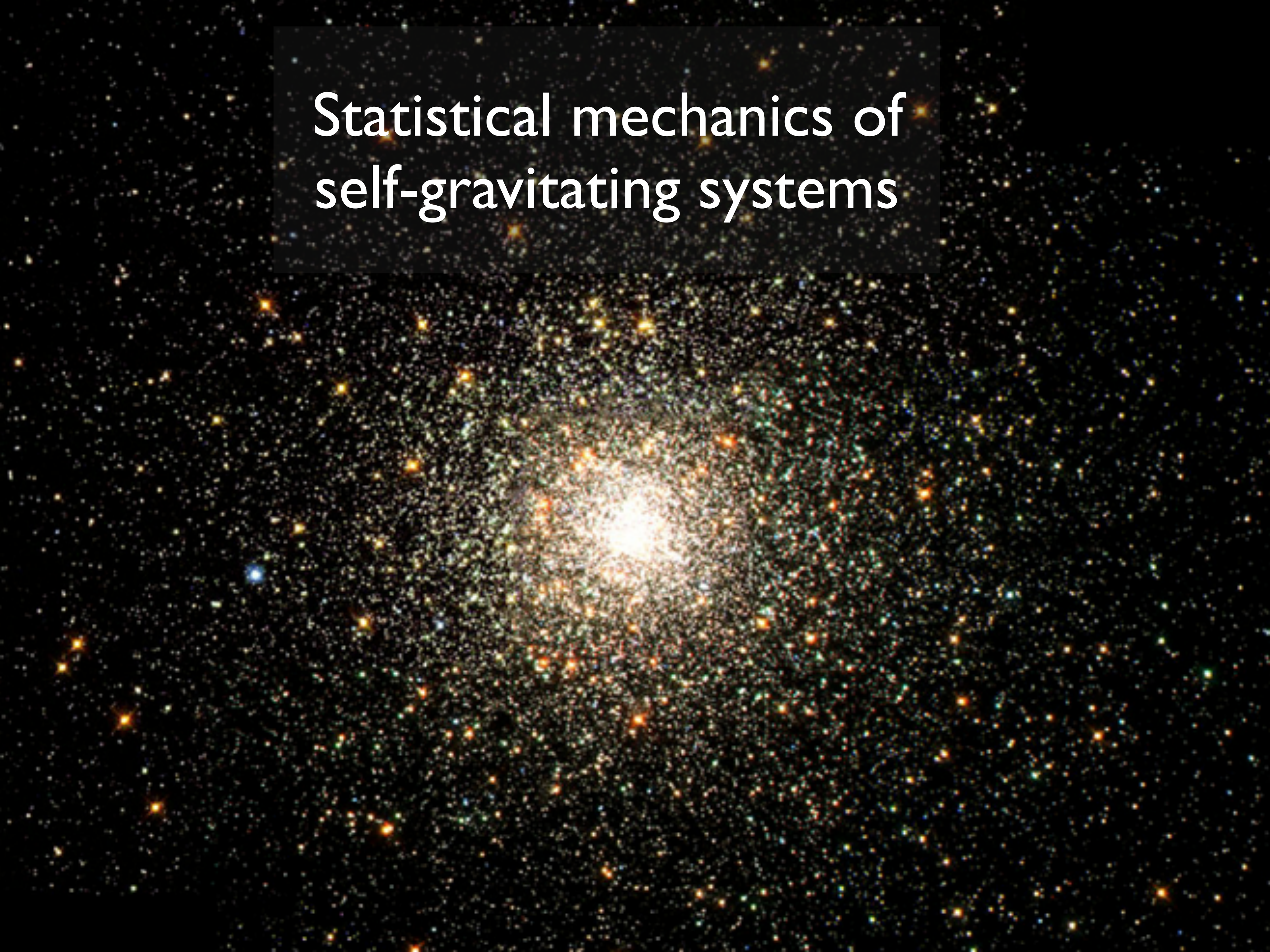
Statistical mechanics of self-gravitating systems



gas in a box	stellar system
molecules, m ~ 10^{-24} g	stars, m ~ 10^{33} g
Avogadro's number, N ~ 10^{23}	N ~10^5 (globular clusters), ~10^5-10^{11} (stars in galaxies)
short-range forces	long-range forces (gravity)
confined in a box	confined by self-gravity
mean free path << system size	mean free path >> system size
heat capacity > 0	heat capacity < 0

Virial theorem

K = kinetic energy
W = potential energy
E = K+W = total energy

In a steady-state system governed by gravity

$2K + W = E + K = 0$ or $E = -K$.

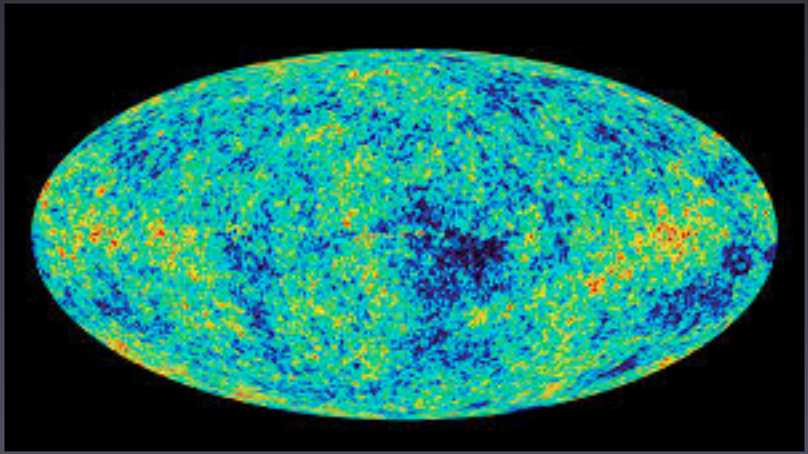
In an isothermal gas $K=3/2\ NkT$ so heat capacity is

$C = dE/dT = -3/2\ Nk$

which is *negative*

- heat capacity of isolated self-gravitating systems is negative…despite the common claim that "heat capacity is always positive" (e.g., Landau & Lifshitz, Statistical Physics)
- systems with long-range interactions
  - are not additive
  - are not extensive
  - cannot be treated using the canonical ensemble
  - can have negative heat capacity
  - may have no thermodynamic equilibrium state
- thermodynamic equilibrium state for self-gravitating systems exists only if
  - they are enclosed in a box, to prevent escape, and
  - the box is sufficiently small, to prevent core collapse
  - the inter-particle potential is softened

- there is no thermodynamic equilibrium state for self-gravitating systems unless they are enclosed in a sufficiently small box
- there is no “heat death” of the Universe



Statistical mechanics of self-gravitating systems

1. Radial profiles of galactic disks (Herpich, ST, & Rix 2017)
2. Multi-planet systems (ST 2015)
3. Nuclear star clusters (Kazandjian, Touma, & ST in prep)
4. Kinematics of the solar neighborhood (Jeans 1928)

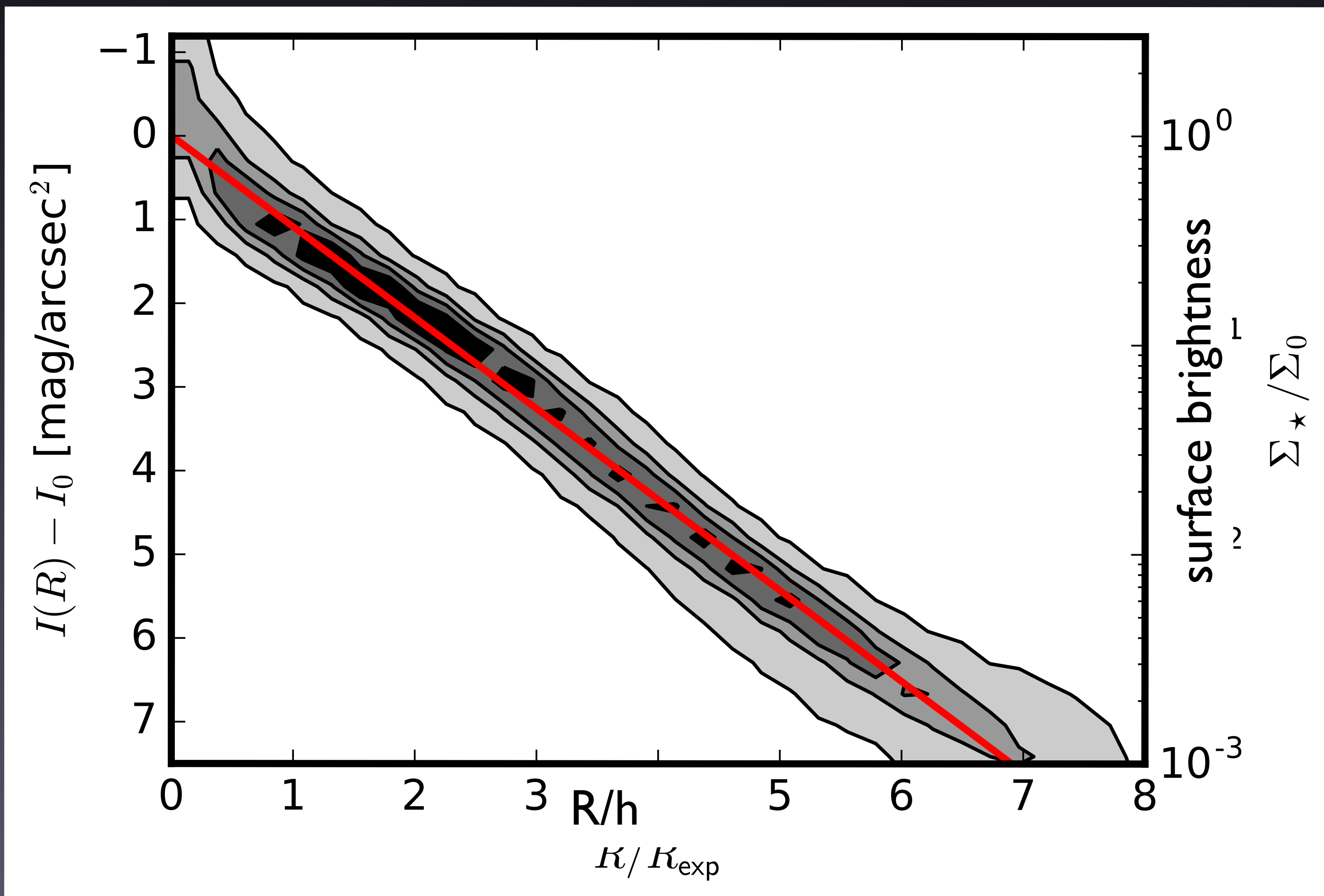
Statistical mechanics of self-gravitating systems

