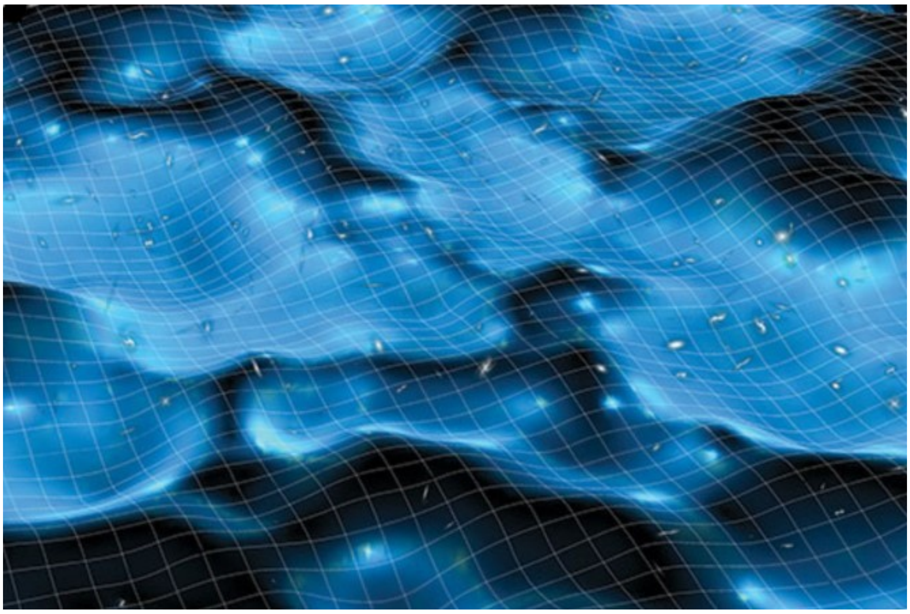

MPA Lectures on Gravitational Waves in Cosmology

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Abstract

Almost a century ago, Albert Einstein predicted the existence of gravitational waves, ripples in the spacetime, as a possible solution of the linearized general relativity. Surprisingly, shortly after that, Einstein himself changed his mind, and since after, like many other physicists of his time, he believed that they are not physical but an artifact of linearization. It was not until Herman Bondi's 1957 Nature paper that the first mathematically precise definition of gravitational waves in the full Einstein equations has been discovered. Seventeen years later, Hulse and Taylor made the first indirect detection of this mysterious waves. And finally, in another historic event, only three years ago, these waves were first detected directly by LIGO. Apart from the astrophysical gravitational waves, a stochastic background of relic gravitational waves is also highly expected. These tensor perturbations are the only missing (key) prediction of the inflation paradigm which has not been detected yet. These primordial gravitational waves, either vacuum or sourced by relic particles, are free-streaming since inflation and can teach us a lot about the physics at its highest possible energy scales. This talk has two parts. In the first part, I will talk about the gravitational waves in asymptotically flat spacetimes and will explain the Bondi's brilliant formalism to prove the physicality of the gravity waves. In the second part, I focus on the gravitational waves in our expanding cosmological universe. I explain the spin-2 fluctuations generated during inflation and the resulting stochastic gravitational background.

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Part I
Asymptotically flat spacetimes

1 Linearized GR and Gravitational Waves

Shortly after proposing general relativity, Albert Einstein linearized his field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}, \tag{1}$$

and realized that general relativity admits solutions in which the fluctuations of the Minkowski space-time are plane waves traveling with the speed of light. In the following, we explain some basic features of these linear solutions of general relativity.

1.1 The weak-field metric

A weak gravitational field corresponds to a region of the spacetime that is weakly curved. In other words, throughout such region, there exist coordinate systems in which the spacetime metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}, \tag{2}$$

in which $\tilde{h}_{\mu\nu}$ and all its partial derivatives are small

$$\tilde{h}_{\mu\nu} \ll 1 \quad \text{and} \quad \partial^n_\lambda \tilde{h}_{\mu\nu} \ll 1 \quad (n > 1). \tag{3}$$

Note that we can well consider small perturbations about some other background metric, such that $g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}$. In particular, we will use a similar weak field approach for cosmological perturbations around FRW metric in section 3.

In the weak-field general relativity, we expand the field equations in powers of $h_{\mu\nu}$ and keep the linear terms. In fact, from the Einstein equation in (1), we find the linearized gravitational field equation for $h_{\mu\nu}$

$$\partial_\alpha\partial^\alpha \tilde{h}_{\mu\nu} + \partial_\mu\partial_\nu \tilde{h} - \partial_\nu\partial_\lambda \tilde{h}^\lambda_\mu - \partial_\mu\partial_\lambda \tilde{h}^\lambda_\nu - \eta_{\mu\nu}(\partial_\alpha\partial^\alpha \tilde{h} - \partial_\sigma\partial_\lambda \tilde{h}^{\lambda\sigma}) = -16\pi G T_{\mu\nu}, \tag{4}$$

where $\tilde{h}$ is the trace of the field, $\tilde{h} = \tilde{h}^\mu_\mu$. Although linearized, the above equation does not look like a wave-equation. However, this messy equation can be simplified in terms of the field redefinition

$$h_{\mu\nu} \equiv \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}, \tag{5}$$

which is the trace-reverse cousin of $\tilde{h}$, *i.e.* $h = -\tilde{h}$. Moreover, these fields are defined only up to gauge transformations, and in this case, the most convenient gauge is the Lorenz gauge

$$\partial_\mu h^{\mu\nu} = 0. \tag{6}$$

The linearized field equation in terms of $h_{\mu\nu}$ in the Lorenz gauge takes the most simple form below

$$\partial_\alpha\partial^\alpha h_{\mu\nu} = -16\pi G\ T_{\mu\nu}. \tag{7}$$

which in the vacuum has the desired wave-like form.

