \in W^*$. We assume that the parameter mapping functions S_μ are all in W^*. Let $\mathbf{W}$ be the Banach space $\mathbf{W} := (W \times \mathbb{C} \times \mathbb{R})^m$ and let $\mathbf{W}^*$ be defined in the same way but with W^* instead of W. The state of the iteration is described by the functions S_μ and by the circle parameters z_μ, R_μ. We interpret this as a point $\mathbf{S} \in \mathbf{W}^*$

The solutions of the conjugation problems (23), (25), (31) and (34) are in W provided the right hand sides v_μ and F_μ are in W. But this is satisfied whenever the functions η_μ are three times differentiable and the S_μ are in W^*. Then it follows that δS_μ is in W. Therefore, the iteration (28) can be described by

$$\mathbf{S}^{(k+1)} = \mathbf{S}^{(k)} + \mathbf{A}(\mathbf{S}^{(k)}) \tag{40}$$

with a nonlinear operator $\mathbf{A}$ which calculates from $\mathbf{S} \in \mathbf{W}^*$ the changes $\mathbf{A}(\mathbf{S}) = (\delta S_\mu, \delta z_\mu, \delta R_\mu) \in \mathbf{W}$ according to (27) or (39).

Theorem 2 *Let $\tilde{H}$ be the circular region corresponding to G and let the conformal mapping $\tilde{\Phi}$ from $\tilde{H}$ to G be described by $\tilde{\mathbf{S}} \in \mathbf{W}^*$. Assume that all η_μ are three times differentiable, and that the third derivatives are Lipschitz continuous.*

Then there exists an $\epsilon_0 > 0$ such that for all $||\mathbf{S}_0 - \tilde{\mathbf{S}}|| < \epsilon_0$ the sequence $\mathbf{S}_k$ defined by (40) converges to $\tilde{\mathbf{S}}$ as $k \to \infty$. Convergence is locally quadratic, *i.e., there exist positive constants ϵ_1 and M such that for $||\mathbf{S}_k - \tilde{\mathbf{S}}|| < \epsilon_1$ the next iterate satisfies*

$$||\mathbf{S}_{k+1} - \tilde{\mathbf{S}}|| < M||\mathbf{S}_k - \tilde{\mathbf{S}}||^2. \tag{41}$$

Proof: The proof uses the theorem of Ostrowski on stationary iteration processes. We consider first the case of an unbounded region

1) We show that $\tilde{\mathbf{S}}$ is a stationary point.

For $\tilde{\mathbf{S}}$ the boundary correspondence equation (6) is satisfied. Differentiation with respect to t yields:

$$\tilde{R}_\mu \tilde{\Phi}'_{|\mu} = ie^{it}\dot{\eta}_\mu \tilde{S}'_\mu. \tag{42}$$

The argument

$$\arg \tilde{\Phi}'_{|\mu} = \theta_\mu + t + \pi/2 = v_\mu \pmod{2\pi} \tag{43}$$

is a 2π periodic function on each C_μ. Since $\tilde{\Phi}' \neq 0$ in $\tilde{H}$ and $\tilde{\Phi}'(z) = 1 + O(1/z^2)$ near ∞ and $\log(\tilde{\Phi}')$ is univalent on each boundary circle of $\tilde{H}$, the logarithm $Y := \log(\tilde{\Phi}')$ is defined as a univalent function in $\tilde{H}$ and behaves like $O(1/z^2)$ near infinity. This implies that the residue $\mathrm{Res}_\infty Y = 0$. This function Y solves (23) with $\lambda_\mu = 0 \pmod{2\pi}$. Using

$$e^{w_\mu} = |\tilde{\Phi}'_{|\mu}| = r_\mu \tilde{S}'_\mu / \tilde{R}_\mu \tag{44}$$

and (42) the functions F_μ can be transformed to

$$F_\mu = \frac{r_\mu}{e^{i\lambda_\mu} e^{w_\mu}} \frac{\eta_\mu}{\dot{\eta}_\mu} = \frac{\eta_\mu \tilde{R}_\mu}{\tilde{S}'_\mu \dot{\eta}_\mu} = ie^{it} \frac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}}. \tag{45}$$

Therefore, the solution of the general conjugation (25) is given by

$$\Xi = \frac{\tilde{\Phi}}{\tilde{\Phi}'}, \quad \alpha_\mu = 0, \quad a_\mu = 0, \quad g_\mu = -\mathrm{Re} F_\mu. \tag{46}$$

The normalization (1) ensures that the function $\Xi = \tilde{\Phi}/\tilde{\Phi}'$ has near ∞ the correct behaviour prescribed by (26). Insertion of (46) into (27) yields

$$\delta S_\mu = 0, \quad \delta R_\mu = 0, \quad \delta z_\mu = 0, \tag{47}$$

i.e., $\tilde{\mathbf{S}}$ is a stationary point of the iteration.