1. Radial profiles of galactic disks — less speculative
2. Multi-planet systems
3. Nuclear star clusters — more speculative
4. Kinematics of the solar neighborhood — wrong

- surface-brightness profiles of galaxy disks are approximately exponential (Freeman 1970)

$$I(R) \sim \exp(-R/h)$$

with h = 2-6 kpc



stacked, rescaled photometry of 300 disk galaxies from Courteau (1996)

- surface-brightness profiles of galaxy disks are approximately exponential (Freeman 1970)

$$I(R) \sim \exp(-R/h)$$

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Two classes of theory for formation of exponential disks:

- the disk profile is determined by the angular-momentum distribution of mass in the halo (Fall & Efstathiou 1980, Dalcanton + 1997, Dutton 2009, etc.)
- the disk profile is determined by internal dynamical processes in the disk after it forms (Lin & Pringle 1987, Yoshii & Sommer-Larsen 1989, Ferguson & Clarke 2001, Elmegreen & Struck 2013, 2016)

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Modern simulations of both isolated disk galaxies and galaxies forming in a cosmological context show that disk surface-density profiles evolve toward exponentials *after* the disks form (Governato + 2007, Guedes + 2011, Stinson + 2013, Berrier & Sellwood 2015, etc.)

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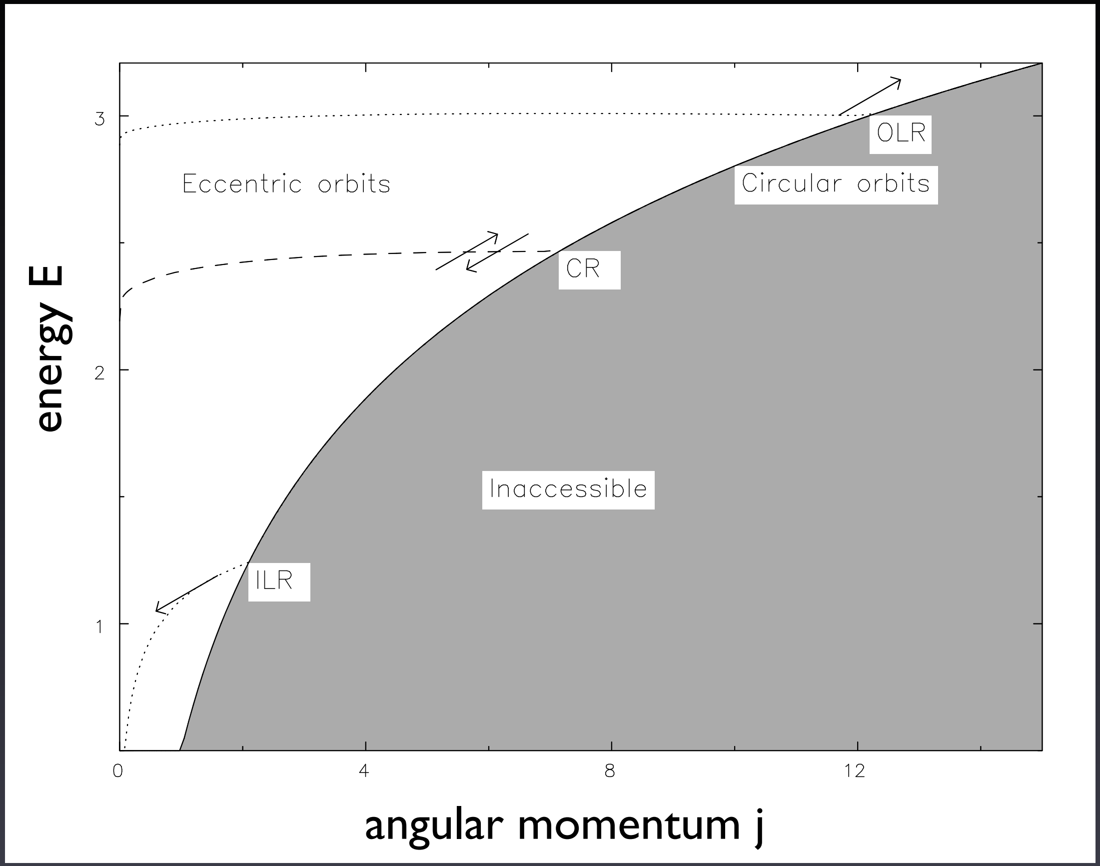
Rearranging the mass distribution in a disk



- stars can change angular momentum and radius through interactions with a bar, spiral structure, sub-halos, giant molecular clouds, etc.

Sellwood (2010)

- interaction with a non-axisymmetric potential rotating with pattern speed Ω_p changes energy and angular momentum in the relative amounts

$$dE/dj = \Omega_p$$

- interaction is strong only at three distinct radii: the inner Lindblad resonance (ILR), the outer Lindblad resonance (OLR), and the corotation resonance (CR)

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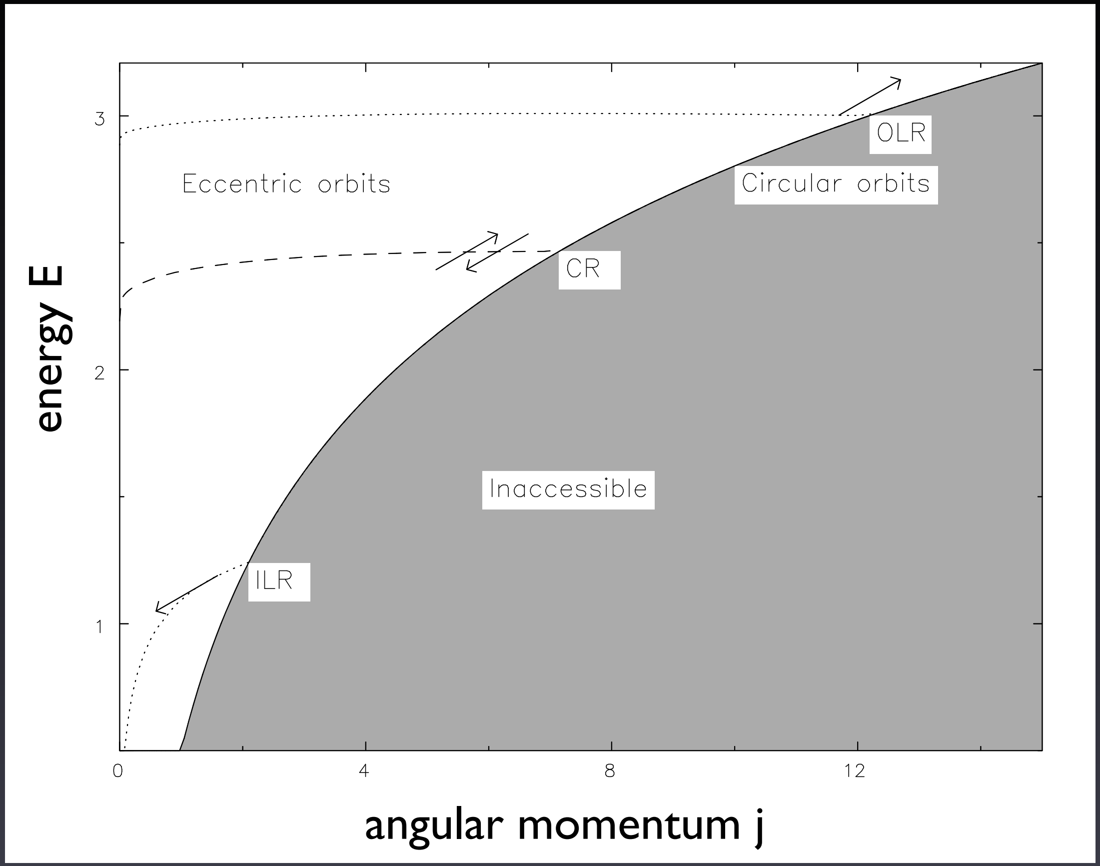