1.2 General solution of the linearized field equation

The linearized equation in vacuum has plane wave solutions of the form

$$h_{\mu\nu} = A_{\mu\nu}(k^\lambda)\exp(ik_\lambda x^\lambda), \tag{8}$$

where $A_{\mu\nu}$ are constant components of a symmetric tensor and k_μ is the wave vector. Equation (7) implies that the wave vector is null, $k_\mu k^\mu = 0$. Therefore, the homogeneous linear Einstein gravity has the following general solution

$$h_{\mu\nu} = \int d^3k \Big(A_{\mu\nu}(\vec{k})\exp(ik_\mu x^\mu) + A^*_{\mu\nu}(\vec{k})\exp(-ik_\mu x^\mu) \Big), \tag{9}$$

which is the superposition of all possible plane waves.

In the presence of a source with nonzero $T_{\mu\nu}$, we need to solve the inhomogeneous equation with the Green's equation

$$\partial_\alpha \partial^\alpha G(x^\mu - y^\mu) = \delta^4(x^\mu - y^\mu), \tag{10}$$

where the required retarded Green's function for $x^0 > y^0$ is

$$G(x^\mu - y^\mu) = \frac{1}{(4\pi)|\vec{x} - \vec{y}|}\delta(x^0 - y^0 - |\vec{x} - \vec{y}|). \tag{11}$$

Therefore, the sourced part of the linear gravitational equation is

$$h_{\mu\nu}(\vec{x}, t) = -\frac{4G}{c^4}\int d^3y\, \frac{T_{\mu\nu}(\vec{y}, ct - |\vec{x} - \vec{y}|)}{|\vec{x} - \vec{y}|}. \tag{12}$$

In fact, the gravitational field at an event point $(t, \vec{x})$ is the integral over the past lightcone of the event point occupied by the source.

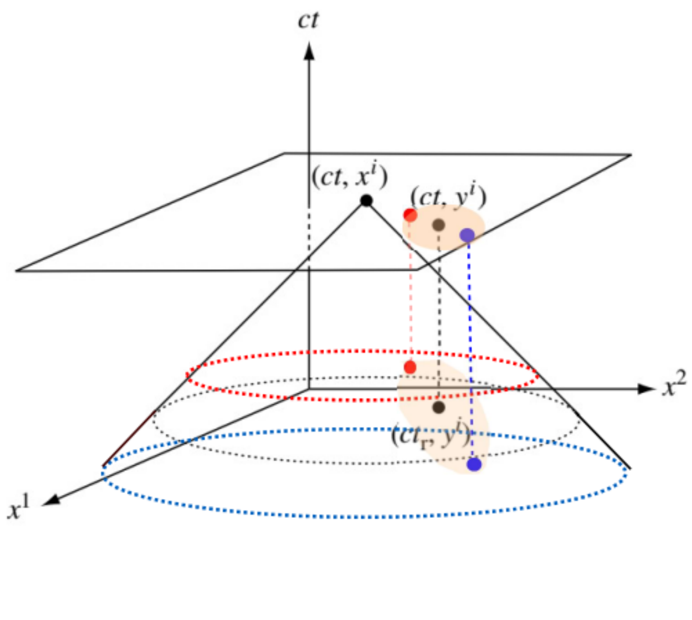

Figure 1: The change in the gravitational field at an event point $(t, \vec{x})$ is the sum of the effects of the source's $T_{\mu\nu}$ at the point $((t - |\vec{x} - \vec{y}|), \vec{y})$ on the past lightcone.

1.3 The compact-source

For astrophysical purposes, the gravitational source has a spatial size much smaller than the distance to the point of the observation. In such cases, it is sufficient to consider the first term in the Taylor expansion

$$\frac{1}{|\vec{x} - \vec{y}|} = \frac{1}{r} + \frac{y^i x_i}{r^3} + y^i y^j \left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right) + ..., \tag{13}$$

where $r = |\vec{x}|$ is the spatial distance from the origin to the field point. In particular, the linear solution (12) has the far-field approximation

$$h_{\mu\nu}(\vec{x}, t) = -\frac{4G}{c^4 r} \int d^3y T_{\mu\nu}(\vec{y}, ct - r), \tag{14}$$

which decays as $1/r$. The physical meaning of each component in the above is as follows:

- $\int T^{00} d^3y = Mc^2$ is the total energy of source particles,
- $\int T^{0i} d^3y = P^i c$ is the total momentum of source particles in the x^i-direction,
- $\int T^{ij} d^3y = \Pi^{ij}$ is the integrated internal stresses in the source.

