The impact of modest stellar rotation on the asymmetric explosion of massive stars

How much rotation can make a difference on the explosion threshold and NS birth?

Why is the prograde mode of SASI destabilized by rotation?

What is the interplay between SASI and the corotation 'low T/|W|' instability?

Is the corotation instability similar to isolated NS with differential rotation?

Should we expect different GW signatures from these two instabilities?
How much rotation is needed to affect the explosion?

**slow rotation** ($j = 10^{15} \text{ cm}^2/\text{s}$): spiral SASI

Retrograde pulsar spin is possible at birth

**Very fast rotation** ($\Omega_0=2\text{ rad/s}, j = 4 \times 10^{16} \text{ cm}^2/\text{s}$):

low-$T/|W|$ instability but no effect for $\Omega_0=1\text{ rad/s}, j = 2 \times 10^{16} \text{ cm}^2/\text{s}$

\[ \Delta L/\nu \sim 10\% \]

for $j=5 \times 10^{15} \text{ cm}^2/\text{s}$ (~ms pulsar)

modest effect compared to the rotational kinetic energy involved

**Slow rotation ($j = 10^{15} \text{ cm}^2/\text{s}$):**

- 3x10^{16}
- 6x10^{15}

\[ \dot{\mathcal{M}} [M_\odot \text{ s}^{-1}] \]

\[ \dot{L}_\nu [10^{52} \text{ erg s}^{-1}] \]

\[ \frac{E_{\text{rot}}}{1.5 \times 10^{52} \text{ erg}} \approx \frac{M_{ns}}{1.5 M_\odot} \left( \frac{10 \text{ km}}{R_{ns}} \right)^2 \left( \frac{j}{5 \times 10^{15} \text{ cm}^2/\text{s}} \right)^2 \]
Spin-up or spin-down of the neutron star?

For a strong rotation rate, the corotation instability decelerates the neutron star by less than 30%.

Range of NS spin at birth

(R = \frac{R_{sh}}{R_{NS}})
Physical insight from an experimental analogue of SASI

Adiabatic gas

\[ c_s^2 \equiv \frac{\gamma P}{\rho} \]
\[ \Phi \equiv -\frac{GM_{\text{ns}}}{r} \]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \]
\[ \frac{\partial v}{\partial t} + (\nabla \times v) \times v + \nabla \left( \frac{v^2}{2} + \frac{c_s^2}{\gamma - 1} + \Phi \right) = \frac{c_s^2}{\gamma} \nabla S \]

Inviscid shallow water is analogue to an isentropic gas \( \gamma = 2 \)

St Venant

\[ c_{sw}^2 \equiv gh \]
\[ \Phi \equiv gH \Phi \]

\[ \frac{\partial H}{\partial t} + \nabla \cdot (H v) = 0 \]
\[ \frac{\partial v}{\partial t} + (\nabla \times v) \times v + \nabla \left( \frac{v^2}{2} + c_{sw}^2 + \Phi \right) = 0 \]

Acoustic waves & shock wave pressure \( \frac{t_{sh}}{t_{fp}} \equiv \left( \frac{r_{sh}}{r_{fp}} \right) \left( \frac{r_{sh} g H_{jp}}{GM_{\text{NS}}} \right)^{\frac{1}{2}} \sim 10^{-2} \)

Expected scaling

Shock radius \( \times 10^{-6} \)

Oscillation period \( \times 10^2 \)

200 km \( \rightarrow \) 20 cm

30 ms \( \rightarrow \) 3 s
Increasing the rotation rate: continuous transition from SASI to the corotation instability

- the rotation period is gradually decreased (205 s → 62 s)
- the flow rate is gradually decreased (1.1 L/s → 0.59 L/s)

→ the gravitational wave signature of the low \( T/|W| \) instability may be hard to disentangle from the SASI oscillations (Kuroda+14)
Unexpectedly robust spiral shock driven at the corotation radius when the inner rotation rate reaches 20% Kepler (low $T/|W|=0.02$)

Radial accretion enforces differential rotation

$$\frac{\Omega}{\Omega_{NS}} \propto \left( \frac{R_{NS}}{R} \right)^2$$

Analogue to the "low $T/|W|$ instability" of a neutron star rotating differentially (Shibata+02,03, Saijo+03,06, Watts+05, Corvino+10, Passamonti & Andersson 15, Yoshida & Saijo 17)