2) We calculate the derivative of $\mathbf{A}$ at $\tilde{\mathbf{S}}$ with respect to $\mathbf{S}$.

Let $DS_\mu, Dz_\mu, DR_\mu \in W \times \mathbb{C} \times \mathbb{R}$ be the components of a change in $\mathbf{S}$. Then the changes of Y, w and λ are determined by the equation

$$DY_{|\mu} + Y'_{|\mu} \left(Dz_\mu + DR_\mu \, e^{-it}\right) = Dw_\mu + iDv_\mu + iD\lambda_\mu. \tag{48}$$

Here we need the smoothness properties of the functions η_μ to ensure that the mapping $S_\mu \to \theta_\mu \circ S_\mu$ as a mapping from W^* to W is Frechet differentiable (see [15],Lemma 3). Therefore, the differential Dv_μ in (48) makes sense.

The analytic function DY is subject to the normalization

$$DY(\infty) = 0. \tag{49}$$

With the derivative

$$DF_\mu = r_\mu e^{-w_\mu} DS_\mu - F_\mu(Dw_\mu + iDv_\mu + iD\lambda_\mu) \tag{50}$$

we get in the general conjugation

$$\begin{aligned} e^{it} D\Xi_{|\mu} \quad & + e^{it}\Xi'_{|\mu} \left(Dz_\mu + DR_\mu \, e^{-it}\right) \\ & + Da_\mu e^{it} + D\alpha_\mu = \mathrm{Im} DF_\mu + iDg_\mu \end{aligned} \tag{51}$$

with an analytic function $D\Xi$ normalized by

$$D\Xi(\infty) = 0. \tag{52}$$

The derivative of $\mathbf{A}$ at $\tilde{\mathbf{S}}$ is described by the formulas:

$$\begin{aligned} D\delta S_\mu &= -\tfrac{e^{w_\mu}}{r_\mu} \left(Dg_\mu + \mathrm{Re} DF_\mu\right), \\ D\delta z_\mu &= Da_\mu - \mathrm{Res}_\infty DY, \quad D\delta R_\mu = D\alpha_\mu. \end{aligned} \tag{53}$$

We insert (48) into (50), and replace F and Y' using (45) and $Y' = \tilde{\Phi}''/\tilde{\Phi}'$. This gives the equation:

$$\begin{aligned} DF_\mu \quad &= r_\mu e^{-w_\mu} DS_\mu - ie^{it} \tfrac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}} \, DY_{|\mu} \\ & - ie^{it} \tfrac{\tilde{\Phi}_{|\mu} \tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}} (Dz_\mu + DR_\mu \, e^{-it}). \end{aligned} \tag{54}$$

It follows from (46) that $\Xi' = 1 - \tilde{\Phi}\tilde{\Phi}''/\tilde{\Phi}'^2$. We insert this into (51) and get

$$\begin{aligned} e^{it} D\Xi_{|\mu} \quad & + e^{it}(Dz_\mu + DR_\mu \, e^{-it}) \\ & - e^{it} \tfrac{\tilde{\Phi}_{|\mu} \tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}} (Dz_\mu + DR_\mu \, e^{-it}) \\ & + Da_\mu e^{it} + D\alpha_\mu = \mathrm{Im} DF_\mu + iDg_\mu. \end{aligned} \tag{55}$$

We rearrange this equation, add on both sides $e^{it}\tilde{\Phi}_{|\mu}\ DY_{|\mu}/\tilde{\Phi}'_{|\mu}$ and distribute $\mathrm{Res}_\infty DY$

$$\begin{aligned} e^{it} \quad & \left(D\Xi_{|\mu} + \frac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}} DY_{|\mu} + \mathrm{Res}_\infty DY\right) \\ & + e^{it}(Dz_\mu + Da_\mu - \mathrm{Res}_\infty DY) + DR_\mu + D\alpha_\mu \\ & = e^{it}\frac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}}(Dz_\mu + DR_\mu\ e^{-it}) + \mathrm{Im}DF_\mu \\ & + e^{it}\frac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}} DY_{|\mu} + iDg_\mu. \end{aligned} \tag{56}$$