For an isolated source, the conservation of energy-momentum tensor reads

$$\partial_\mu T^{\mu\nu} = 0.$$

Therefore, in the linear theory, M and P^i are conserved. Furthermore, for a compact object, P^i can be written as

$$P^i = \frac{\partial}{\partial y^0} \left[\int T^{00}(y^0, \vec{y}) y^i d^3y \right]_{y^0 = ct - r}, \tag{15}$$

which can always set to zero by choosing the origin of the coordinate system at the source's center of mass. Therefore, we can always work in a coordinate system associated to the center of momentum frame of the source particles in which $P^i = 0$.

Finally, in the center of momentum coordinate, we have the desired $h_{\mu\nu}$ components as

$$h_{00} = -\frac{4GM}{c^2 r}, \tag{16}$$

$$h_{0i} = 0, \tag{17}$$

$$h_{ij} = -\frac{4G}{c^4 r} \int T_{ij}(ct - r, \vec{y}) d^3y. \tag{18}$$

It is possible to further simplify the form of h_{ij} for the compact-source case. We can write [1]

$$\int T^{ij} d^3y = -\frac{1}{2} \int \left(\partial_k (T^{ik}) y^j + \partial_k (T^{jk}) y^i \right) d^3y. \tag{19}$$

Moreover, the energy-momentum conservation leads to

$$\int \partial_k (T^{ik}) y^j d^3y = -\frac{d}{dy^0} \left[\int T^{i0} y^j d^3y \right], \tag{20}$$

By using the energy-momentum conservation once again and dropping the total derivative term, we have

$$\int \left(T^{i0} y^j + T^{j0} y^i \right) d^3y = \frac{d}{dy^0} \left[\int T^{00} y^i y^j d^3y \right]. \tag{21}$$

Thus, in terms of the quadruple-moment tensor of the source

$$I_{ij}(y^0) = \int T_{00}(y^0, \vec{y}) y^i y^j d^3y, \tag{22}$$

[1] Note that enclosing the integral outside the source, we dropped the total derivative term in the RHS, $\int \partial_k (T^{ik} y^j) d^3y = 0$.

we can express h_{ij} in the far-field limit as

$$h_{ij}(t,\vec{x}) = -\frac{2G}{c^6 r}\frac{d^2}{dy^{02}}\Big[I_{ij}(y^0)\Big]_{y^0=ct-r}. \tag{23}$$

The above can be decomposed into the trace and a traceless tensor, γ_{ij}, as

$$h_{ij}(t,\vec{x}) = h\delta_{ij} + \gamma_{ij}, \tag{24}$$

where γ_{ij} is

$$\gamma_{ij} = -\frac{2G}{c^6 r}\frac{d^2}{dy^{02}}\Big[J_{ij}(y^0)\Big]_{y^0=ct-r}, \tag{25}$$

and J_{ij} is the reduced quadrupole-moment tensor of the source distribution

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I^k_k. \tag{26}$$

1.4 Polarization states and effect on free particles

In the previous section, we saw that similar to the Maxwell's equation which predicts electromagnetic waves, the linearized GR also suggests the existence of gravitational waves. In this section, we discuss the propagation of gravitational radiation in flat space.

Let us consider a general gravitational perturbation satisfying the empty-space linearized field equation and the Lorenz gauge condition (see (9)). Under a coordinate (gauge) transformation of the form

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu, \tag{27}$$

the transformed perturbation

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu, \tag{28}$$

is still in the Lorenz gauge condition provided that the 4-vector ξ^μ satisfies $\partial_\nu\partial^\nu\xi^\mu = 0$. Also the trace-reverse field tensor in (5) transforms as

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \eta_{\mu\nu}\partial_\alpha\xi^\alpha. \tag{29}$$

We can use the above gauge transformation to set any four linear combinations of $h_{\mu\nu}$ to zero. In particular, the transverse-traceless gauge, so called TT gauge, is defined by choosing

$$h^{TT}_{0i} = 0 \quad \text{and} \quad h^{TT} = 0. \tag{30}$$

Moreover, we are still in the Lorenz gauge which in the TT gauge gives

$$\partial^0 h^{TT}_{00} = 0 \quad \text{and} \quad \partial^i h^{TT}_{ij} = 0. \tag{31}$$

Notice that for non-stationary (time-dependent) gravitational fields, as for a general gravitational wave disturbance, it implies that h^{TT}_{00} also vanishes. As a result, for those cases, we are left with the spatial components of a symmetric, traceless and transverse tensor, i.e. we have only two dynamical degrees of freedom.

For a given gravitational wave with a spatial wave-vector $\vec{k} = k\hat{n}$ in an arbitrary coordinate system, we can read the form of the wave in the TT gauge as

$$A_{ij}^{TT} = (P_i^k P_j^l - \frac{1}{2} P^{kl} P_{ij}) A_{kl}, \tag{32}$$

where P_{ij} is the spatial projection tensor

$$P_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j, \tag{33}$$

which projects tensors to the 2 dimensional surface normal to $\hat{n}$. Similar to the electromagnetic waves, the two dynamical modes in the A_{ij}^{TT} can be decomposed in terms of two polarization states. Writing $\hat{n}$ in terms of polar and azimuthal angles, θ and ϕ, as