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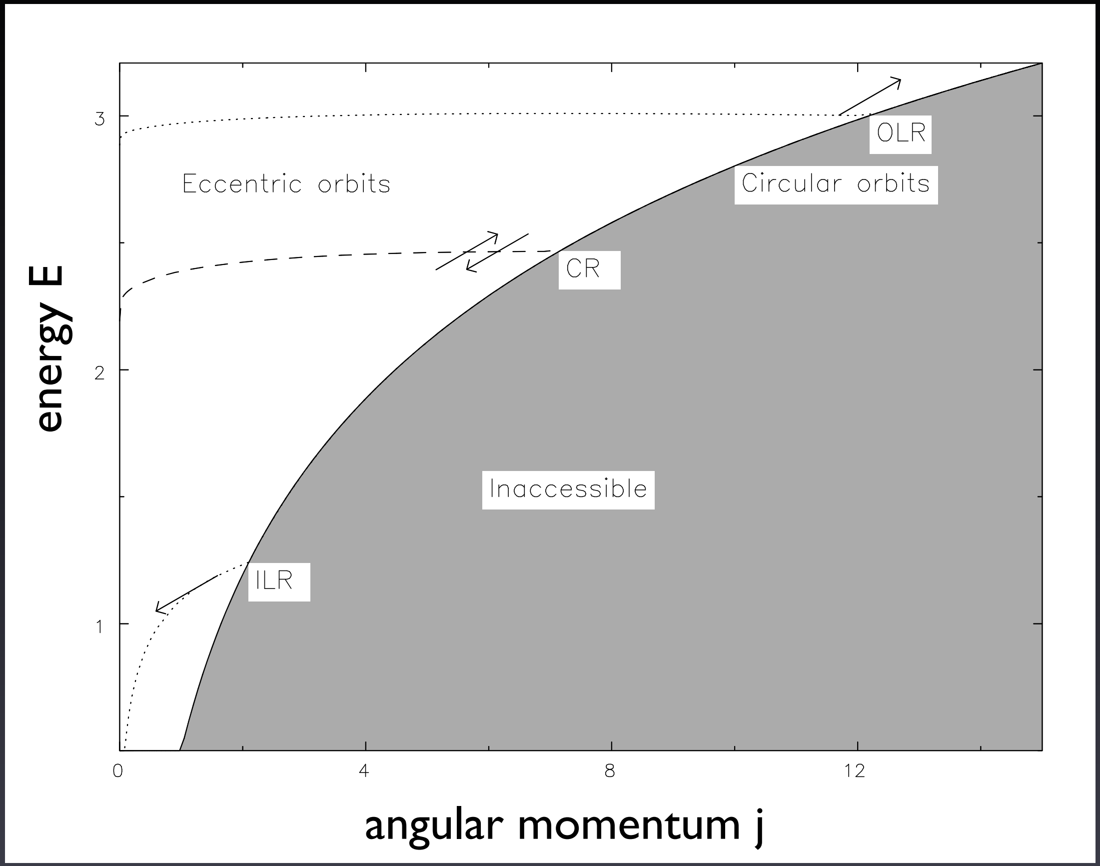
- interactions at the Lindblad resonances excite eccentricities, while interactions at corotation do not



Rearranging the mass distribution in a disk

- stars can change angular momentum and radius through interactions with a bar, spiral structure, sub-halos, giant molecular clouds, etc.
- transient spiral patterns can (i) change angular momenta of stars over a wide range of radii, while (ii) not exciting the stellar eccentricities

⇒ stellar migration (Sellwood & Binney 2002)

- there is strong circumstantial evidence for stellar migration in the Milky Way:
  - spread in metallicity of stars in solar neighborhood increases with age
  - solar metallicity is similar to present metallicity of interstellar gas
  - very low metallicities of some nearby giant molecular clouds

Rearranging the mass distribution in a disk

What is the final state of the disk after migration?

- since the orbits remain circular the phase space has only one degree of freedom, described by canonical coordinates which are the orbital phase φ and the angular momentum j.
- the only conserved quantity is the total angular momentum J.
- the equilibrium distribution function is such that the distribution of stars on the surface of constant total angular momentum in 2N-dimensional phase space is uniform (like the microcanonical ensemble, but for angular momentum, not energy)
- this occurs when

$$f(\varphi, j) \sim \exp(-\beta j)$$

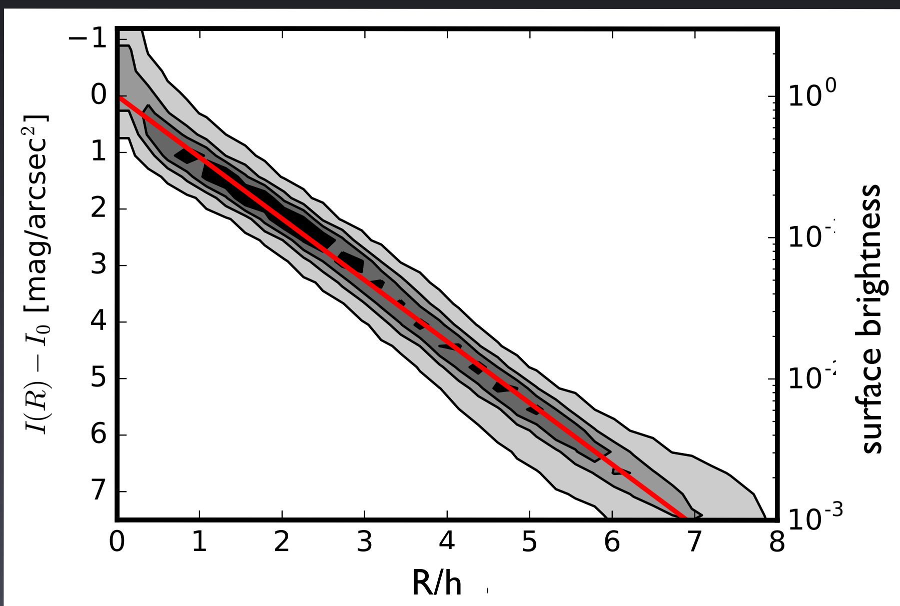
i.e., mass of stars per unit j is an exponential function of j

Freeman disk

surface brightness is an exponential function of radius

$$I(R) \propto \exp(-R/h)$$

Comparison to data needs photometry

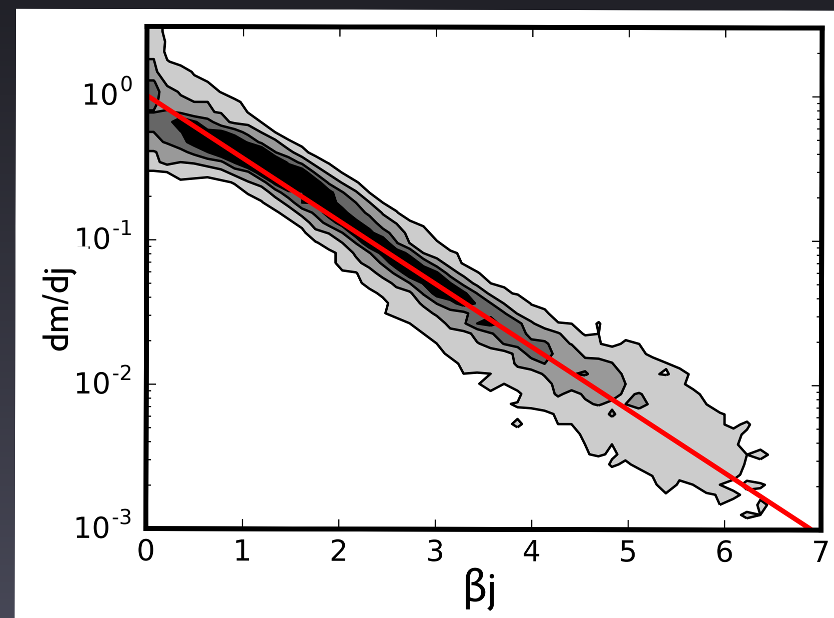


Maximum-entropy disk

mass per unit j is an exponential function of j

$$dm/dj \propto \exp(-\beta j)$$

Comparison to data needs photometry and rotation curve and assumption M/L = constant

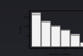


photometry and rotation curves from Courteau (1996)

