CHALLENGES OF COSMOLOGICAL FUZZY DARK MATTER SIMULATIONS

Simon May
simon.may@mpa-garching.mpg.de

Max-Planck-Institut für Astrophysik

29th January 2020

MAX-PLANCK-GESELLSCHAFT

What is “fuzzy dark matter”?

- F(C)DM, BECDM, ULDM, ELBDM, ψDM, quantum-wave DM, (ultra-light) axion(-like) DM (ULA, ALP)...
- New **extremely light scalar** particle ($m \approx 10^{-22}$ eV!)
- Non-thermal production mechanism (thus not ultra-hot)
- Aggregations of bosons can form a **Bose–Einstein condensate**
- Quantum effects counteract gravity at **small scales** (uncertainty principle), erase structure
- Tiny mass
  $\Rightarrow$ large de Broglie wavelength ($\lambda \sim 1/m$)
  $\Rightarrow$ **macroscopic quantum effects** on kpc scales

Motivation for fuzzy dark matter

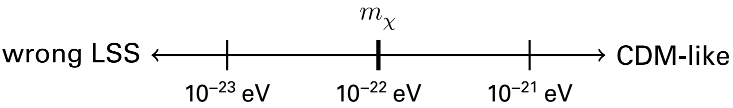
Particle physics perspective:

- Original concept – strong CP problem:
  Why doesn't QCD violate CP symmetry?
- Solved by Peccei–Quinn $\mathrm{U}(1)$ symmetry
  and (pseudo-)scalar field (*axion*!)
  Peccei and Quinn (1977)!
- Fuzzy dark matter is **not** the QCD axion, but axion-like particles are a common feature of early-universe theories

Astrophysics perspective:

- Small-scale challenges (cusp–core, missing satellites, ...)

→ Ultra-light scalars: WIMP alternative, could improve this



- **No sign of (WIMP) CDM**

Motivation for fuzzy dark matter

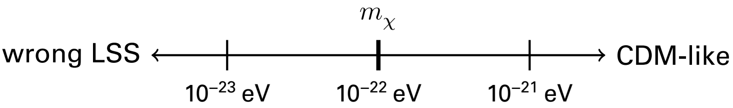
Particle physics perspective:

- Original concept – strong CP problem: Why doesn't QCD violate CP symmetry?
- Solved by Peccei–Quinn $\mathrm{U}(1)$ symmetry and (pseudo-)scalar field (*axion*!) Peccei and Quinn (1977)!
- Fuzzy dark matter is **not** the QCD axion, but axion-like particles are a common feature of early-universe theories

Astrophysics perspective:

- Small-scale challenges (cusp–core, missing satellites, ...)

→ Ultra-light scalars: WIMP alternative, could improve this



- **No sign of (WIMP) CDM**

Derivation of the fuzzy dark matter equations

- Start with simple scalar field action

$$S = \frac{1}{\hbar c^2} \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 - \frac{\lambda}{\hbar^2 c^2} \phi^4 \right)$$

  Note: corresponds to superfluid dark matter without self-interaction ($\lambda = 0$ or $T \to 0$)
  - QCD axion case: originates from periodic potential $V(\phi) \sim \Lambda^4 (1 - \cos(\phi/f_a))$ for $\phi \ll f_a$
- Rewrite

$$\phi = \sqrt{\frac{\hbar^3 c}{2m}} \frac{1}{2} \mathrm{Re}\left(\psi e^{-i c^2/\hbar \, m t}\right) = \sqrt{\frac{\hbar^3 c}{2m}} \left(\psi e^{-i c^2/\hbar \, m t} + \psi^* e^{i c^2/\hbar \, m t}\right)$$

  and take non-relativistic limit with perturbed FRW metric $\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - a(t)^2 \left(1 - \frac{2\Phi}{c^2}\right) \mathrm{d}\vec{x}^2$
- Result: Schrödinger equation

$$i\hbar \left(\partial_t \psi + \frac{3}{2} H \psi \right) = -\frac{\hbar^2}{2m} \nabla^2 \psi + m \Phi \psi$$

Derivation of the fuzzy dark matter equations

- Start with simple scalar field action

$$S = \frac{1}{\hbar c^2} \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 - \cancel{\frac{\lambda}{\hbar^2 c^2} \phi^4} \right)$$

  Note: corresponds to superfluid dark matter without self-interaction ($\lambda = 0$ or $T \to 0$)
  - QCD axion case: originates from periodic potential $V(\phi) \sim \Lambda^4 (1 - \cos(\phi/f_a))$ for $\phi \ll f_a$
- Rewrite

$$\phi = \sqrt{\frac{\hbar^3 c}{2m}} \frac{1}{2} \mathrm{Re}\left(\psi e^{-i c^2/\hbar\, mt}\right) = \sqrt{\frac{\hbar^3 c}{2m}} \left(\psi e^{-i c^2/\hbar\, mt} + \psi^* e^{i c^2/\hbar\, mt}\right)$$

  and take non-relativistic limit with perturbed FRW metric $\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - a(t)^2 \left(1 - \frac{2\Phi}{c^2}\right) \mathrm{d}\vec{x}^2$
- Result: Schrödinger equation

$$i\hbar \left(\partial_t \psi + \frac{3}{2} H \psi \right) = -\frac{\hbar^2}{2m} \nabla^2 \psi + m \Phi \psi$$

Derivation of the fuzzy dark matter equations

$$i\hbar\left(\partial_t\psi + \frac{3}{2}H\psi\right) = -\frac{\hbar^2}{2m}\nabla^2\psi + m\Phi\psi$$