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{34}$$

one can define the complex polarization vectors

$$e^{\pm}(\hat{n}) = \frac{1}{\sqrt{2}}(\hat{\theta} \pm i\hat{\phi}), \tag{35}$$

where $\hat{\theta}$ and $\hat{\phi}$ are orthogonal unit vectors in the plane perpendicular to $\hat{n}$, in the directions of increasing θ and ϕ, respectively. In terms of $e^{\pm}$ and $(\hat{\theta}, \hat{\phi})$, we can construct two types of polarization states:

i) the linear, *plus* and *cross*, polarization tensors,

$$e_{ij}^{\text{Plus}}(\hat{n}) \equiv \frac{1}{\sqrt{2}}\Big(\hat{\theta}_i\hat{\theta}_j - \hat{\phi}_i\hat{\phi}_j\Big) \quad \text{and} \quad e_{ij}^{\text{Cross}}(\hat{n}) \equiv \frac{1}{\sqrt{2}}\Big(\hat{\theta}_i\hat{\phi}_j + \hat{\phi}_i\hat{\theta}_j\Big), \tag{36}$$

and ii) the circular, *right*- and *left*-handed, polarization tensors

$$e_{ij}^{R,L}(\hat{n}) \equiv e_i^{\pm} e_j^{\pm}. \tag{37}$$

Notice that these two pair of tensors are related as

$$e^{R,L}(\hat{n}) = \frac{1}{\sqrt{2}}\Big(e^{Plus} \pm ie^{Cross}\Big) \tag{38}$$

For $\hat{n} = \hat{x}^3$, the P=plus and C=cross polarization tensors can be written in terms of the components

$$e_{11}^P = -e_{22}^P = e_{12}^C = e_{21}^C = \frac{1}{\sqrt{2}}, \quad \text{and} \quad e_{12}^P = e_{21}^P = e_{11}^C = e_{22}^C = e_{i3}^{P,C} = 0, \tag{39}$$

while the circular ones are given as

$$e_{11}^{R,L} = -e_{22}^{R,L} = \frac{1}{\sqrt{2}}, \quad e_{12}^{R,L} = e_{21}^{R,L} = \pm\frac{i}{\sqrt{2}} \quad \text{and} \quad e_{3i}^{R,L} = e_{i3}^{R,L} = 0. \tag{40}$$

In order to have a feeling about the effect of each of the above polarization states of gravitational waves, it is useful to consider their effect on the geodesic deviation of free-falling particles. Consider two nearby (non-interacting) free-falling particles initially at rest with separation vector

$$\xi_0^\mu = (0, \xi_0^i). \tag{41}$$

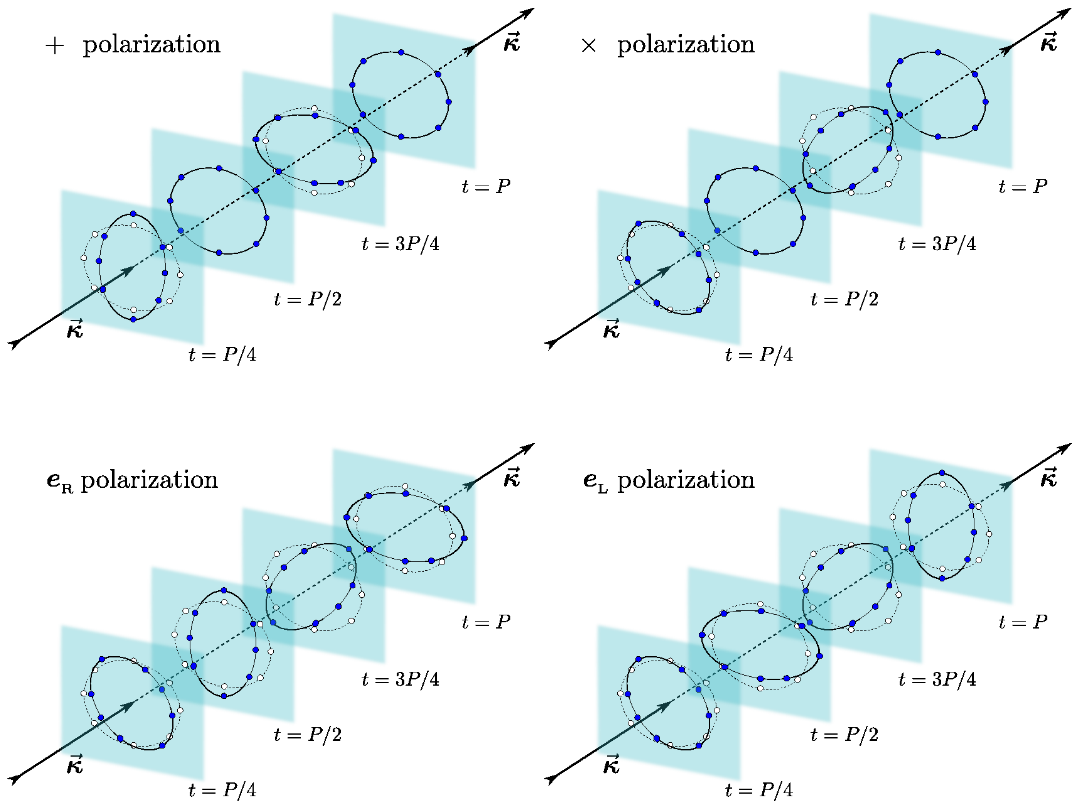

Figure 2: Effect of gravitational wave in different polarization states on a ring of freely-falling particles. The continuous lines and the dark filled dots show the positions of the particles at different times, while the dashed lines and the open dots show the unperturbed positions. This illustration is barrowed from [1].