It follows from (54) that

$$\begin{aligned} \mathrm{Im}DF_\mu \quad & = -\mathrm{Re}\left(e^{it}\frac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}} DY_{|\mu}\right) \\ & -\mathrm{Re}\left(e^{it}\frac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}}(Dz_\mu + DR_\mu\ e^{-it})\right). \end{aligned} \tag{57}$$

After inserting (57) into (56) the real parts on the right hand side of (56) cancel. It remains a general conjugation problem

$$e^{it}\Psi_{|\mu} + b_\mu e^{it} + \beta_\mu = i\phi_\mu \tag{58}$$

for the analytic function $\Psi := D\Xi + \tilde{\Phi}\ DY/\tilde{\Phi}' + \mathrm{Res}_\infty DY$ with zero real part on the right hand side. Since $\tilde{\Phi}(z)/\tilde{\Phi}'(z) = z + O(1/z)$ near infinity the function $\tilde{\Phi}\ DY/\tilde{\Phi}'$ approaches the value $-\mathrm{Res}_\infty DY$ as $z \to \infty$. Together with the normalization (52) this implies that Ψ vanishes at ∞. Since the general conjugation problem has a unique solution, the solution of (58) must be zero. This implies for the left hand side of (56)

$$D\Xi = -\frac{\tilde{\Phi}}{\tilde{\Phi}'}DY - \mathrm{Res}_\infty DY, \tag{59}$$

$$Da_\mu - \mathrm{Res}_\infty DY = -Dz_\mu, \quad D\alpha_\mu = -DR_\mu.$$

Since the imaginary part of the right hand side of (56) vanishes, Dg_μ can be calculated

$$\begin{aligned} Dg_\mu \quad & = -\mathrm{Im}\left(e^{it}\frac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}} DY_{|\mu}\right) \\ & -\mathrm{Im}\left(e^{it}\frac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}}(Dz_\mu + e^{-it}DR_\mu)\right). \end{aligned} \tag{60}$$

We insert this into (54) and get

$$\mathrm{Re}DF_\mu = r_\mu e^{-w_\mu}\ DS_\mu - Dg_\mu \tag{61}$$

and after insertion into (53)

$$D\delta S_\mu = -\frac{e^{w_\mu}}{r_\mu}\left(Dg_\mu + \mathrm{Re}DF_\mu\right) = -DS_\mu. \tag{62}$$

The derivatives of the circle parameters are obtained from (59) and (53):

$$D\delta z_\mu = -Dz_\mu, \quad D\delta R_\mu = -DR_\mu. \tag{63}$$

This means that the derivative $D\mathbf{A}$ at the stationary point $\tilde{\mathbf{S}}$ is equal to $-\mathbf{I}$, the negative identity operator $\mathbf{I}$. Since the iteration is described in (40) by the operator $\mathbf{A}_1 = \mathbf{I} + \mathbf{A}$, the derivative $D\mathbf{A}_1$ of the iteration operator $\mathbf{A}_1$ at the stationary point $\tilde{\mathbf{S}}$ is the zero operator. The conclusions of the theorem follow now from Ostrowski's theorem (see e.g. [9],p.303).

3) The proof for bounded regions is along the same lines with a few modifications. First we show that $\tilde{\mathbf{S}}$ is a stationary point.

For $\tilde{\mathbf{S}}$ the boundary correspondence equation (6) is satisfied. Differentiation with respect to t yields:

$$\tilde{R}_\mu \tilde{\Phi}'_{|\mu} = \mp i e^{\mp it} \dot{\eta}_\mu \tilde{S}'_\mu. \tag{64}$$

The argument

$$\arg \tilde{\Phi}'_{|\mu} = \theta_\mu \mp t \mp \pi/2 = v_\mu \pmod{2\pi} \tag{65}$$

is a 2π-periodic function on each C_μ. Since $\tilde{\Phi}' \neq 0$ in $\tilde{H}$ and $\arg\tilde{\Phi}'$ is univalent on the boundary curves, the logarithm $Y := \log(\tilde{\Phi}')$ is defined as a univalent analytic function in $\tilde{H}$. The solution of (31) and (32) is of the form $Y = \log(\tilde{\Phi}') + \omega$ with a real number ω and $\lambda_\mu = 0 \pmod{2\pi}$. Using