Statistical mechanics of planetary systems

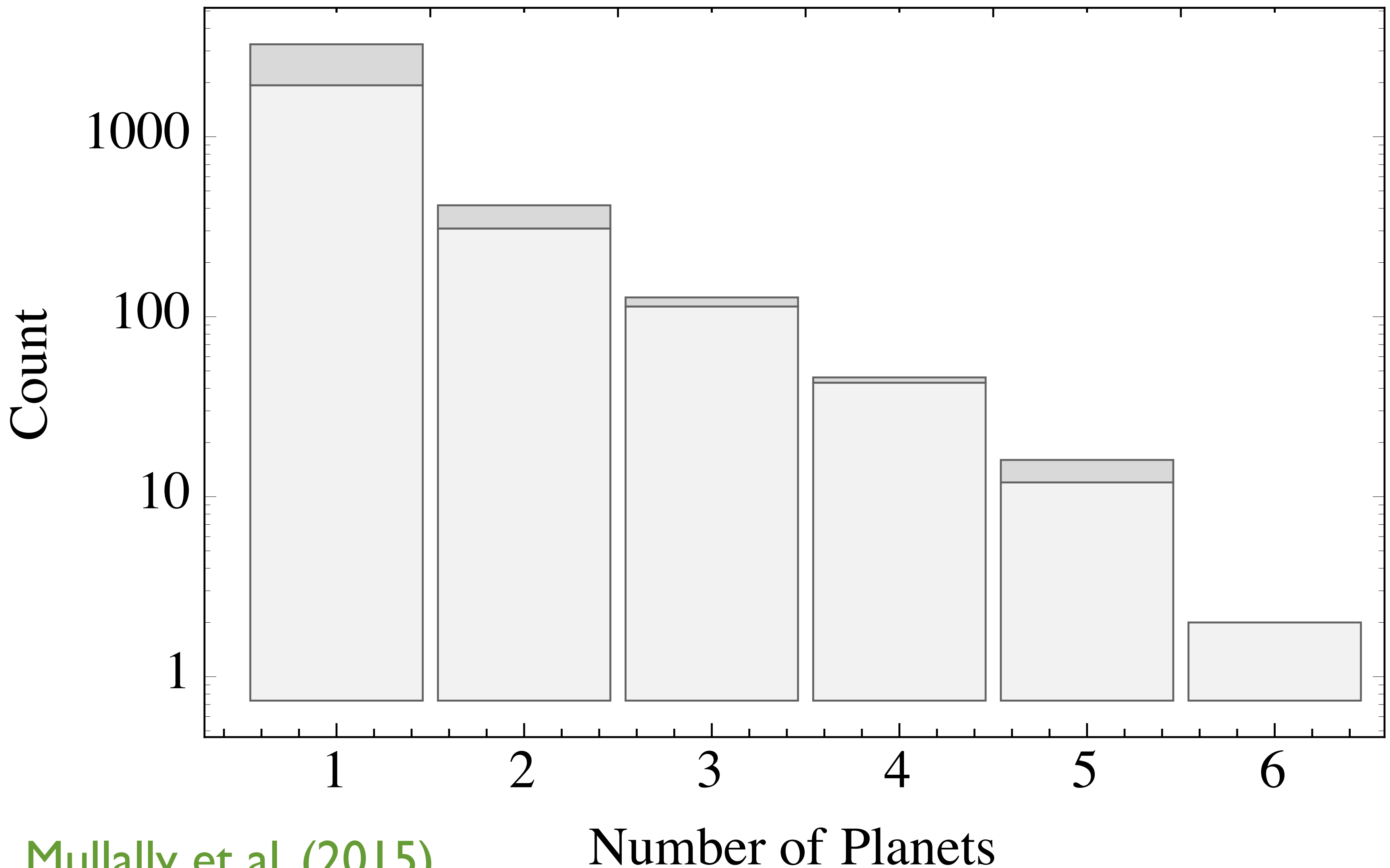
There are many bad examples of attempts to explain the spacing and other properties of planetary orbits from first principles

Nevertheless there are reasons to try again:

- *Kepler* has provided a large statistical sample of multi-planet systems 
- N-body integrations can routinely follow the evolution of systems for 100 Myr
- there are hints of interesting behavior from studies of the solar system:
  - the orbits of the planets in the solar system are chaotic, with Liapunov (e-folding) times of $\sim 10^7$ yr (Sussman & Wisdom 1988, 1992, Laskar 1989, Hayes 2008)
  - there is a 1% chance that Mercury will be lost from the solar system before the end of the Sun's life in ~ 7 Gyr

These suggest that some properties of planetary systems might be determined by the statistical mechanics of orbital chaos

Multi-planet systems



Mullally et al. (2015)

Statistical mechanics of planetary systems

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The last stages of terrestrial planet formation

- the accretion of planetesimals leads to a few dozen "planetary embryos" of similar size
- eccentricities of the embryos remain small because they are damped by dynamical friction from residual population of small planetesimals
- eventually the small planetesimals disappear
- surviving embryos gradually excite one another's eccentricities until their orbits cross and they collide
- through collisions, the number of surviving bodies slowly declines until we are left with a small number of planets on well-separated, stable orbits (the giant-impact phase)
- maybe the giant-impact phase tends to produce an ensemble of planetary systems with statistically similar properties

Statistical mechanics of planetary systems

The range of strong interactions from a planet of mass m orbiting a star of mass M in a circular orbit of radius a is the Hill radius

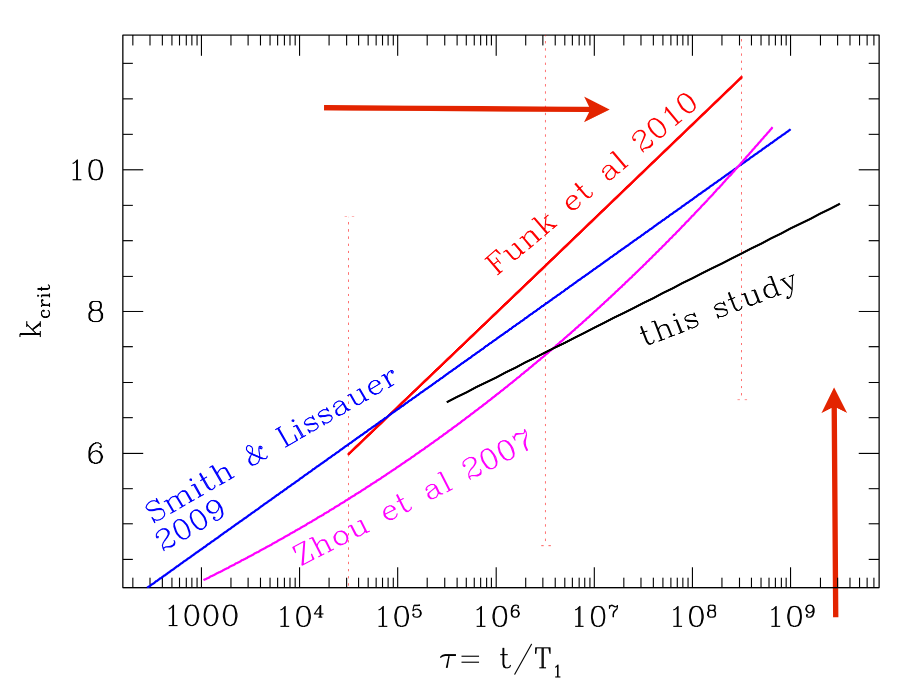
$$r_H = a\left(\frac{m}{3M}\right)^{1/3}.$$

Numerical integrations show that planets of mass m, m' with semi-major axes a, a', a < a' are stable for N orbital periods if closest approach exceeds k Hill radii, or

$$a'(1-e') - a(1+e) > k(N) r_H$$

typically $k(10^{10}) \simeq 11 \pm 1$

Pu & Wu (2014)



A model system

1. Use the sheared sheet approximation, which replaces usual Keplerian disk by a rectangular box with shear (not essential, but eliminates spatial gradients and other complications)

2. Since the number of planets per system is small, work with the grand canonical ensemble, i.e., assume each planetary system is a subsystem with variable number of planets

3. <u>Ansatz</u>: planetary systems fill uniformly the region of phase space allowed by stability (~ ergodic approximation)

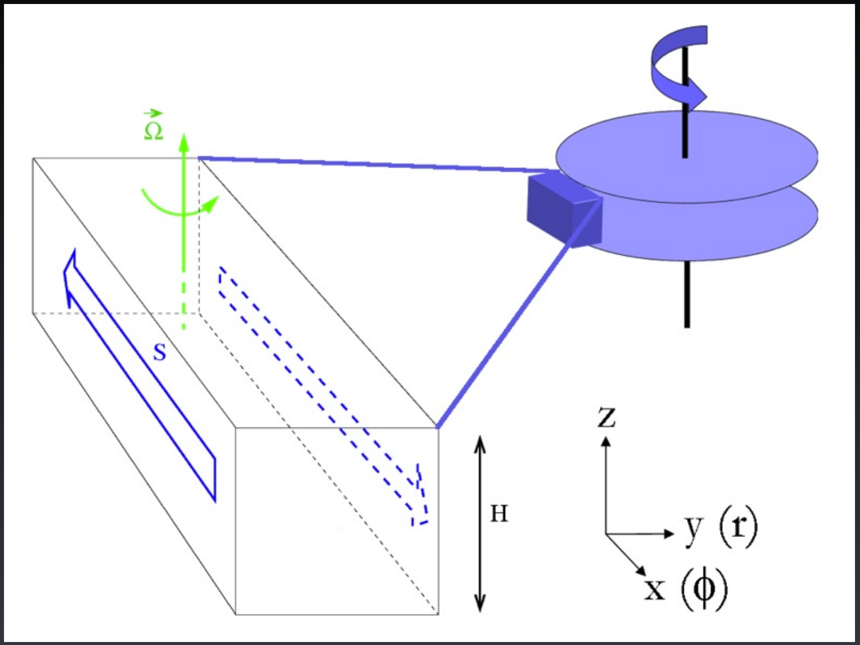


image credit: P.-Y. Longaretti

Statistical mechanics of planetary systems

1. use the sheared sheet approximation
2. work with the grand canonical ensemble
3. assume planetary systems fill the region of phase space allowed by stability

Leads to an N-planet distribution function

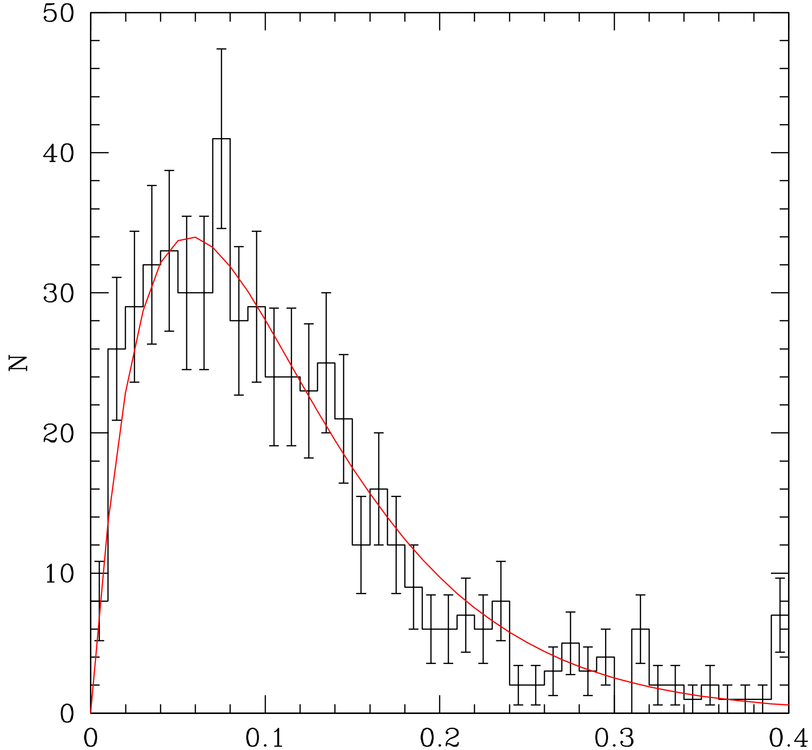
$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} \overbrace{da_i de_i^2}\, H(a_{i+1} - a_i - \overline{a}(e_{i+1} + e_i) - k r_H)$$

where $H(\cdot)$ is the step function, $k \simeq 9$, and $r_H = \overline{a}(m_i + m_{i+1})^{1/3}/(3M)^{1/3}$.