The arrival of a gravitational wave will perturb the geodesic motion of the two particles and produce a nonzero contribution to the geodesic-deviation equation. We recall the geodesic-deviation equation, the changes in the separation four-vector X^μ between two geodesic trajectories with tangent four-vector u^μ, is

$$\frac{D^2X^\mu}{D\tau^2} = -R^\mu{}_{\nu\lambda\sigma}u^\nu u^\lambda X^\sigma, \tag{42}$$

where

$$\frac{D}{D\tau} \equiv u^\mu \nabla_\mu,$$

is the covariant time derivative along the geodesic of a particle. Therefore, the geodesic deviation of the nearly particles in the presence of the GW is

$$\frac{D^2X^i}{D\tau^2} = -R^i{}_{0j0}X^j. \tag{43}$$

In the rest frame of particle A, around the particle the connection vanishes and in the TT gauge the coordinate time and the particle's proper time coincides at the leading order ($t = \tau + \mathcal{O}((h^{TT}_{\mu\nu})^2)$), and we simply have

$$\frac{d^2X^i(t)}{dt^2} = \frac{1}{2}\partial_t^2 h^{TT}_{ij}\xi^j, \tag{44}$$

which gives the separation vector as

$$X^i(t) = \xi^j \left(\delta_{ij} + \frac{1}{2} h_{ij}^{TT}(t) \right). \tag{45}$$

In figure 2, we present this effect by each of the 4 possible polarization states of GWs on a ring of freely-falling particles.

2 Physicality of Gravitational radiation?!

Shortly after proposing general relativity, Albert Einstein linearized it and predicted the existence of gravitational waves as a possible solution of his theory. However, he mistakenly thought they are not the solutions of the full nonlinear theory and therefore unphysical. He advanced arguments against the existence of gravity waves, which stopped the development of the subject for decades. It was only after his death that the actual physical nature of GWs was understood. In this section, first, I briefly mention the ambiguity and fundamental problems which led to this mistake. Then, I will explain the mathematically and geometrically precise definition of GWs by H. Bondi, F. Pirani, I. Robinson, and A. Trautman which proved the existence of GWs as the solutions of the full theory.

2.0.1 Theoretical issues of linear gravity

Unlike Maxwell's equations, gravitation is a non-linear theory. That is the direct result of the fact that gravity gravitates. More precisely, any energy-momentum acts as a source of gravity, including its energy-momentum tensor. That is unlike the electromagnetic field (photons in QED) which are not charged. So an important question that one should address before taking the linear solutions too seriously is:

Do the fully nonlinear Einstein equations admit solutions that can be described as gravitational waves? If yes, are they coincide with the linear solutions far from the source? If everything is fine, then we need to solve some other subproblems, including;

i) What is the definition of a plane gravitational wave in the full theory?

ii) Do they carry energy and angular momentum? Etc.

In 1958, H. Bondi formulated the first mathematically precise definition of GWs, which followed by papers by H. Bondi, F. Pirani, I. Robinson, and A. Trautman proved the existence of GWs as the solutions of the full theory and confirmed that they do carry energy. Here, I briefly review their stunning results. For a nice and recent review on the mathematics of GWs and the history behind it, see [2]. More details about the asymptotically flat spacetimes, BMS group, and the IR gravity can be found in [3].

2.1 Geometric definition for GWs

Even before Bondi's Nature paper, Felix Pirani used a Brilliant idea as the first attempt at a purely geometric definition of spacetime with gravitational waves. Pirani's intuition was based on using the algebraic form of the Weyl tensor, the so-called Petrov classification to see if spacetime has radiative regions. That intuition makes sense since far from the source the Ricci tensor vanishes, and the Riemann tensor reduces to Weyl. Here we briefly discuss this idea and the resulting geometrical description of radiative spacetimes.

In 4d, the Riemann curvature tensor has 20 independent components in which 10 independent components are associated to the Ricci tensor. The other 10 independent components can be assigned to the (traceless) Weyl tensor which is a conformal tensor

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - (g_{\mu[\lambda}R_{\sigma]\nu} - g_{\nu[\lambda}R_{\sigma]\mu}) + \frac{1}{3}R\, g_{\mu[\lambda}g_{\sigma]\nu}. \tag{46}$$

More precisely, it is unchanged under a conformal transformation of the metric [2]

$$g_{\mu\nu} \quad \mapsto \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{47}$$

$$C_{\mu\nu\lambda}{}^{\sigma} \quad \mapsto \quad C'_{\mu\nu\lambda}{}^{\sigma} = C_{\mu\nu\lambda}{}^{\sigma}. \tag{48}$$

More intuitively, the Weyl tensor expresses the tidal forces that a free-falling body feels along a geodesic. However, unlike the Ricci tensor, it does not have information about the change of the volume, but only how the shape of the body is distorted by the tidal forces (see illustration in figure 3).