- Mean field approximation: interpretation as the single macroscopic wave function of Bose-Einstein condensate with density $\rho = m|\psi|^2$
- "FDM equations" are the nonlinear Schrödinger–Poisson system of equations

$$i\hbar\partial_t\psi_\mathrm{c} = -\frac{\hbar^2}{2ma^2}\nabla_\mathrm{c}^2\psi_\mathrm{c} + m\Phi\psi_\mathrm{c}$$
$$\nabla_\mathrm{c}^2\Phi = \frac{4\pi G}{a}m(|\psi_\mathrm{c}|^2 - |\bar{\psi}_\mathrm{c}|^2)$$

- Only a single scale, determined by $\frac{\hbar}{m}$ ($\to$ wavelength)

Approaches to fuzzy dark matter simulations

I. Schrödinger–Poisson equations

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + m\Phi\psi$$
$$\nabla^2\Phi = 4\pi G m(|\psi|^2 - \bar{|\psi|^2})$$

II. Madelung formulation (fluid dynamics representation)

$$\partial_t\rho + \nabla\cdot\rho\vec{v} = 0$$
$$\partial_t\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{m}\nabla\Bigg(\underbrace{-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}}_{=Q} + \Phi\Bigg)$$
$$\nabla^2\Phi = 4\pi G(\rho - \bar{\rho})$$

$$\psi = \sqrt{\frac{\rho}{m}}e^{i\alpha}$$
$$\rho = m|\psi|^2$$
$$\vec{v} = \frac{\hbar}{m}\nabla\alpha$$

- Phase is undefined for $\rho = 0$
  $\Rightarrow$ significant effects on overall evolution

Approaches to fuzzy dark matter simulations

I. Schrödinger–Poisson equations

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + m\Phi\psi$$
$$\nabla^2\Phi = 4\pi G m(|\psi|^2 - \bar{|\psi|^2})$$

II. Madelung formulation (fluid dynamics representation)

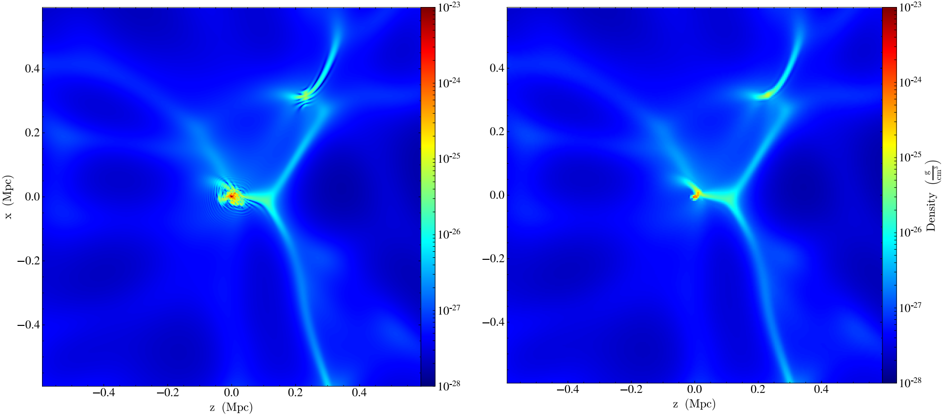
$$\partial_t\rho + \nabla\cdot\rho\vec{v} = 0$$
$$\partial_t\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{m}\nabla\Bigg(\underbrace{-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}}_{=Q} + \Phi\Bigg)$$
$$\nabla^2\Phi = 4\pi G(\rho - \bar{\rho})$$

$$\psi = \sqrt{\frac{\rho}{m}}e^{i\alpha}$$
$$\rho = m|\psi|^2$$
$$\vec{v} = \frac{\hbar}{m}\nabla\alpha$$

"quantum potential"
"quantum pressure"

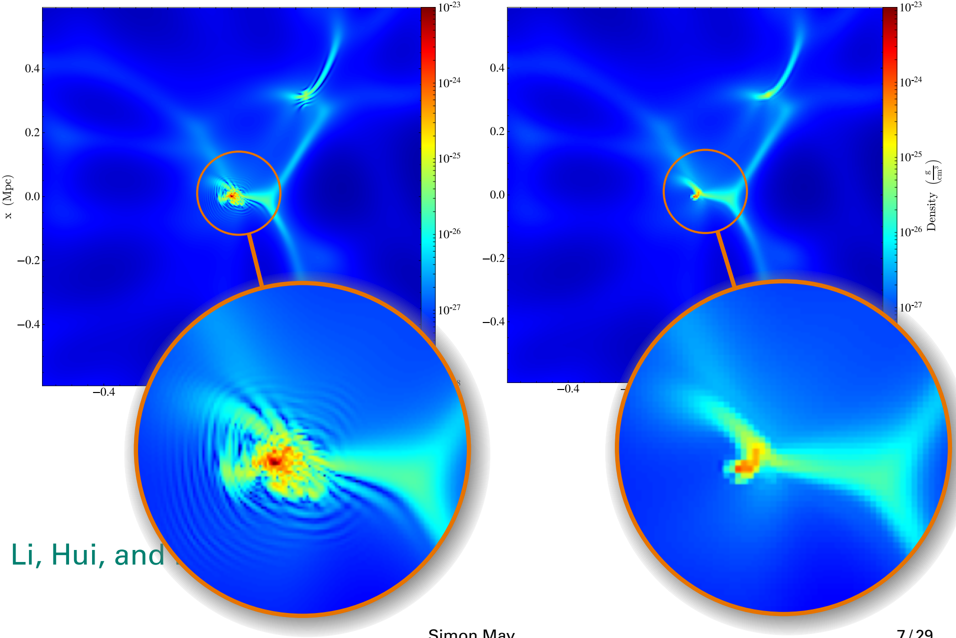
- Phase is undefined for $\rho = 0$
  $\Rightarrow$ significant effects on overall evolution