$$e^{w_\mu} = e^{\omega}|\tilde{\Phi}'_{|\mu}| = e^{\omega} r_\mu \tilde{S}'_\mu / \tilde{R}_\mu \tag{66}$$

and (64) the functions F_μ can be transformed to

$$F_\mu = \frac{r_\mu}{e^{i\lambda_\mu} e^{w_\mu}} \frac{\eta_\mu}{\dot{\eta}_\mu} = \frac{\eta_\mu \tilde{R}_\mu}{e^{\omega} \tilde{S}'_\mu \dot{\eta}_\mu} = \mp i e^{\mp it} \frac{\tilde{\Phi}_{|\mu}}{e^{\omega} \tilde{\Phi}'_{|\mu}}. \tag{67}$$

Therefore, the solution of the general conjugation (35) is given by

$$\Xi = \frac{\tilde{\Phi}}{e^{\omega}\tilde{\Phi}'}, \quad \alpha_\mu = 0, \quad a_\mu = 0, \quad g_\mu = -\mathrm{Re}F_\mu. \tag{68}$$

This function Ξ satisfies also the interpolation conditions (36). Insertion of (68) into (39) yields

$$\delta S_\mu = 0, \quad \delta R_\mu = 0, \quad \delta z_\mu = 0, \tag{69}$$

i.e., $\tilde{\mathbf{S}}$ is a stationary point of the iteration.

4) We calculate the derivative of $\mathbf{A}$ at $\tilde{\mathbf{S}}$ with respect to $\mathbf{S}$.

Let $DS_\mu, Dz_\mu, DR_\mu \in W \times \mathbb{C} \times \mathbb{R}$ be the components of a change in $\mathbf{S}$. The outer circle remains fixed: $Dz_0 = 0, DR_0 = 0$. The changes of Y, w and λ are determined by the equation

$$DY_{|\mu} + Y'_{|\mu}\,(Dz_\mu + DR_\mu\, e^{\pm it}) = Dw_\mu + iDv_\mu + iD\lambda_\mu \tag{70}$$

and the constraints $D\lambda_0 = 0$ and $Dw_0(0) = 0$.

The derivative DF_μ is as in (50). With this we get in the general conjugation

$$\begin{aligned} e^{\mp it} D\Xi_{|\mu} \quad & +e^{\mp it}\Xi'_{|\mu}\,(Dz_\mu + DR_\mu\, e^{\pm it}) \\ & +Da_\mu e^{\mp it} + D\alpha_\mu = \mp \mathrm{Im} DF_\mu \mp iDg_\mu \end{aligned} \tag{71}$$

with $Da_0 = 0, D\alpha_0 = 0$ and an analytic function $D\Xi$ subject to the interpolation conditions

$$\mathrm{Im}(e^{-it_j} D\Xi(e^{it_j})) = -\frac{r_0(S_0(t_j))}{\exp(w_0(t_j))} DS_0(t_j) + \mathrm{Re} DF_0(t_j) \tag{72}$$

for $j = 1, 2, 3$. The derivative of $\mathbf{A}$ at $\tilde{\mathbf{S}}$ is described by the formulas:

$$\begin{aligned} D\delta S_\mu &= -\tfrac{e^{w_\mu}}{r_\mu}\,(Dg_\mu + \mathrm{Re} DF_\mu)\,, \\ D\delta z_\mu &= BDa_\mu, \quad D\delta R_\mu = BD\alpha_\mu \end{aligned} \tag{73}$$

with $B = e^{\omega}$. We insert (70) into (50), and replace F and Y' using (67) and $Y' = \tilde{\Phi}''/\tilde{\Phi}'$. This gives the equation:

$$\begin{aligned} DF_\mu \quad & = r_\mu e^{-w_\mu} DS_\mu \pm i e^{\mp it} e^{-\omega} \frac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}}\, DY_{|\mu} \\ & \pm i e^{\mp it} e^{-\omega} \frac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}} (Dz_\mu + DR_\mu\, e^{\pm it}). \end{aligned} \tag{74}$$

It follows from (68) that $\Xi' = e^{-\omega}(1 - \tilde{\Phi}\tilde{\Phi}''/\tilde{\Phi}'^2)$. We insert this into (71) and get

$$\begin{aligned} e^{\mp it} D\Xi_{|\mu} \quad & +e^{-\omega} e^{\mp it} (Dz_\mu + DR_\mu\, e^{\pm it}) \\ & -e^{-\omega} e^{\mp it} \frac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}} (Dz_\mu + DR_\mu\, e^{\pm it}) \\ & +Da_\mu e^{\mp it} + D\alpha_\mu = \mp \mathrm{Im} DF_\mu \mp iDg_\mu. \end{aligned} \tag{75}$$