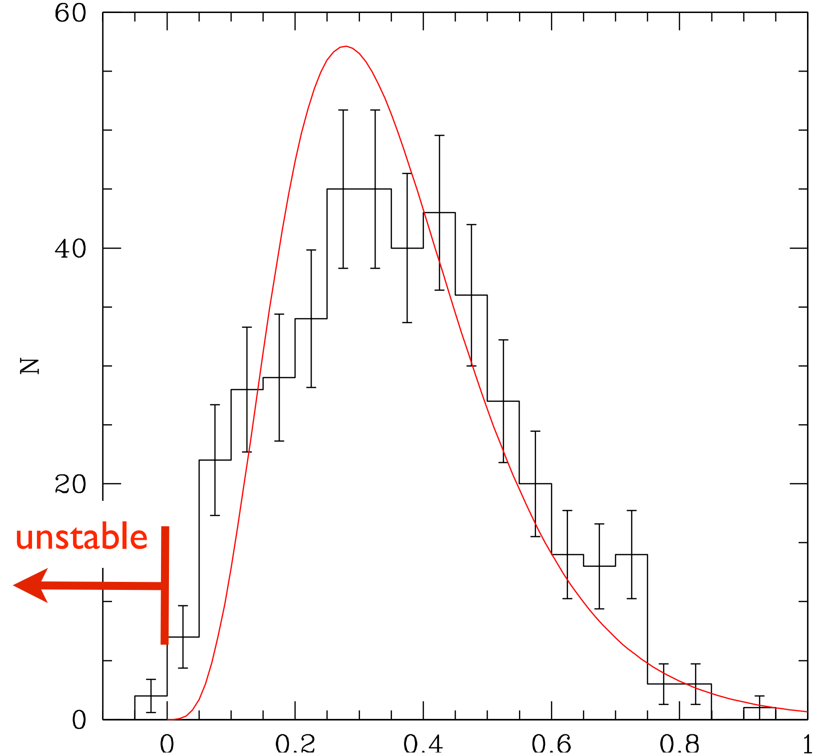
For comparison the distribution function for a one-dimensional gas of hard rods of length L (Tonks 1936) is

$$p(a_1, \ldots, a_N) \propto \prod_{i=1}^{N} da_i H(a_{i+1} - a_i - L).$$

compare to N-body simulations of planet growth by Hansen & Murray (2013)



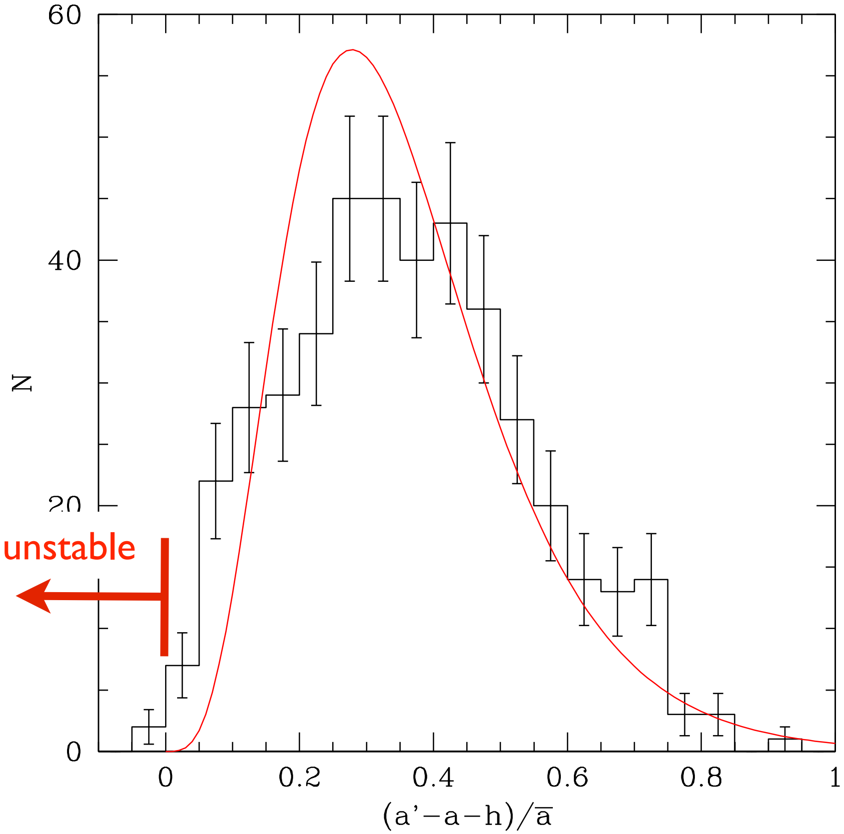
distribution of eccentricities



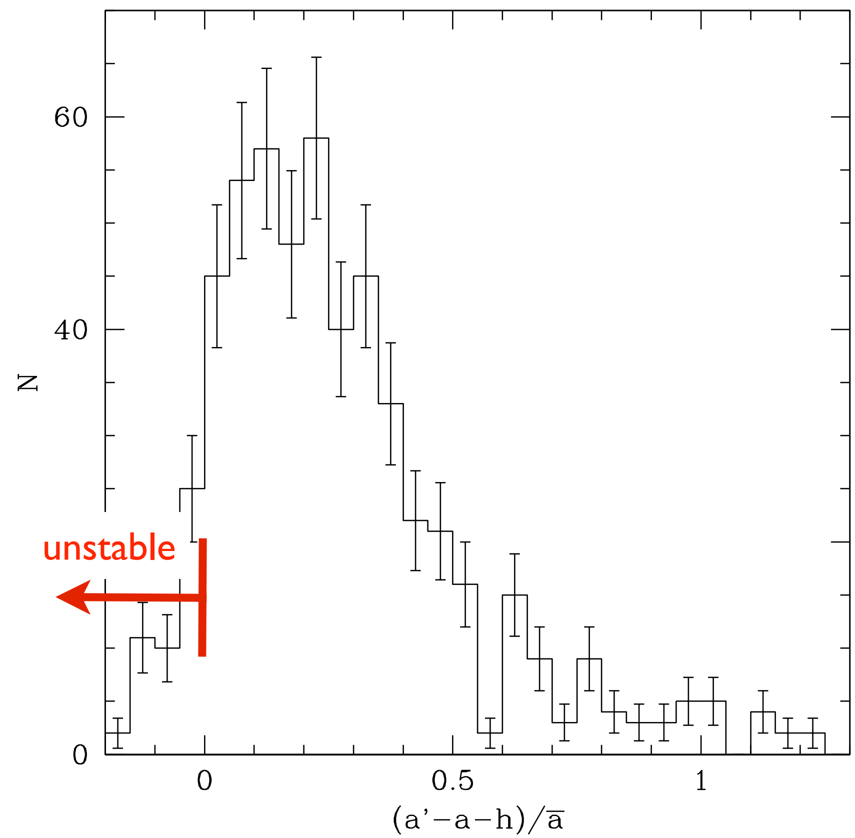
distribution of separations of nearest neighbors

one free parameter for two fits

Hansen & Murray (2013) simulations



Kepler planets, using Weiss & Marcy (2014) mass-radius relation:



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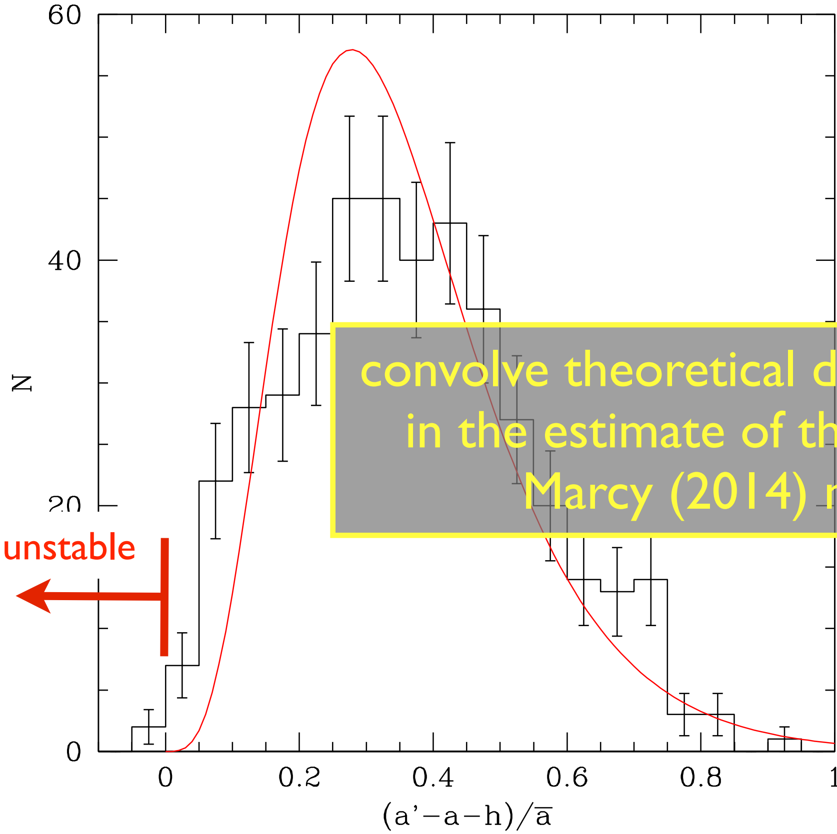

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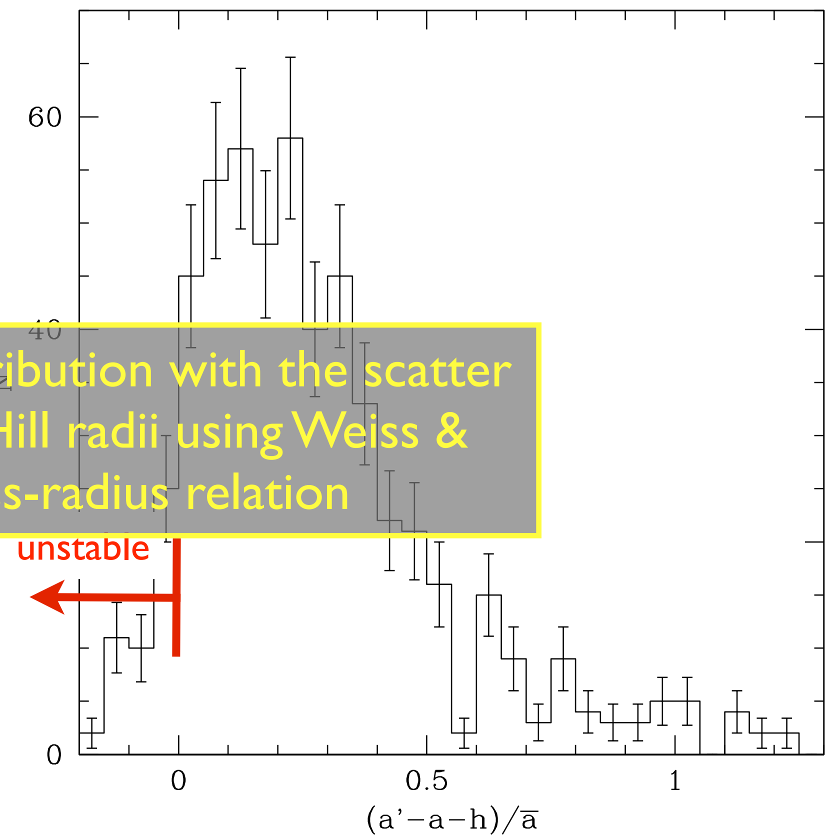
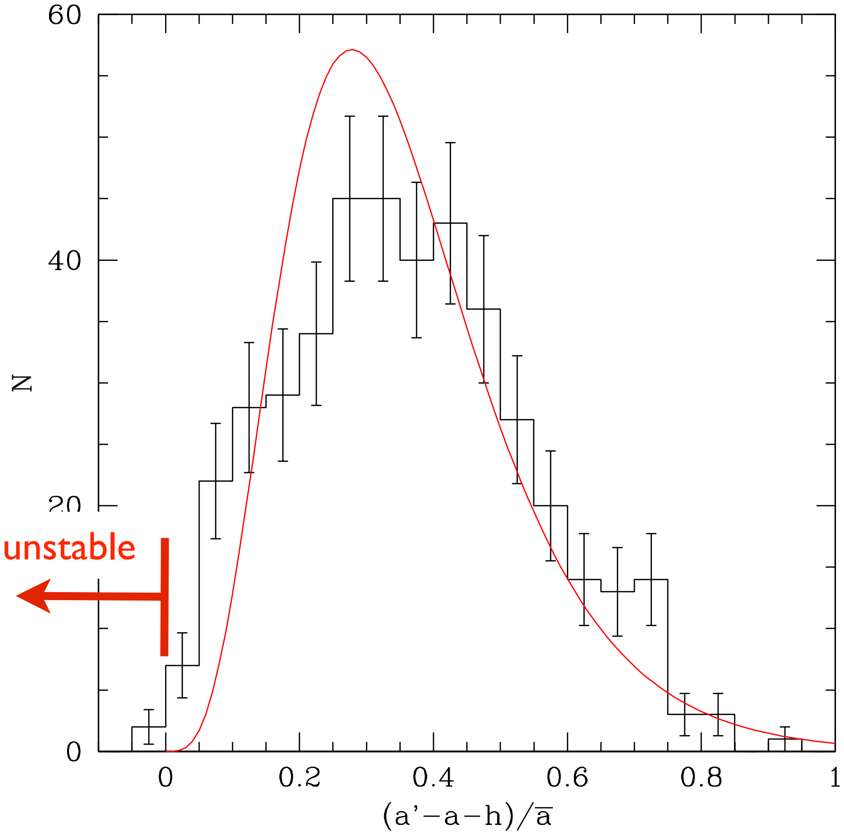
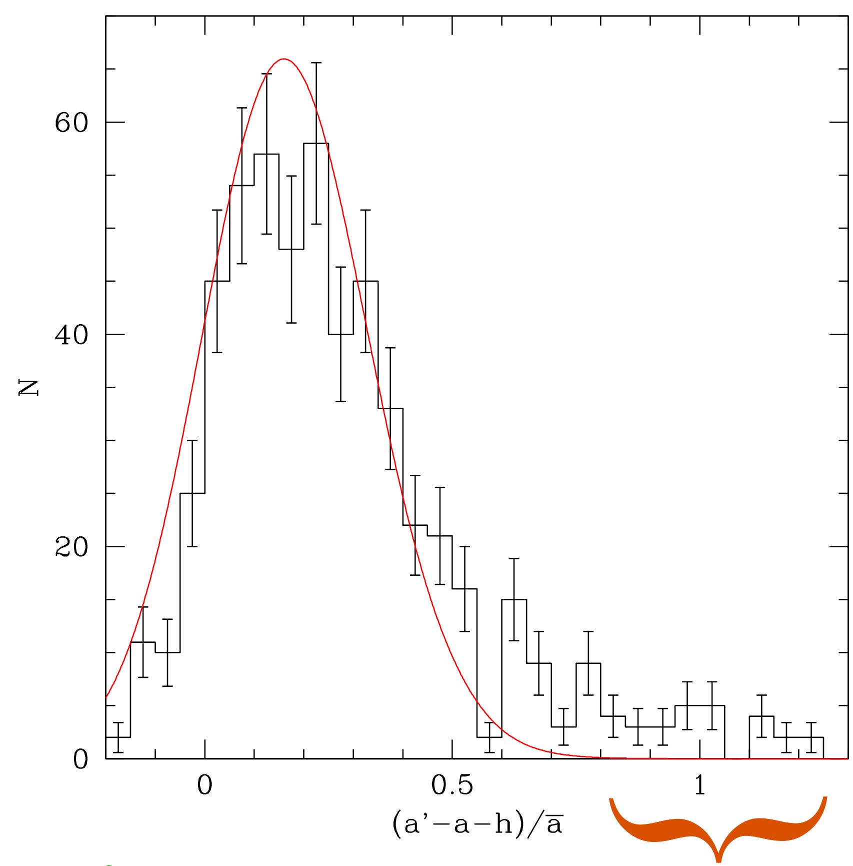


convolve theoretical distribution with the scatter in the estimate of the Hill radii using Weiss & Marcy (2014) mass-radius relation