Schrödinger–Poisson vs. Madelung formulation



Li, Hui, and Bryan (2018)

Schrödinger–Poisson vs. Madelung formulation



Li, Hui, and

Schrödinger–Poisson vs. Madelung formulation

The choice

Madelung formulation:

- Simple to add to existing hydrodynamics codes
- Hydrodynamic interpretation

But:

- Validity unclear
- Quantum potential/pressure is computationally challenging (third derivative of density!)
- Schrödinger–Poisson implementation actually not so difficult (depending on the method)

$\rightarrow$ stick with Schrödinger–Poisson wave function

Soliton solutions to the FDM equations

- Spherical symmetry & time-independent density profile:

$$\psi(t,\vec{x}) \to e^{i\beta t} f(r), \qquad \Phi(t,\vec{x}) \to \varphi(r) \quad (\tilde{\varphi}(r) = \varphi(r) + \beta)$$

- SP equations reduce to

$$0 = -\frac{1}{2} f''(r) - \frac{1}{r} f'(r) + \tilde{\varphi}(r) f(r)$$
$$0 = \tilde{\varphi}''(r) + \frac{2}{r} \tilde{\varphi}'(r) - 4\pi f(r)^2$$

  All constants can be "absorbed"!
- Solve numerically using $f(0) = \alpha$, $f'(0) = \tilde{\varphi}'(0) = 0$, $\varphi(r) \xrightarrow{r\to\infty} -\frac{c}{r}$
- Given solution $e^{i\beta t} f(r)$, $g(r) = e^{i\alpha\beta t} \alpha f(\sqrt{\alpha} r)$ is also a solution

$$\psi(t,\vec{x}) = \alpha f(\sqrt{\alpha}|\vec{x} - \vec{v}t|) e^{i(\alpha\beta t + \vec{v}\cdot\vec{x} - 1/2\, v^2 t + \delta)}$$

Properties of solitonic cores

$$\psi(t,\vec{x}) = \alpha f(\sqrt{\alpha}|\vec{x}-\vec{v}t|)e^{i(\alpha\beta t+\vec{v}\cdot\vec{x}-1/2\,v^2t+\delta)}$$

- Scaling symmetry of SP equations:
  $(t,x,\Phi,\psi) \to (\lambda^{-2}t, \lambda^{-1}x, \lambda^2\Phi, \lambda^2\psi)$
- Solitons only have a single parameter: α
  Can be expressed through soliton mass or radius
- $M_\mathrm{c} \sim \frac{1}{m^2 r_\mathrm{c}} \Rightarrow$ more massive cores are smaller
- Profile approximated by analytical expression

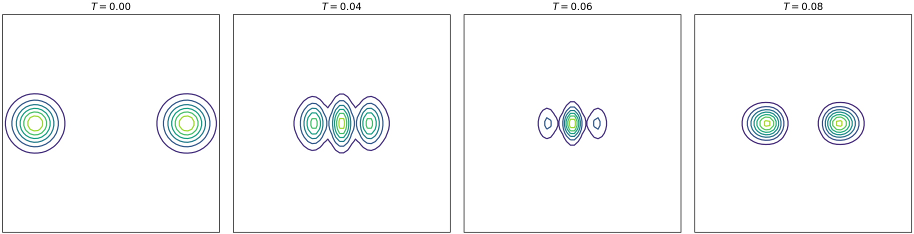
$$\rho_\mathrm{s}(r) \approx \rho_0(m,r_\mathrm{c})\left(1+0.091\left(\frac{r}{r_\mathrm{c}}\right)^2\right)^{-8} \xrightarrow{r\to 0} \text{const.} \quad \xrightarrow{r\to\infty} r^{-16}$$

- Virialized FDM halos form soliton(-like) cores

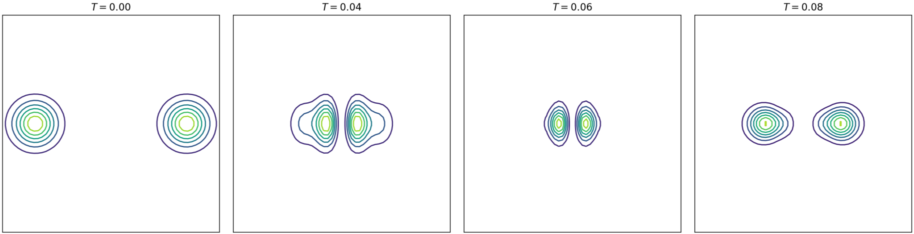
Simple soliton simulations (→ `PyUltraLight`)

Soliton collision interference pattern

Phase difference $= 0$



Phase difference $= \pi$



Using pseudo-spectral methods to simulate FDM (in AREPO)

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + m\Phi\psi$$
$$\nabla^2\Phi = 4\pi G m(|\psi|^2 - \bar{|\psi|}^2)$$

- Symmetrized split-step Fourier method ("kick–drift–kick")
- Small time step Δt:

$$\begin{aligned}\psi(t+\Delta t,\vec{x}) &= \mathcal{T} e^{-i\int_t^{t+\Delta t}\left(-\frac{\hbar}{2m}\nabla^2+\frac{m}{\hbar}\Phi(t',\vec{x})\right)\mathrm{d}t'}\,\psi(t,\vec{x}) \\ &\approx e^{i\frac{\Delta t}{2}\left(\frac{\hbar}{m}\nabla^2-\frac{m}{\hbar}\Phi(t+\Delta t,\vec{x})-\frac{m}{\hbar}\Phi(t,\vec{x})\right)}\psi(t,\vec{x}) \\ &\approx \underbrace{e^{-i\frac{m}{\hbar}\frac{\Delta t}{2}\Phi(t+\Delta t,\vec{x})}}_{\text{"kick"}}\ \underbrace{e^{i\frac{\hbar}{m}\frac{\Delta t}{2}\nabla^2}}_{\text{"drift"}}\ \underbrace{e^{-i\frac{m}{\hbar}\frac{\Delta t}{2}\Phi(t,\vec{x})}}_{\text{"kick"}}\ \psi(t,\vec{x})\end{aligned}$$