Spiral instability with a weak shock

Spiral instability with subsonic accretion

Instability mechanism: interaction of a corotation radius with acoustic waves (Papaloizou & Pringle 84, Goldreich & Narayan 85)
Experimental growth rate and oscillation period compared to shallow water modelling: a hint for an advective mechanism

- excellent modelling of the oscillation frequency limited by the measured radial width of the hydraulic jump

- systematic offset of the experimental growth rate expected phase mixing of the dragged vorticity

The vertically averaged vorticity is damped by a factor \( Q \)

\[
Q \sim \int_0^H \frac{dz}{H} \cos \left( \frac{\omega_{\text{SASI}} \Delta R}{v(z)} \right) \sim 0.27 \text{ (laminar)}
\]

\[
\sim 0.52 \text{ (turbulent)}
\]

\( R_{\text{jump}} \sim 20 \text{cm} \)

\( R_{\text{NS}} \)

\( \Rightarrow \) at odds with the idea of a transition to an acoustic corotation instability?
The St Venant system of shallow water equations is a simpler framework to understand the coupling of SASI with rotation and the transition to the low T/|W| instability: 
- adiabatic equations (no neutrino cooling) 
- no buoyancy effects
Compact formulation of the perturbative problem with rotation

Differential system for the linearized perturbations

\[
\begin{align*}
\frac{\partial \mathbf{\mathbf{v}}}{\partial t} + \mathbf{w} \times \mathbf{\mathbf{v}} + \nabla \left( \frac{|\mathbf{\mathbf{v}}|^2}{2} + gH + \Delta \Phi \right) &= 0, \\
\frac{\partial H}{\partial t} + \nabla \cdot (H \mathbf{\mathbf{v}}) &= 0,
\end{align*}
\]

mass conservation

Euler equation

Doppler shifted frequency

\[
\omega' = \omega - \frac{mL}{r^2},
\]

same as in a cylindrical flow (Yamasaki & Foglizzo 08)

Conserved specific vorticity

new variable \(X\)

phase shifted velocity

acoustic equation with a source term similar to Foglizzo 09 without rotation

same changes of variables as in a plane parallel flow (Foglizzo 09)

\[
\begin{align*}
\frac{dX}{dr} &= \frac{v_r}{1 - Fr^2}, \\
\frac{dX}{dr} &= \frac{v_r}{1 - Fr^2}, \\
\delta \tilde{v}_\theta &= e^{\int_{sh} i \omega' \frac{dX}{r}} \delta v_\theta.
\end{align*}
\]

+ boundary conditions from conservation equations

- at the shock/jump

- at the inner boundary

\[
\begin{align*}
(r \delta v_\theta)_{sh} &= -i m \frac{1 - v_{sh} \frac{v_1}{v_1}}{v_1} \Delta \zeta \left[ -i \omega' \frac{v_1}{v_1} \left( 1 - \frac{v_{sh}}{v_1} \right) + \frac{\partial \Phi}{\partial r} - \frac{v_{sh} v_{sh}}{r_{sh}} - \frac{L^2}{r_{sh}^3} \right], \\
\frac{d}{dX} (r \delta v_\theta)_{sh} &= -i \frac{m}{v_1^2} \frac{1 - v_{sh} \frac{v_1}{v_1}}{v_1} \Delta \zeta \left[ -i \omega' \frac{v_1}{v_1} \left( 1 - \frac{v_{sh}}{v_1} \right) + \frac{\partial \Phi}{\partial r} - \frac{v_{sh} v_{sh}}{r_{sh}} - \frac{L^2}{r_{sh}^3} \right], \\
v_1^2 \frac{d}{dX} (r \delta v_\theta)_{sh} &= i \omega' \frac{Fr_1^2}{v_1} (r \delta v_\theta)_{sh} + \left( 1 - Fr_1^2 \right) (r v_r \delta \tilde{w}).
\end{align*}
\]
Wronskien resolution: convolution of the acoustic solution with the source term

acoustic solution satisfying the inner boundary condition

\[
\left\{ \frac{d^2}{dX^2} + \frac{\omega'^2 - \frac{m^2}{r^2} (c^2 - v_r^2)}{v_r^2 c^2} \right\} r \delta v_0 = 0,
\]