Hansen & Murray (2013) simulations



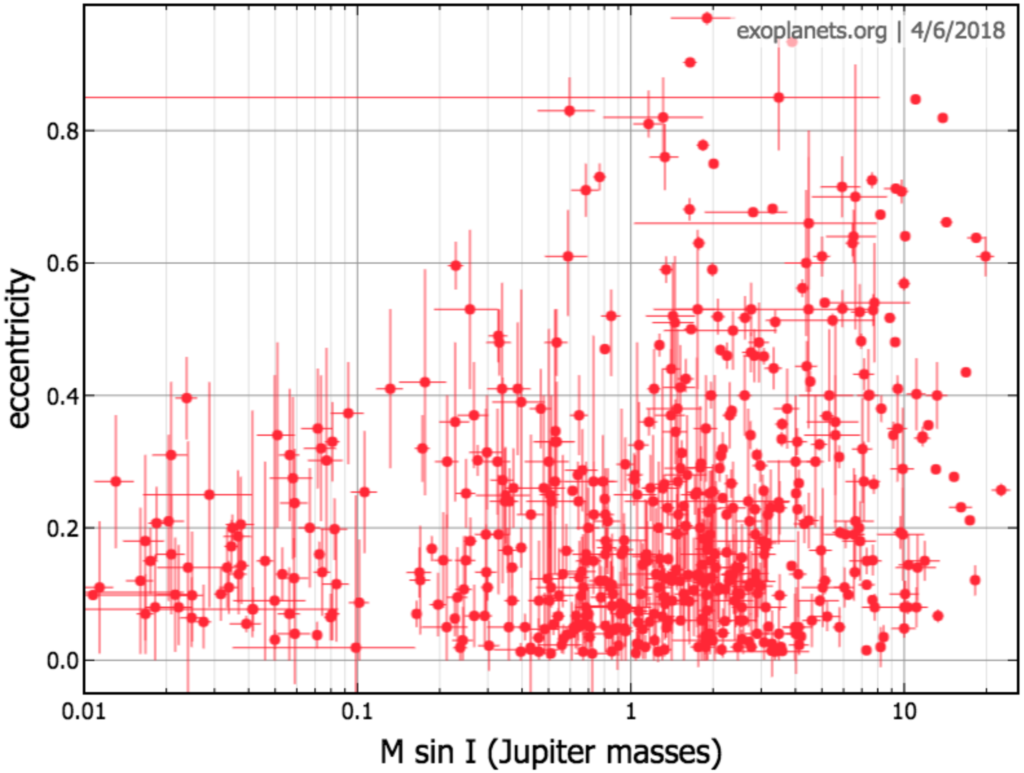
Kepler planets, using Weiss & Marcy (2014) mass-radius relation corrected for scatter



each fit has one free parameter

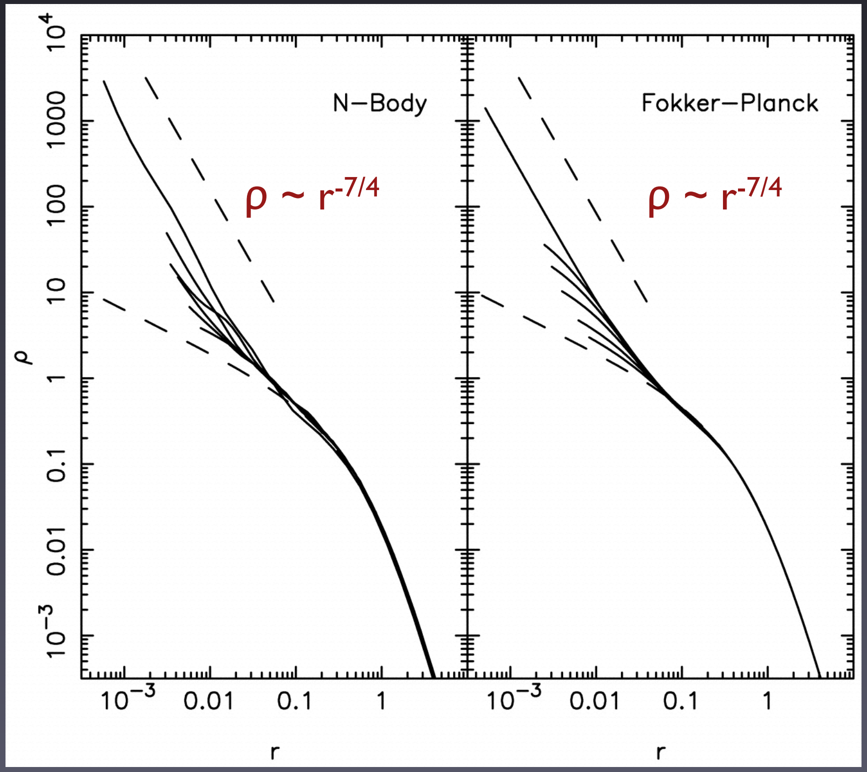
missing planets?

- statistical model predicts <e>=0.06
  - <e>≃ 0.02 - 0.03 (Hadden & Lithwick 2014)
  - <e>≃ 0.03 (Fabrycky + 2014)
  - <e>≃ 0.05 - 0.08 (van Eylen & Albrecht 2015)
  - <e> < 0.07 (Xie + 2016)
  - <e>≃ 0.07 (Shabram + 2015)
  - <e>≃ a few percent or less (Hadden & Lithwick 2017)
- statistical model predicts no correlation between mass and eccentricity in a given system



- black holes in the centers of galaxies are surrounded by dense nuclear star clusters in which the relaxation time can be as short as ~ 1 Gyr on scales ~ 1 pc
- what is the nature of thermodynamic equilibrium in this region?
- in potential Φ ~ - GM/r the equilibrium density is

$$n(r) \propto \exp\left(-\frac{m\Phi}{kT}\right) = \exp\left(-\frac{GMm}{kTr}\right)$$

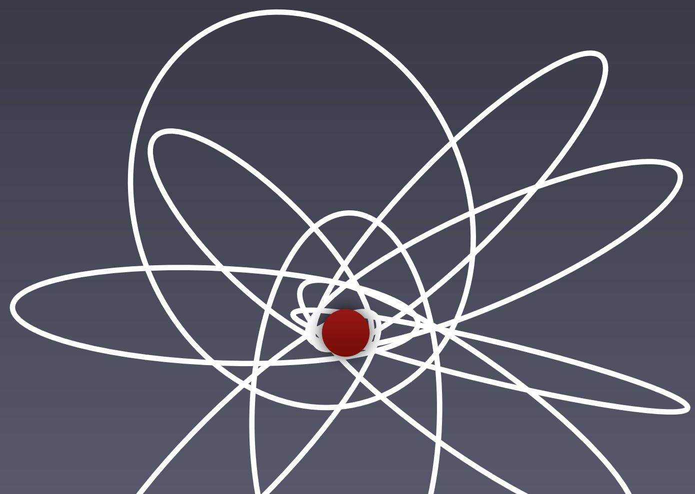


- this doesn't apply because stars are eaten by the black hole
- must instead find constant-flux solution with absorbing boundary conditions at r=0. For a single stellar mass (Peebles 1972, Bahcall & Wolf 1974)

$$n(r) \propto r^{-7/4}$$

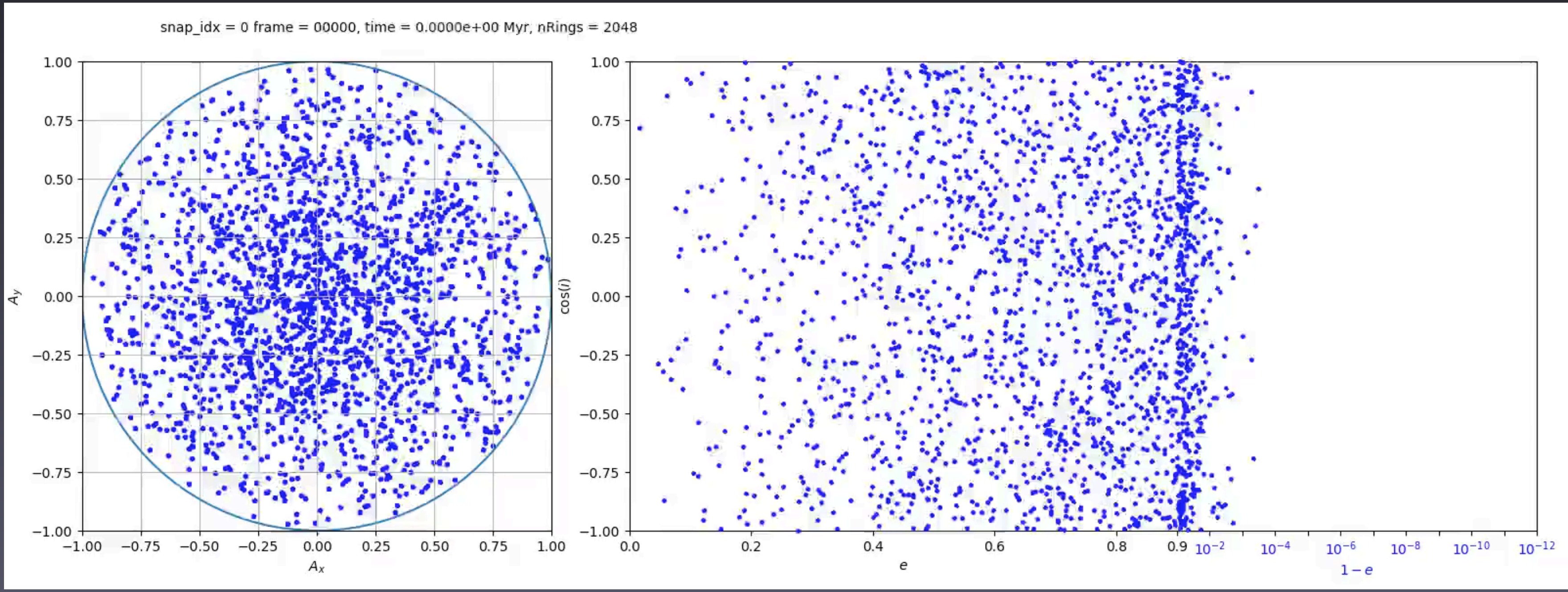
Resonant relaxation

- in the central ~ 1 pc gravitational potential is dominated by the central black hole
- on timescales much longer than the orbital period, time-averaged density of each orbit looks like an eccentric wire
- each wire exerts a torque on the others which leads to relaxation of angular momentum (but not energy)
- phase-space distribution becomes uniform on each energy surface but there is no mixing between different energies ("resonant relaxation")
- resonant relaxation is faster than relaxation due to two-body encounters by $O(M_{bh}/M_{stars})$



Resonant relaxation

- nuclear star clusters in which the gravitational potential is dominated by the central black hole are dynamically stable and thermodynamically stable under resonant relaxation (Tremaine 2005)
- resonant relaxation can be simulated using a wire-wire code (Touma + 2009)
- simulations show a robust lopsided instability that develops on the relaxation time (Touma + 2018); verified by MCMC simulations
- could affect star cluster shape, tidal disruption event rate, black hole feeding, LISA rates, etc.