$\Rightarrow$ Algorithm:

$$\psi \leftarrow e^{-i\frac{m}{\hbar}\frac{\Delta t}{2}\Phi}\psi \qquad \text{kick}$$
$$\psi \leftarrow \mathrm{IFFT}\left(e^{-i\frac{\hbar}{m}\frac{\Delta t}{2}k^2}\,\mathrm{FFT}(\psi)\right) \qquad \text{drift}$$
$$\Phi \leftarrow \mathrm{IFFT}\left(-\frac{1}{k^2}\,\mathrm{FFT}\left(4\pi G m(|\psi|^2 - \bar{|\psi|}^2)\right)\right) \qquad \text{update potential}$$
$$\psi \leftarrow e^{-i\frac{m}{\hbar}\frac{\Delta t}{2}\Phi}\psi \qquad \text{kick}$$

Pseudo-spectral methods for FDM

Advantages

Algorithm:

$\psi \leftarrow e^{-i \frac{m}{\hbar} \frac{\Delta t}{2} \Phi} \psi$	kick
$\psi \leftarrow \mathrm{IFFT}\Big(e^{-i \frac{\hbar}{m} \frac{\Delta t}{2} k^2}\, \mathrm{FFT}(\psi)\Big)$	drift
$\Phi \leftarrow \mathrm{IFFT}\left(-\frac{1}{k^2}\, \mathrm{FFT}\big(4\pi G m(\lvert\psi\rvert^2 - \lvert\bar{\psi}\rvert^2)\big)\right)$	update potential
$\psi \leftarrow e^{-i \frac{m}{\hbar} \frac{\Delta t}{2} \Phi} \psi$	kick

Choice of time step: $\Delta t \leq \min\Big(\frac{1}{\pi} \frac{m}{\hbar} a^2 \Delta x^2 \ \ , \ \frac{\hbar}{m} a \frac{1}{|\Phi_{\mathrm{max}}|}\Big)$

- "Exact" solution
- Automatic conservation of mass
- Can adapt existing PM code
- Simple implementation

Pseudo-spectral methods for FDM

Advantages

Algorithm:

$$\psi \leftarrow e^{-i \frac{m}{\hbar} \frac{\Delta t}{2} \Phi} \psi \qquad \text{kick}$$

$$\psi \leftarrow \mathrm{IFFT}\Big(e^{-i \frac{\hbar}{m} \frac{\Delta t}{2} k^2}\, \mathrm{FFT}(\psi)\Big) \qquad \text{drift}$$

$$\Phi \leftarrow \mathrm{IFFT}\left(-\frac{1}{k^2}\, \mathrm{FFT}\big(4\pi G m (|\psi|^2 - |\bar{\psi}|^2)\big)\right) \qquad \text{update potential}$$

$$\psi \leftarrow e^{-i \frac{m}{\hbar} \frac{\Delta t}{2} \Phi} \psi \qquad \text{kick}$$

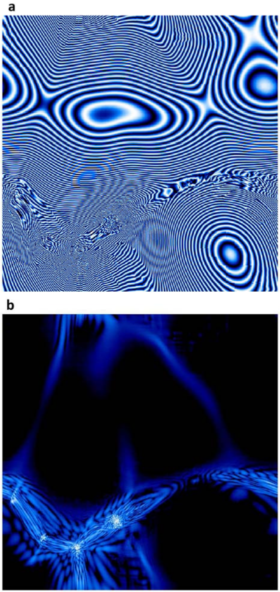
Choice of time step: $\Delta t \leq \min\Big(\frac{1}{\pi} \frac{m}{\hbar} a^2 \boldsymbol{\Delta x}^2,\ \frac{\hbar}{m} a \frac{1}{|\Phi_{\max}|}\Big)$

- “Exact” solution
- Automatic conservation of mass
- Can adapt existing PM code
- Simple implementation

Why is it hard to simulate FDM?

Computational challenges

- Tiny mass $\leftrightarrow$ macroscopic quantum effects, de Broglie wavelength of galactic scale
- Both large scales and de Broglie scale must be resolved for correct evolution (sub-kpc cores can form)
- **Time step criterion:** $\Delta t \sim \Delta x^2$ (seems to be approach-independent)
- Tooling: Hydrodynamics codes are designed for N-body simulations



Schive, Chiueh, and Broadhurst (2014)

Plans for future numerical methods

- **Disadvantage** of pseudo-spectral method: uniform mesh, lacks adaptivity
  $\rightarrow$ large high-resolution simulations are expensive/infeasible
- **Plan** for improved methods:
  - Investigate hybrid methods
  - Perhaps adapt AREPO's moving mesh for FDM
    $\rightarrow$ need second derivative on irregular mesh with minimal noise
- **Goal:** full cosmological simulations with baryons (hydrodynamics)

Correspondence of CDM and FDM initial conditions

Constructing a wave function ψ from a phase space distribution function f:

$$\psi(\vec{x}) \sim \sum_{\vec{v}} \sqrt{f(\vec{x}, \vec{v})} e^{im/\hbar\, \vec{x}\cdot\vec{v} + R_{\vec{v}}}$$