We rearrange this equation and add on both sides

$e^{-\omega}e^{\mp it}\tilde{\Phi}_{|\mu}\, DY_{|\mu}/\Phi'_{|\mu}$

$$
\begin{aligned}
e^{\mp it} \quad & \left(D\Xi_{|\mu} + e^{-\omega}\tfrac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}}DY_{|\mu}\right) \\
& + e^{\mp it}(e^{-\omega}Dz_\mu + Da_\mu) + e^{-\omega}DR_\mu + D\alpha_\mu \\
& = e^{-\omega}e^{\mp it}\tfrac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}}(Dz_\mu + DR_\mu\, e^{\pm it}) \\
& \mp \mathrm{Im}DF_\mu + e^{-\omega}e^{\mp it}\tfrac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}}DY_{|\mu} \mp iDg_\mu.
\end{aligned}
\tag{76}
$$

It follows from (74) that

$$
\begin{aligned}
\mathrm{Im}DF_\mu \quad & = \pm\mathrm{Re}\left(e^{-\omega}e^{\mp it}\tfrac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}}DY_{|\mu}\right) \\
& \pm\mathrm{Re}\left(e^{-\omega}e^{\mp it}\tfrac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}}(Dz_\mu + DR_\mu\, e^{\pm it})\right).
\end{aligned}
\tag{77}
$$

After inserting (77) into (76) the real parts on the right hand side of (76) cancel. It remains a general conjugation problem

$$
e^{\mp it}\Psi_{|\mu} + b_\mu e^{\mp it} + \beta_\mu = i\phi_\mu \tag{78}
$$

for the analytic function $\Psi := D\Xi + e^{-\omega}\tilde{\Phi}\, DY/\tilde{\Phi}'$ with zero real part on the right hand side and $b_0 = 0, \beta_0 = 0$. We derive the interpolation conditions

$$
\mathrm{Im}(e^{-it_j}\Psi(e^{it_j})) = 0 \tag{79}
$$

for $j = 1, 2, 3$. We have used (72) to get the values for the $D\Xi$ part of Ψ. The second part $e^{-\omega}\tilde{\Phi}\, DY/\tilde{\Phi}'$ is transformed using (67) and (50).

The general conjugation problem has a unique solution for each right hand side. Therefore, the solution of (78) must be zero. This implies for the left hand side of (76)

$$
D\Xi = -\frac{\tilde{\Phi}}{e^\omega\tilde{\Phi}'}DY, \quad Da_\mu = -e^{-\omega}Dz_\mu, \quad D\alpha_\mu = -e^{-\omega}DR_\mu. \tag{80}
$$

Since the imaginary part of the right hand side of (76) vanishes, Dg_μ can be calculated as

$$
\begin{aligned}
Dg_\mu \quad & = \pm e^{-\omega}\mathrm{Im}\left(e^{\mp it}\tfrac{\tilde{\Phi}_{|\mu}}{\tilde{\Phi}'_{|\mu}}DY_{|\mu}\right) \\
& \pm e^{-\omega}\mathrm{Im}\left(e^{\mp it}\tfrac{\tilde{\Phi}_{|\mu}\tilde{\Phi}''_{|\mu}}{\tilde{\Phi}'^2_{|\mu}}(Dz_\mu + e^{\pm it}DR_\mu)\right).
\end{aligned}
\tag{81}
$$

We insert this into (74) and get

$$\mathrm{Re} DF_\mu = r_\mu e^{-w_\mu}\ DS_\mu - Dg_\mu \tag{82}$$

and after insertion into (73)

$$D\delta S_\mu = -\frac{e^{w_\mu}}{r_\mu}\left(Dg_\mu + \mathrm{Re} DF_\mu\right) = -DS_\mu. \tag{83}$$

The derivatives of the circle parameters are obtained from (80) and (73):

$$D\delta z_\mu = -Dz_\mu, \quad D\delta R_\mu = -DR_\mu. \tag{84}$$

This means that the derivative $D\mathbf{A}$ at the stationary point $\tilde{\mathbf{S}}$ is equal to $-\mathbf{I}$. As for unbounded regions this implies locally quadratic convergence. QED