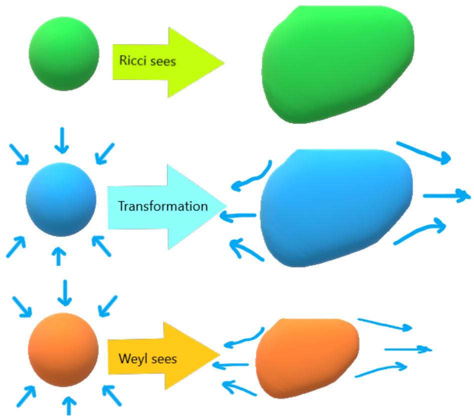

Figure 3: Weyl vs. Ricci. Weyl tensor is blind to the scaling and change of volume while Ricci is blind to the tidal forces!

2.2 Petrov classification for Weyl tensor

The Petrov classification describes the possible algebraic symmetries of the Weyl tensor at each event in a Lorentzian manifold. This classification was found in 1954 by A. Z. Petrov and independently by Felix Pirani in 1957. The classification is based on the observation that an arbitrary asymmetric tensor

$$X_{\mu\nu} = -X_{\nu\mu},$$

under the action of the Weyl tensor transforms to another asymmetric tensor as

$$X_{\mu\nu}C^{\mu\nu}{}_{\alpha\beta} = Y_{\alpha\beta}. \tag{49}$$

Therefore, the natural question is to find the eigen-bivectors and eigenvalues of the above equation, i.e.

$$\frac{1}{2}X_{\mu\nu}C^{\mu\nu}{}_{\alpha\beta} = \lambda X_{\alpha\beta}. \tag{50}$$

[2] Therefore, we have $C'_{\mu\nu\lambda\sigma} = \Omega^2 C_{\mu\nu\lambda\sigma}$.

The Weyl tensor can have at most four linearly independent eigen-bivectors at each given event which are associated with some null vectors in the original spacetime, called the principal null directions (PND). The Petrov classification states that there are precisely six possible types of algebraic symmetries, known as the Petrov types:

- Type I: four simple PNDs,
- Type II: two simple PNDs and two PNDs coincide,
- Type D: two pairs of coinciding PNDs,
- Type III: one triple and one simple PNDs,
- Type N: all PNDs coincide,
- Type O: the Weyl tensor vanishes.

We present these six types of the Weyl tensor in figure 4. Let us take a closer look at the Petrov types D, N, and O.

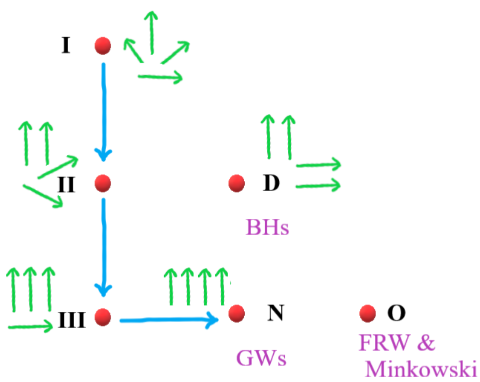

Figure 4: The schematic form of the Petrov classification of the Weyl tensor. The green arrows correspond to each of the principal null directions and the parallel green arrows represent the number of coinciding PNDs. In the O type, the Weyl tensor vanishes.

Type D regions are associated with the gravitational fields of isolated massive objects, e.g., stars and black holes, which is entirely characterized by its mass and angular momentum. The two coincided PNDs present radially ingoing and outgoing null congruences near the object.

Type O regions are conformally flat places with zero Weyl tensor, e.g., exact Minkowski and FRW. In this case, any gravitational effects must be due to the immediate presence of matter or the field energy of some non-gravitational field.

Type N regions are those regions with transverse gravitational radiation. A spacetime region is type N, if and only if there exists a null vector, k_μ, such that

$$C_{\mu\nu\lambda\sigma}k^\sigma = 0. \tag{51}$$

The four coinciding PNDs are given by this null vector which is the wave vector of the propagating gravitational wave.

It is noteworthy to mention that the Petrov type of a spacetime may vary from region to region. For instance, in figure 4, the blue arrow shows the direction of change of Petrov type of Weyl tensor as we approach null infinity in an asymptotically flat spacetime. [3] To summarize, the Weyl tensor of a radiative spacetime must be of type N very far from the sources, i.e. in the asymptotic future.

2.3 Gravitational radiation after Bondi

Herman Bondi in his Nature paper [5] followed by a subsequent paper by Bondi, Pirani and Robinson [6], provided the first mathematically precise definition of gravitational waves in the full Einstein equation. Moreover, he proved that gravitational radiation carries energy. Here, we review that in asymptotically flat spacetimes. In this part, we adopt the Bondi's $(u, r, z, \bar{z})$ coordinate largely because they are used in most of the literature on asymptotically flat spacetimes.

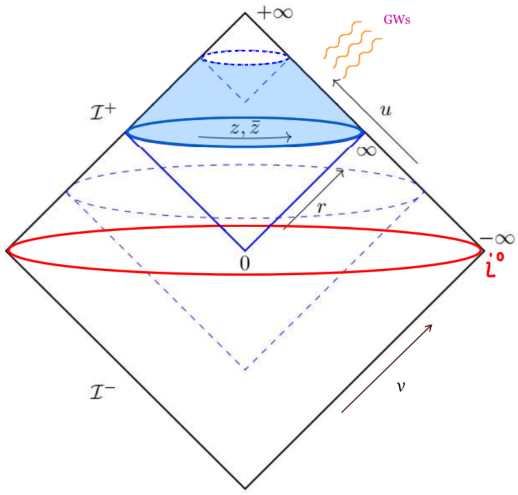

Figure 5: The Penrose diagrame of an asymptotically flat spacetime. The future/past null infinities, $\mathcal{I}^{\pm}$, are parametrized by retarded/advanced Bondi time u/v. The red ring, i_0, is the spatial infinity. The two blue cones in the upper half, present two null surfaces specified with $u = u_1$ and $u = u_2$. The shaded area is the cut in $\mathcal{I}^+$ with $u_1 < u < u_2$.