For “cold”/single-stream distribution function:

$$\psi = \sqrt{\rho} e^{i\alpha}$$

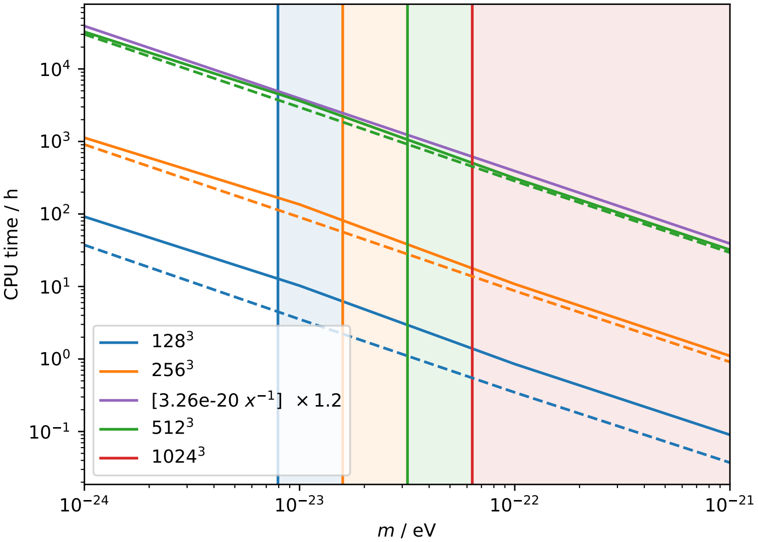
$$\vec{v} = \frac{\hbar}{m} \nabla \alpha$$

Grid discretization implies a maximum velocity which can be represented

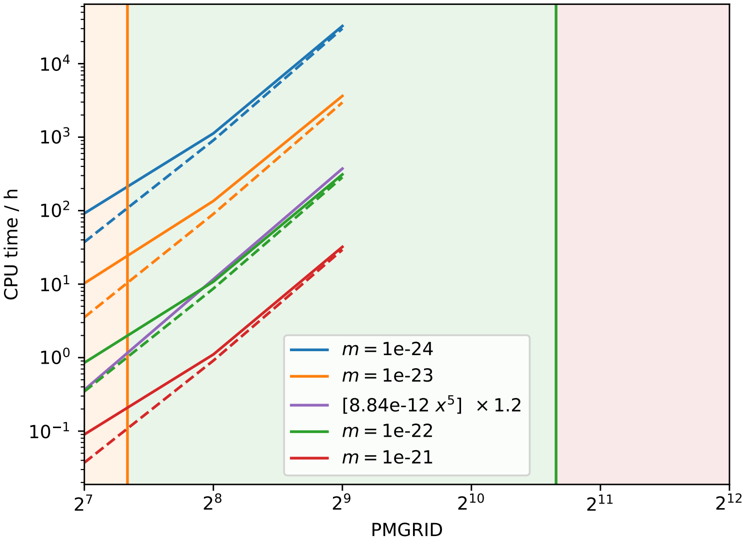
Mocz, Lancaster, et al. (2018), Mocz, Fialkov, et al. (2019)

$$v < \frac{\hbar}{m} \frac{\pi}{\Delta x}$$

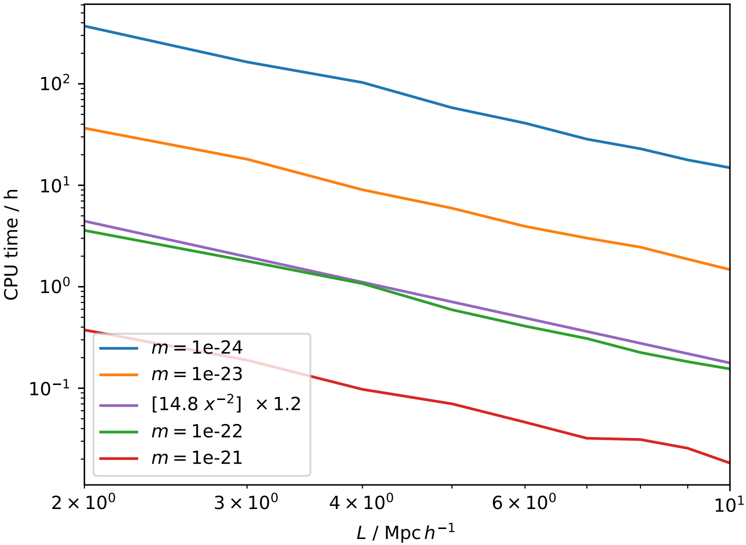
CPU time vs. mass ($L = 1\,\mathrm{Mpc}$)



CPU time vs. resolution ($L = 1\,\mathsf{Mpc}$)

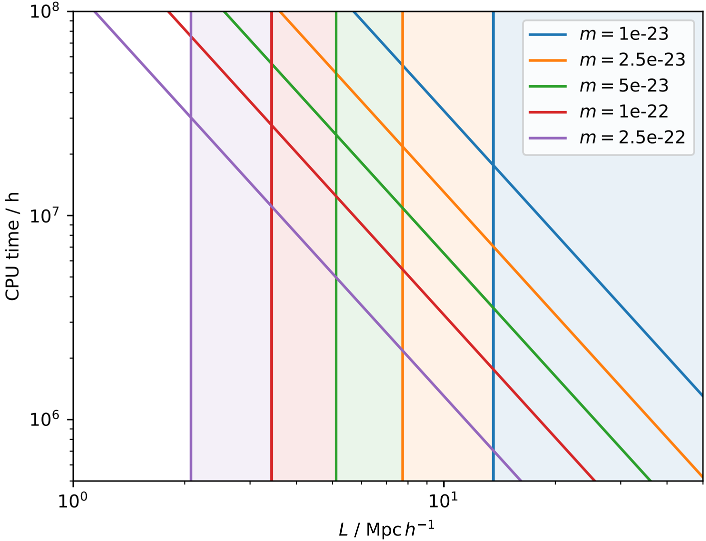


CPU time vs. resolution ($N = 256^3$)