5 Numerical examples

The Riemann-Hilbert problem can be solved numerically by the method of successive conjugation [16]. On the circle C_μ there are n_μ grid points. Each function is represented on these grid points, the Fourier transform is evaluated with the full grid, and the Laurent coefficients of all functions are calculated as far as the necessary information is available. When this straightforward discretization is applied, the method shows the same effect of initial convergence and final divergence which was observed already for simply connected regions [11]. It has been demonstrated in [15] that one can enforce convergence of the numerical method by smoothing the shifts δS. A suitable smoothing procedure is as follows: Calculate the functions δS_μ by (27). Then interpolate each periodic function by a trigonometric polynomial of degree $n_\mu/2$

$$\delta S_\mu = \sum_{j=0}^{n_\mu/2} c_j \cos(jt) + d_j \sin(jt) \tag{85}$$

and throw away the $\sin(jt)$ and $\cos(jt)$ terms with $j > N_\mu$, where $N_\mu < n_\mu/2$ is a suitably chosen number,i.e., replace δS_μ by

$$\delta S_\mu^{smooth} := \sum_{j=0}^{N_\mu} c_j \cos(jt) + d_j \sin(jt) \tag{86}$$

in the iteration (28). This smoothing is successful also in the multiply connected case.

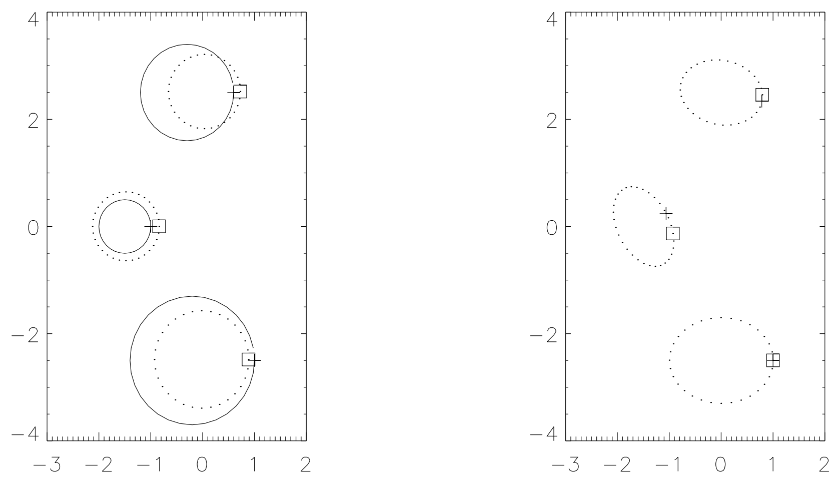

Figure 1: The first example: On the left side the circular regions at the start (solid) and at the end of the iteration (dotted). The right side shows the image points on the boundary of G of the grid points on the circles C_μ. The point for $t = 0$ is marked on each curve: by a cross for the start, and by a quadrangle for the end of the iteration.

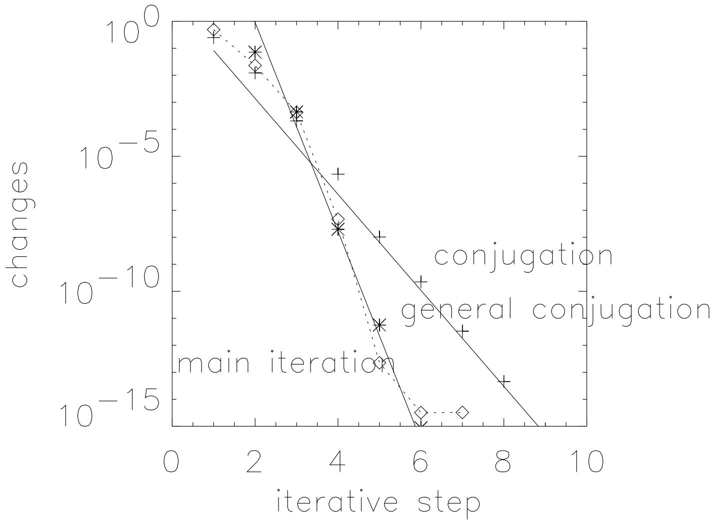

Figure 2: The changes in each iterative step for conjugation (crosses), for general conjugation (stars) and for the main iteration (diamonds). The solid lines are the curves Cq_l^k with the theoretical convergence rates q_l calculated for the final circular region.