Far from the source, the spacetime is very close to flat Minkowski space

$$ds^2 = -du^2 - 2dudr + 2r^2\eta_{z\bar{z}}dzd\bar{z}, \tag{53}$$

and throughout we use the Bondi retarded coordinate $(u, r, z, \bar{z})$ which in terms of the spherical coordinate (t, r, θ, ϕ) are given as

$$u = t - r, \quad z = \cot\frac{\theta}{2}e^{i\phi} \quad \text{and} \quad \bar{z} = \cot\frac{\theta}{2}e^{-i\phi}. \tag{54}$$

[3] This effect is due to the peeling theorem in general relativity which describes the asymptotic behavior of the Weyl tensor as one goes to null infinity. Let γ be a null geodesic from a point p to null infinity, with affine parameter λ. Then the theorem states that, as λ approaches infinity

$$C_{abcd} = \frac{C^{(1)}_{abcd}}{\lambda} + \frac{C^{(2)}_{abcd}}{\lambda^2} + \frac{C^{(3)}_{abcd}}{\lambda^3} + \frac{C^{(4)}_{abcd}}{\lambda^4} + \ldots, \tag{52}$$

where $C^{(1)}_{abcd}$ is type N, $C^{(2)}_{abcd}$ is type III, $C^{(3)}_{abcd}$ is type II and $C^{(4)}_{abcd}$ is type I [4].

Here u is the Bondi (retarded) time while the advanced time (in Minkowski) is

$$v = t + r. \tag{55}$$

- Definition I. *Asymptotic flatness:* An asymptotically flat spacetime is a Lorentzian manifold in which, the curvature vanishes at large distances from some region so that the geometry becomes indistinguishable from Minkowski. Outside the source, the Ricci tensor is zero and therefore, the asymptotic flatness impose some asymptomatic falloff conditions on the Weyl tensor.
- Definition II. *Future null infinity:* $\mathcal{I}^+$ is defined as endpoints of all future-directed null geodesics along which $r \to \infty$. This null surface is the product of S^2 with a null line u taking values in $\mathbb{R}$. Each null hypersurface, σ_{u_0}, intersects $\mathcal{I}^+$ in a 2-sphere with $u = u_0$. (See figure 5)

Now, we want to study gravitational theories in which the metric is asymptotic, but not exactly equal to, the flat metric, and we abbreviate $\Theta^A = (z, \bar{z})$. Choosing the Bondi gauge

$$g_{rr} = g_{rA} = 0, \tag{56}$$

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0, \tag{57}$$

the most general four-dimensional metric has the form

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + g_{AB}\left(d\Theta^A + \frac{1}{2}U^A du\right)\left(d\Theta^B + \frac{1}{2}U^B du\right). \tag{58}$$

Notice that the gauge condition (56) completely fixed the local diffeomorphisms. Moreover, it implies that r is the luminosity distance. Up to now, we just wrote a general metric in the specific gauge (56). Any geometry can be described locally by the metric (58). Imposing the asymptotic flatness condition at large r with fixed $(u, z, \bar{z})$ leads to falloff conditions on the metric components. For the natural choice made by Bondi, van der Burg, Metzner, and Sachs (BMS) [7], the large-r structure of the metric is constrained to be

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\eta_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + r\gamma_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^z\gamma_{zz}dudz + D^{\bar{z}}\gamma_{\bar{z}\bar{z}}dud\bar{z} \\ &+ \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(\gamma_{zz}\gamma^{zz})\right)dudz + c.c. + \dots, \end{aligned} \tag{59}$$

where D_z is the covariant derivative with respect to $\eta_{z\bar{z}}$ while γ_{zz}, m_B and N_z are r independent and functions of $(u, z, \bar{z})$. The first three terms in (59) are simply the flat Minkowski metric, and the remaining terms are the leading corrections:

- The first quantity $m_B(u, z, \bar{z})$ is the Bondi mass aspect (for Kerr BH, $m_B = GM$),
- The next one is $N_z(u, z, \bar{z})$ which is the Bondi angular momentum aspect (for a Kerr BH $N_z = 2GmL$),
- The last term is $\gamma_{z\bar{z}}$ which describes the gravitational waves. This quantity is transverse to $\mathcal{I}^+$ and r^{-1}-suppressed comparing to the dominant orders. The Bondi news tensor is defined as

  $$N_{z\bar{z}} = \partial_u \gamma_{z\bar{z}}, \tag{60}$$

  which is the gravitational analogue of the field strength in gauge field theories, i.e. $F_{uz} = \partial_u Az$.