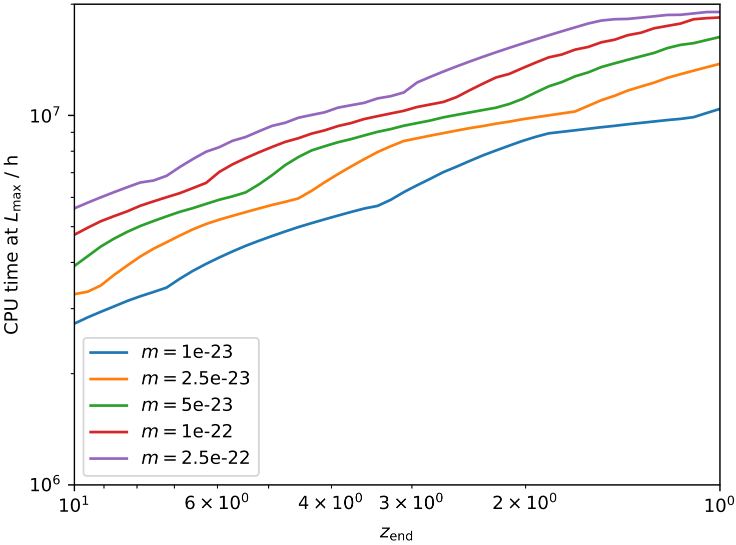
Velocity criterion

$$v < \frac{\hbar}{m} \frac{\pi}{\Delta x}$$



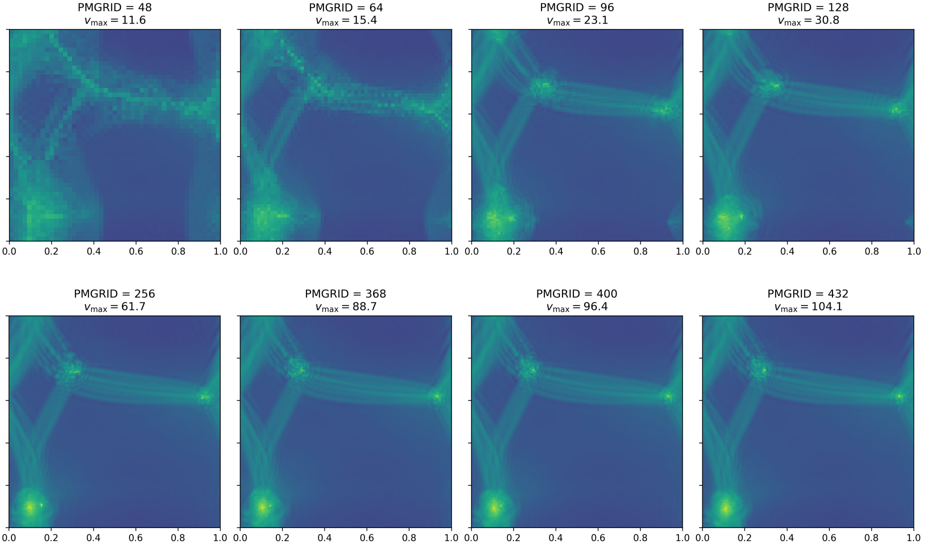
10^7 CPU h $\triangleq$ two full months of entire MPA cluster (FREYA)!

Velocity criterion



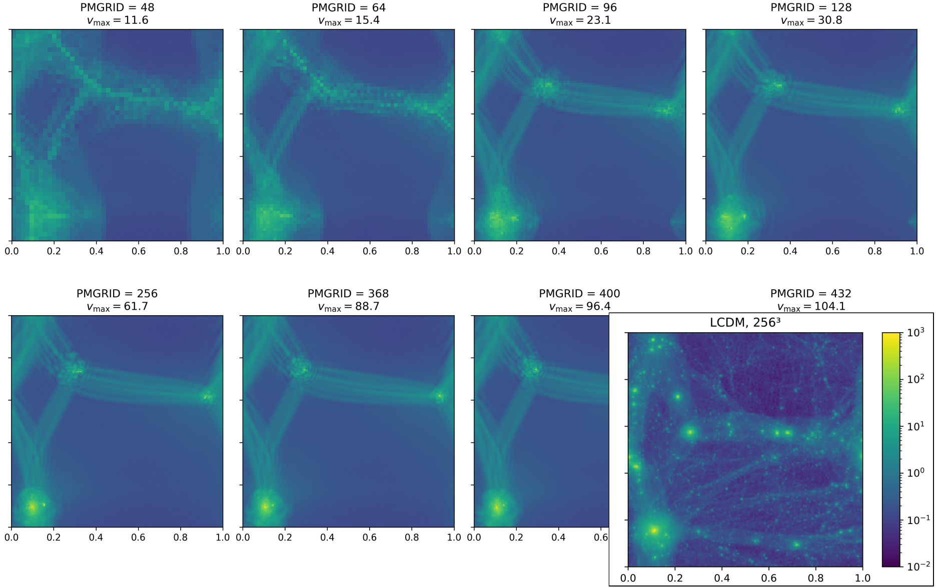
Cosmological 1 Mpc boxes

$z = 0, m = 2.5 \times 10^{-23}\,\mathrm{eV}, v_{\Lambda\mathrm{CDM}} = 97.2\,\mathrm{km/s}$