Finally we show several examples. The boundary of the region G consists of ellipses. All examples are calculated with $n_\mu = 32$ gridpoints on each circle.

Example 1 The boundary of G consists of three well separated ellipses (Figure 1). The changes in each iteration are shown in Figure 2. The theoretical rates of convergence for the conjugations have been calculated by the method described in [16]. They are $q_0 = 1.67e-2$ and $q_1 = 1.35e-4$, respectively. Therefore, each conjugation requires 9, and each general conjugation 6 iterations to get an accuracy of $1.e-15$. The main iteration converges quadratically and reaches the final state after 6 iterations.

Example 2 A region bounded by four ellipses which are closer together (Figure 3). This makes successive conjugation slower convergent (Figure 4).. The theoretical rates are $q_0 = 2.39e-1$ and $q_1 = 1.27e-2$, respectively. Each conjugation needs 20 iterations, and each general conjugation 9 iterations to reach the final state. The main iteration reaches the goal again in only 6 iterations.

Multiply connected bounded regions are not very popular among conformal mappers. Though it is conceivable, that most of the known mapping methods can be adapted to bounded regions, there are few if any examples

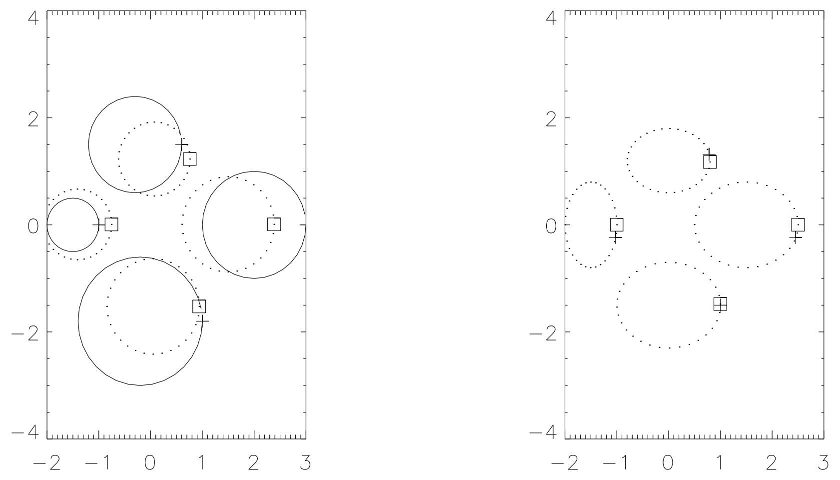

Figure 3: The same as figure 1 for the second example.

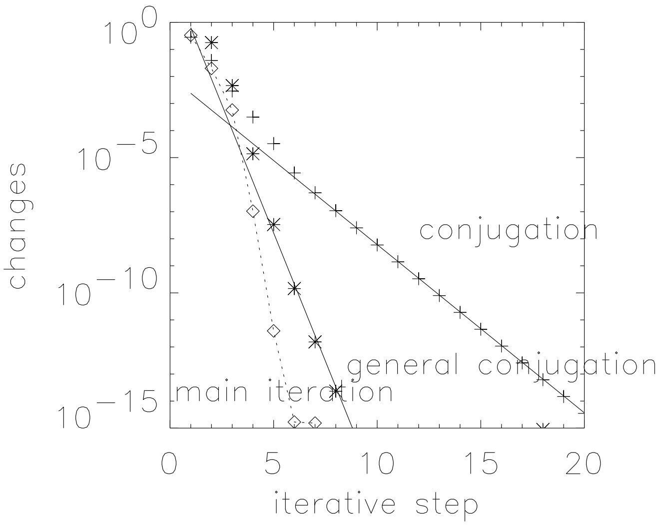

Figure 4: The same as figure 2 for the second example.