inner boundary condition

definition of the perturbed mass flux

\[
v^2 \frac{d}{dX} \delta (r \delta v_0) = i \omega' \frac{F_{*}^3}{r} (r \delta v_0),
\]

advected phase of the source term

\[
\begin{align*}
\int_0^\infty \left( h^0 + \frac{\omega'}{mc^2} r \delta v_0 \right) e^{i \omega' \frac{1 + Fr^2}{1 - Fr^2} \frac{dr}{v_r}} \frac{dr}{1 - Fr^2} &= - \frac{\omega'}{mv_{sh}} (r \delta v_0)_{sh}^0 + v_1 h_{sh}^0 e^{i \omega' \frac{1 + Fr^2}{1 - Fr^2} \frac{dr}{v_r}} \\
\partial \Phi \frac{1}{dr} - v_1 v_{sh} \frac{L^2}{r_{sh}^2} - i \omega' v_1 \left( 1 - \frac{H_1}{H_{sh}} \right)
\end{align*}
\]

advective-acoustic coupling

shock boundary condition

The Doppler shifted frequency \( \omega' \equiv \omega - \frac{mL}{r^2} \) affects the phase mixing between the source and the acoustic wave.

The frequency of the prograde mode is locally decreased by the doppler shift: the decrease of \( \omega' \tau_\nabla \) is favourable to the advective-acoustic coupling as in Sheck+08 and Foglizzo 09 without rotation.

The corotation condition \( \omega' = 0 \) favours the advective-acoustic coupling: the stationary phase prevents phase mixing.
Analogue of SASI modes without rotation: $Fr_1$, $R_{jp}/R_{ns}$

Eigenfrequencies $\sim$ multiples of $2\pi/\tau_Q$ suggest that the advective-acoustic coupling is dominated by the lower boundary.
Comparison of a shocked rotating flow and a trapped acoustic mode

**normal shock condition: vorticity + pressure perturbation**

- as in Yamasaki & Foglizzo 08, the growth rate of the prograde mode increases with the rotation rate
- a corotation radius can exist for rotation rates as low as 3% $v_{\text{Kepler}}$ at $r_*$ ($T/W \sim 0.05\%$)
- a corotation radius is not a sufficient condition for instability (e.g. $Fr_1=3$, $R=2$)
- the transition from SASI to an instability with a corotation is very smooth

**ad-hoc shock condition: total acoustic reflexion, no vorticity**

- when acoustic reflexion at the shock is total, the existence of the corotation radius is a sufficient condition for instability: similar to differentialy rotating NS (Watts+05, Passamonti & Andersson 15, Yoshida & Saijo 17)
- a corotation radius can exist for rotation rates as low as 6% $v_{\text{Kepler}}$ at $r_*$ ($T/W \sim 0.02\%$)
- however, the growth rate of this corotation instability seems loosely correlated with the growth rate of the shocked flow.
- estimated growth rate: $\frac{\omega_i}{\Omega_{sh}} \propto (2.2 \pm 0.4) \left(\frac{\Omega_{sh}}{\Omega_*}\right)^{\frac{3}{2}} \left[1 - \frac{\Omega_{\text{corot}}}{\Omega_*}\right]$
Conclusions

2D Cylindrical gas dynamics (Kazeroni+17) suggests that
- SASI can account for pulsar rotation periods down to ~50ms
- for rotation rates >100Hz the 'corotation instability' decreases the pulsar spin by <30%

Both instabilities are captured in the supernova fountain experiment
- as the injected angular momentum increases, the prograde spiral mode of SASI seems to connect smoothly to the 'low T/|W|' instability
- the offset growth rate in the experiment suggests advection may play a dominant role even when a corotation is present

The shallow water model offers a simple analytical framework to study the interplay of SASI & 'low T/W'
- equations are both simple and connected to a real experiment
- the rotational destabilization of the prograde mode of SASI can be explained by its lower doppler shifted frequency which benefits to the advective-acoustic coupling
- a classical corotation instability is recovered as a purely acoustic process, despite radial advection, if the shock is artificially replaced by a total acoustic reflection and no advected vorticity
- the existence of a corotation radius is not a sufficient condition for instability in a shocked flow
- the prograde mode of SASI can be more unstable than an acoustic corotation instability: the stationary phase at the corotation radius favours the advective-acoustic coupling

→ A sharp transition between SASI and the 'low T/|W|' instability in a shocked flow is not expected