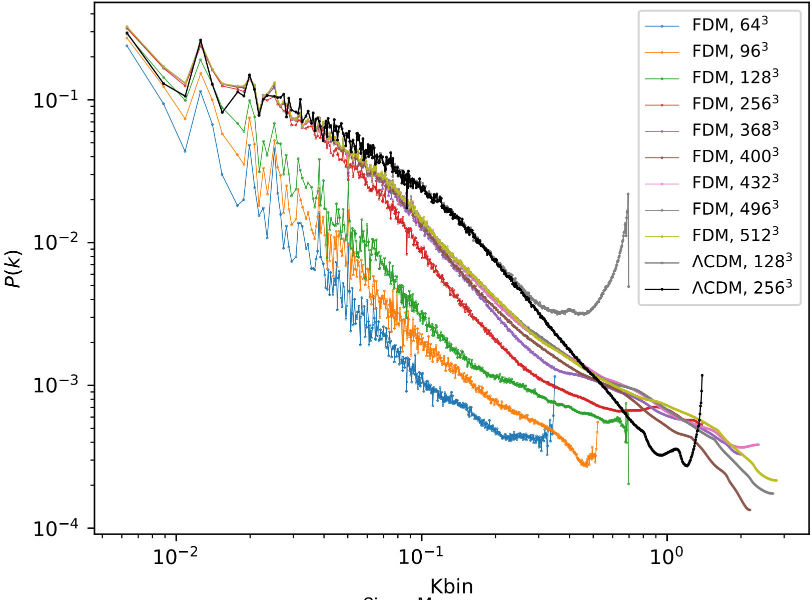
Cosmological 1 Mpc boxes

$z = 0, m = 2.5 \times 10^{-23}\,\text{eV}, v_{\Lambda\text{CDM}} = 97.2\,\text{km/s}$



Matter power spectra

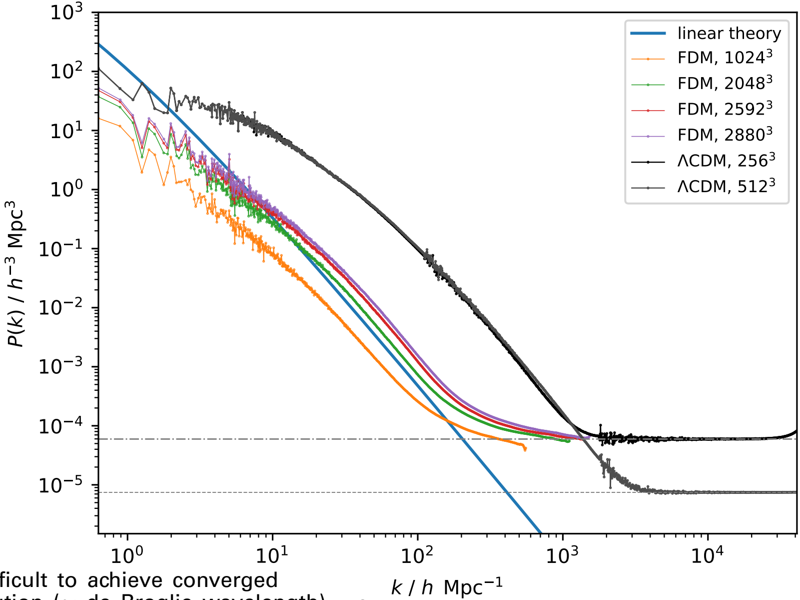
1 Mpc boxes



Matter power spectra

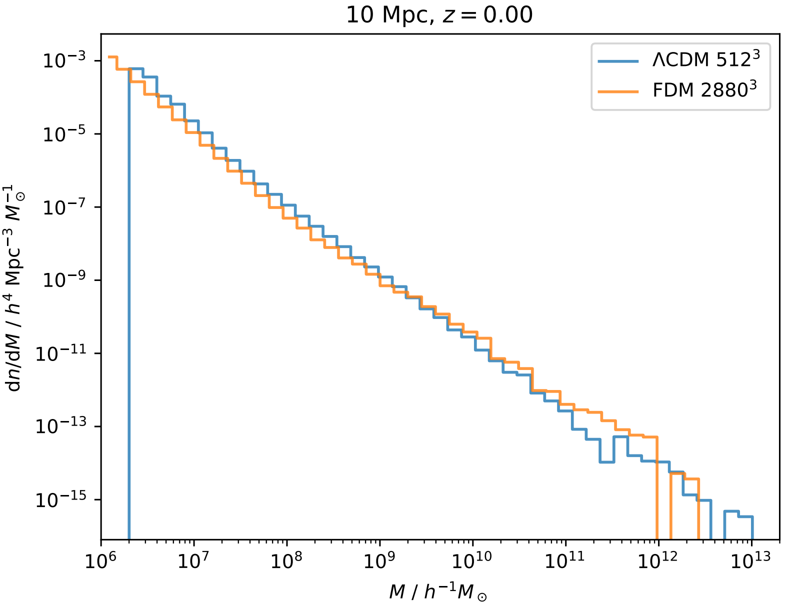
10 Mpc boxes

2880^3 grid already takes $>$ 530 GB RAM!