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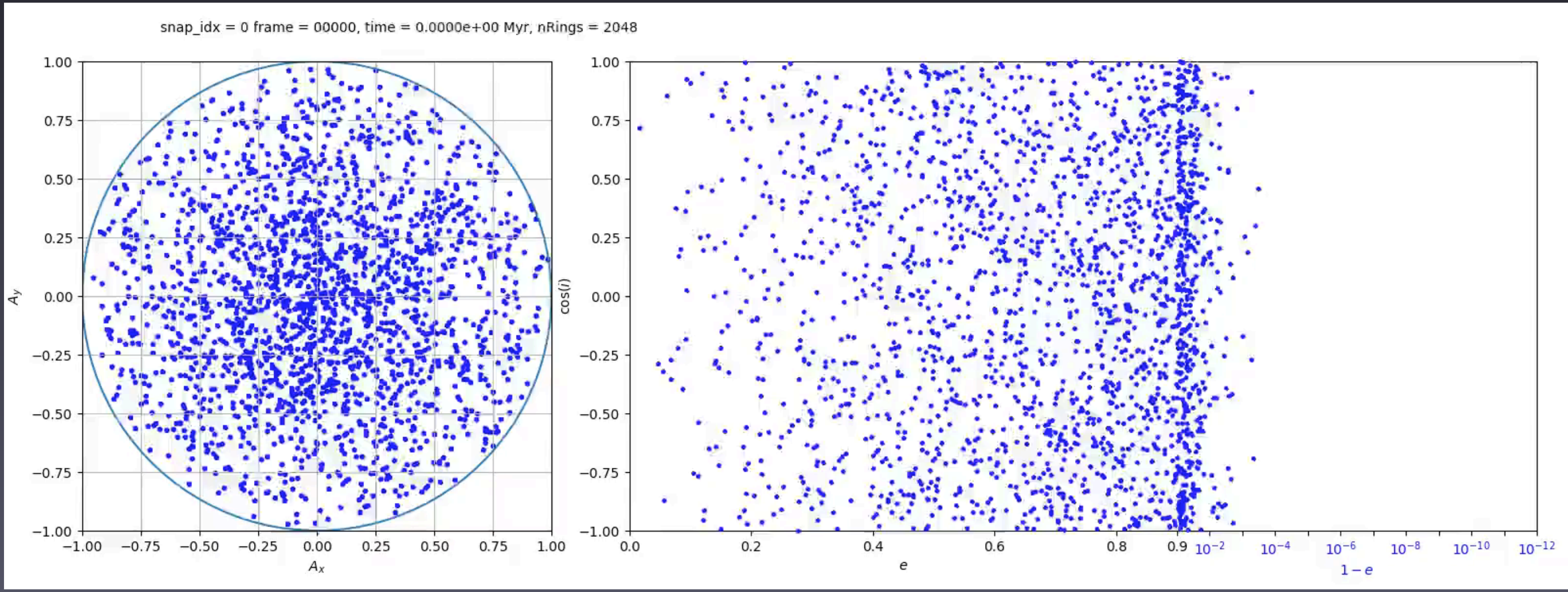
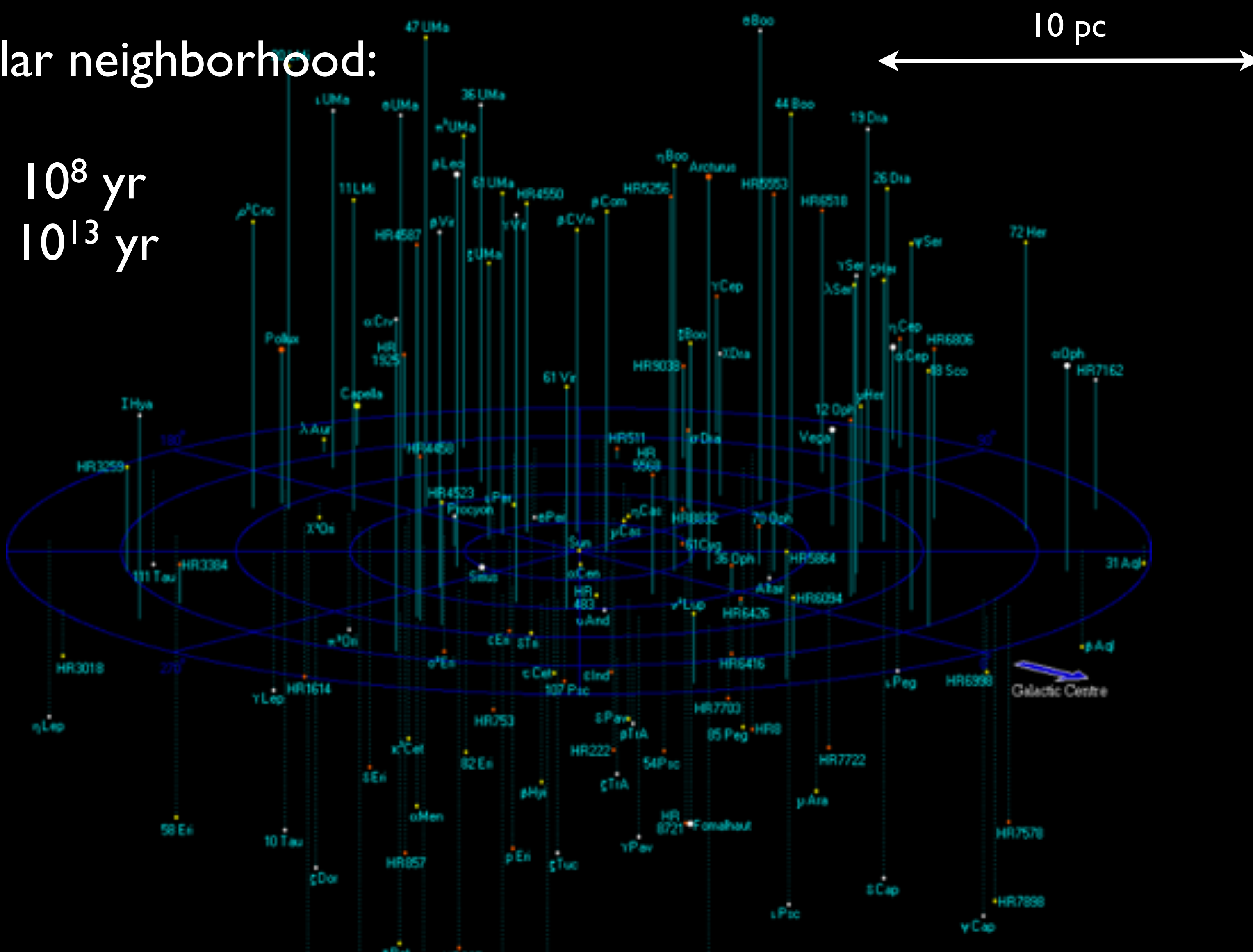


the solar neighborhood:

$t_{cross} \simeq 10^8$ yr

$t_{relax} \simeq 10^{13}$ yr



stars in the solar neighborhood have randomly directed velocities of 5-50 km/s in addition to a common rotational velocity of ~220 km/s

- stars in the solar neighborhood have randomly directed velocities of 5-50 km/s in addition to a common rotational velocity of ~220 km/s
- more massive stars have smaller random velocities, consistent with equipartition
- Jeans (1928): timescale required to reach equipartition due to gravitational encounters between stars is $\sim 10^{13}$ yr $\Rightarrow$ universe must be *at least* this old

TABLE I.—EQUIPARTITION OF ENERGY IN STELLAR MOTIONS.

Type of Star.	Mean Mass, M.	Mean Velocity, C.	Mean Energy, $\frac{1}{2}MC^2$.	Corresponding Temperature.
				Degrees.
Spectral type $B3$.	$19{\cdot}8 \times 10^{33}$	$14{\cdot}8 \times 10^{5}$	$1{\cdot}95 \times 10^{46}$	$1{\cdot}0 \times 10^{62}$
,, $B8{\cdot}5$.	12·9	15·8	1·62	0·8
,, $A0$.	12·1	24·5	3·63	1·8
,, $A2$.	10·0	27·2	3·72	1·8
,, $A5$.	8·0	29·9	3·55	1·7
,, $F0$.	5·0	35·9	3·24	1·6
,, $F5$.	3·1	47·9	3·55	1·7
,, $G0$.	2·0	64·6	4·07	2·0
,, $G5$.	1·5	77·6	4·57	2·2
,, $K0$.	1·4	79·4	4·27	2·1
,, $K5$.	1·2	74·1	3·39	1·7
,, $M0$.	1·2	77·6	3·55	1·7

- stars in the solar neighborhood have randomly directed velocities of 5-50 km/s in addition to a common rotational velocity of ~220 km/s
- more massive stars have smaller random velocities, consistent with equipartition
- Jeans (1928): timescale required to reach equipartition due to gravitational encounters between stars is $\sim 10^{13}$ yr $\Rightarrow$ universe must be *at least* this old
- in fact random velocities arise from gravitational interactions with interstellar clouds and spiral arms, and more massive stars have smaller velocities because they are younger

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“All models are wrong, but some are useful”

Box & Draper (1987)