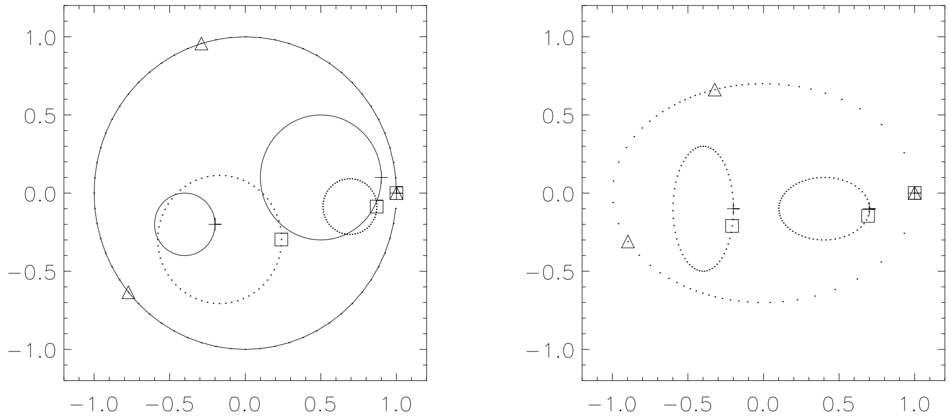

Figure 5: The same as figure 1 for the third example. The points where Φ is prescribed by the normalization (5) are marked by triangles.

in the literature. The bounded case is appreciably more difficult and the outer boundary requires in any case a non negligible amount of extra work.

Example 3 An elliptical region with two elliptical holes (Figure 5). This example is numerically difficult, since "crowding" occurs. The images of equidistant points on the boundary circles are very unevenly distributed on the ellipses. One of the inner circles approaches the outer circle. This makes conjugation slowly converging. Figure 6 shows, that the discrete method does not converge quadratically, but only linearly. This effect has been explained for simply connected regions in [14].

While graphical representations of mappings of unbounded regions are somewhat boring, since the mapping approaches the identity far away from the boundary curves, mappings for bounded regions are more interesting as Figure 7 demonstrates.

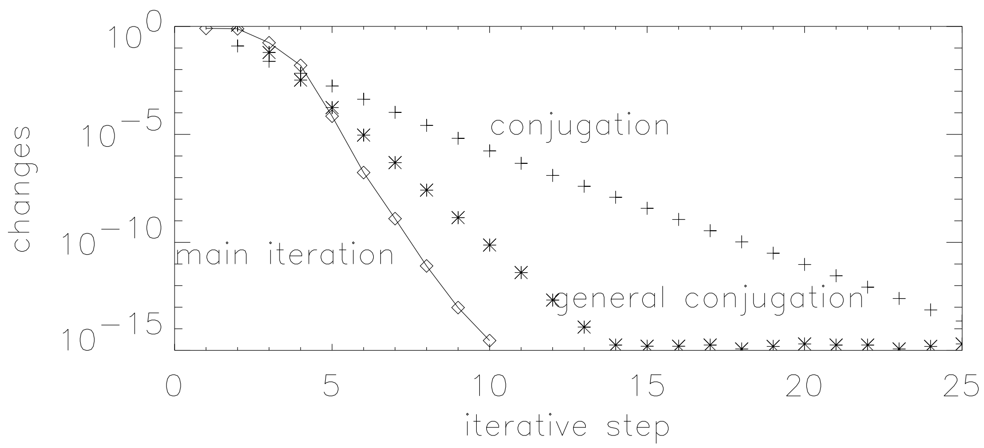

Figure 6: The same as figure 2 for the third example.

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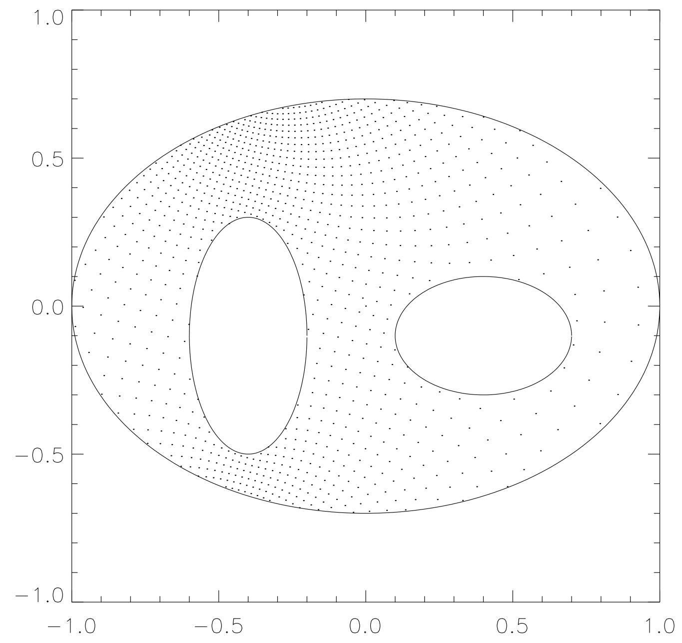

Figure 7: For the third example the image of a rectangular grid in H with grid-size .05

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