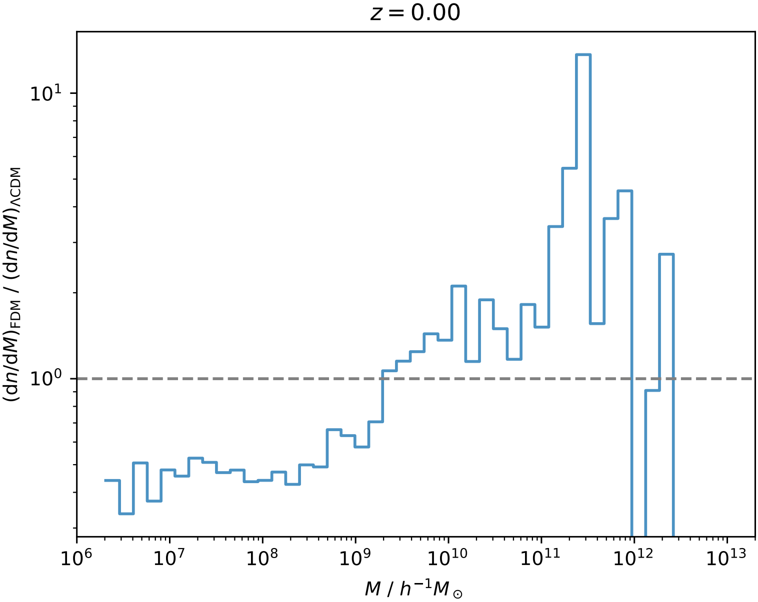


→ difficult to achieve converged resolution (≈ de Broglie wavelength)

Halo finding: halo mass function



Halo finding: halo mass function



First combined simulation of FDM with baryons & stars

Mocz, Fialkov, et al. (2019)

- Uniform pseudo-spectral method
- $L = 1.7h^{-1}$Mpc, $z = 127 \ldots 5.5$
- 1024^3 FDM cells $\Rightarrow \Delta x = 1.66h^{-1}$kpc
- 512^3 baryon cells
- Runtime: $\approx 3 \times 10^6$ CPU h, $20\times$ CDM run with $512^3 + 512^3$ particles/cells
- Baryonic models still calibrated for CDM!

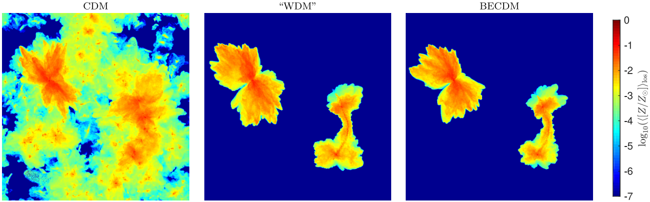
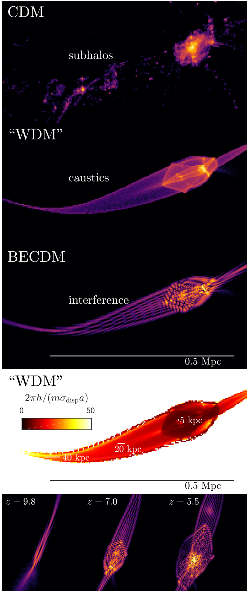


Figure 10. Mass-averaged line-of-sight metallicities in the cosmic volume at $z = 6$ (see colorbar on the right). BECDM/"WDM" shows a significantly more pristine intergalactic medium due to the lack of subhaloes.



Outlook

Conclusions from my own work so far:

- Cosmological FDM simulations seem to give sensible results, in agreement with CDM for limiting cases
- Working with “big” data (simulations) requires time/effort
- Central problem of static pseudo-spectral method: limited resolution/box size!
  - Both memory and run-time cost

Still to do:

- Pushing the pseudo-spectral method to its limits with the largest cosmological FDM simulations so far (current largest: $\approx 2.5\,\mathrm{Mpc}$)
- Surpassing the limitations of uniform grids using hybrid or adaptive schemes?
- Combination with baryons

Short summary of the main problems for simulations

1. **Time integration** $\Delta t \sim \Delta x^2$
2. **Rapid oscillations** even in low-density regions since velocity corresponds to the gradient of the phase ($\rightarrow$ velocity criterion)
3. **Large dynamic range**: “large”-scale structure simulations still require resolving de Broglie wavelength
4. “New” field without decades of experience or refined codes/methods as for CDM

References

Peccei, R. D. and Helen R. Quinn (June 1977). “CP conservation in the presence of pseudoparticles”. In: *Phys. Rev. Lett.* 38, pp. 1440–1443. DOI: 10.1103/PhysRevLett.38.1440.

Li, Xinyu, Lam Hui, and Greg L. Bryan (Oct. 2018). “Numerical and Perturbative Computations of the Fuzzy Dark Matter Model”. In: arXiv: 1810.01915 [astro-ph.CO].

Schive, Hsi-Yu, Tzihong Chiueh, and Tom Broadhurst (July 2014). “Cosmic structure as the quantum interference of a coherent dark wave”. In: *Nature Physics* 10, pp. 496–499. DOI: 10.1038/nphys2996. arXiv: 1406.6586 [astro-ph.GA].

Mocz, Philip, Lachlan Lancaster, et al. (Apr. 2018). “Schrödinger-Poisson-Vlasov-Poisson correspondence”. In: *Phys. Rev. D* 97.8, 083519, p. 083519. DOI: 10.1103/PhysRevD.97.083519. arXiv: 1801.03507 [astro-ph.CO].

Mocz, Philip, Anastasia Fialkov, et al. (Nov. 2019). “Galaxy Formation with BECDM – II. Cosmic Filaments and First Galaxies”. In: *arXiv e-prints*, arXiv:1911.05746, arXiv:1911.05746. arXiv: 1911.05746 [astro-ph.CO].

Zhang, Jiajun, Hantao Liu, and Ming-Chung Chu (Dec. 2018). “Cosmological Simulation for Fuzzy Dark Matter Model”. In: *Frontiers in Astronomy and Space Sciences* 5, 48, p. 48. DOI: 10.3389/fspas.2018.00048. arXiv: 1809.09848 [astro-ph.CO].