- Definition III. The *Bondi mass* at a Bondi time, u_1, is defined as the integral over S^2 (the sphere with u_1 at $\mathcal{I}^+$), as

$$M_B(u_1) = \frac{1}{4\pi G}\int_{S^2} d^2z\eta_{z\bar{z}} m_B(u_1, z, \bar{z}). \tag{61}$$

The Bondi mass is positive and time dependent, such that it is always non-increasing with time. Moreover, in the limit $u \to -\infty$, S^2_u asymptotically approaches the spatial infinity, i^0 and the Bondi mass is equal to the (conserved) ADM mass [8, 9]

$$M_{ADM} = \lim_{u\to-\infty} M_B(u). \tag{62}$$

Moreover, the time evolution of m_B is given by the Einstein equation component G_{uu} at $\mathcal{I}^+$ as

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - \frac{1}{4} N_{zz}N^{zz} - 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right], \tag{63}$$

where T^M_{uu} is the matter field's energy-momentum tensor. Using the Einstein equation in (61) and considering a compact source with $r^2 T^M_{uu} \sim \mathcal{O}(r^{-1})$ at future null infinity, we have

$$M_B(u_2) - M_B(u_1) = -\frac{1}{4}\int_{u_1}^{u_2} du \int d^2z\eta_{z\bar{z}} N_{zz}N^{zz}, \quad (u_2 > u_1), \tag{64}$$

in which the $D_z^2 N^{zz}$ terms vanishes under the S^2 integral. This is the famous Bondi mass-loss formula which measures the amount of mass-loss after some radiation through $\mathcal{I}^+$ (see figure 5). That is zero in case that the Bondi news vanishes. Otherwise, the Bondi mass is decreasing with time in the form of gravitational radiation.

The angular momentum aspect,

$$N_z(u, z, \bar{z}),$$

defined in the above is governed by a constraint equation

$$\partial_u N_z = \frac{1}{4}\partial_z(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u\partial_u\partial_z m_B - \frac{1}{4}\partial_z(C_{zz}N^{zz}) - \frac{1}{2}C_{zz}D_z N^{zz} + 8\pi G \lim_{r\to\infty} r^2 T^M_{uz}. \tag{65}$$

Integrating the above similar to the mass aspect, one can find the amount of angular momentum carried by the GWs.

Part II
Expanding Universe

3 Gravitational Waves in expanding Universe

In the first part, we studied gravitational radiation in asymptotically flat spacetimes. In this part, we want to explore the generation and evolution of gravitational waves in the cosmological background.

Cosmological era	Eq. of state	Scale factor	Hubble	SEC
Cosmic inflation	$w \simeq -1$	$a(t) = Exp(Ht)$	$H(t) = H_{inf}$	No
$(\epsilon = -\frac{\dot{H}}{H} \ll 1)$	$w = -1 + \frac{2}{3}\epsilon$	$a(\tau) = -\frac{1}{H\tau}$	$\mathcal{H}(\tau) = -\frac{1}{\tau}$	
Radiation era	$w = \frac{1}{3}$	$a(t) = \left(\frac{t}{t_I}\right)^{\frac{1}{2}}$	$H(t) = \frac{1}{2t}$	Yes
		$a(\tau) = \frac{\tau}{\tau_I}$	$\mathcal{H}(\tau) = \frac{1}{\tau}$	
Matter era	$w = 0$	$a(t) = \left(\frac{t}{t_J}\right)^{\frac{2}{3}}$	$H(t) = \frac{1}{3t}$	Yes
		$a(\tau) = \left(\frac{\tau}{\tau_J}\right)^2$	$\mathcal{H}(\tau) = \frac{2}{\tau}$	

Table 1: Equation of state, scale factor, and the Hubble parameter for the each cosmological eras. The last column shows the validity or violation of strong energy condition (SEC) during that era, i.e. if $\rho + 3P > 0$. Here $\mathcal{H} \equiv aH$.

3.1 Cosmological background

Modern cosmology is based on two key observational facts: i) the universe is expanding and ii) on large scales ($> 100Mpc$) the matter distribution is homogeneous and isotropic. The average spacetime is then described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2_{\text{FLRW}} = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2), \tag{66}$$

where t is the cosmic time, $a(t)$ is the scale factor, and $k = 0, +1, -1$ describe flat, positively curved and negatively curved spacelike 3-hypersurfaces, respectively. From now on, we restrict our discussion to the case of the flat universe with $k = 0$ which is favored by present observations.

For a perfect fluid with energy density, ρ, and equation of state, w

$$P = w\rho,$$

we have the Friedmann equations as

$$3M^2_{\text{Pl}}(\frac{\dot{a}}{a})^2 = \rho \quad \text{and} \quad M^2_{\text{Pl}}\frac{\ddot{a}}{a} = -\frac{1}{6}(1+3w)\rho. \tag{67}$$

The scale factor is a function of only time and during different cosmological eras is given in the table 1.

The cosmic time, t, is related to the conformal times, τ, as

$$\tau \equiv \int \frac{dt}{a(t)}. \tag{68}$$

Solving for the scale factor in (67), we have $a(t)$ in terms of the cosmic time and conformal time respectively as

$$a(t) = \left(\frac{t}{t_I}\right)^{\frac{2}{(1+3w)+2}} \quad \text{and} \quad a(\tau) = \left(\frac{\tau}{\tau_I}\right)^{\frac{2}{(1+3w)}}, \tag{69}$$

where t_I and τ_I are some positive constants.

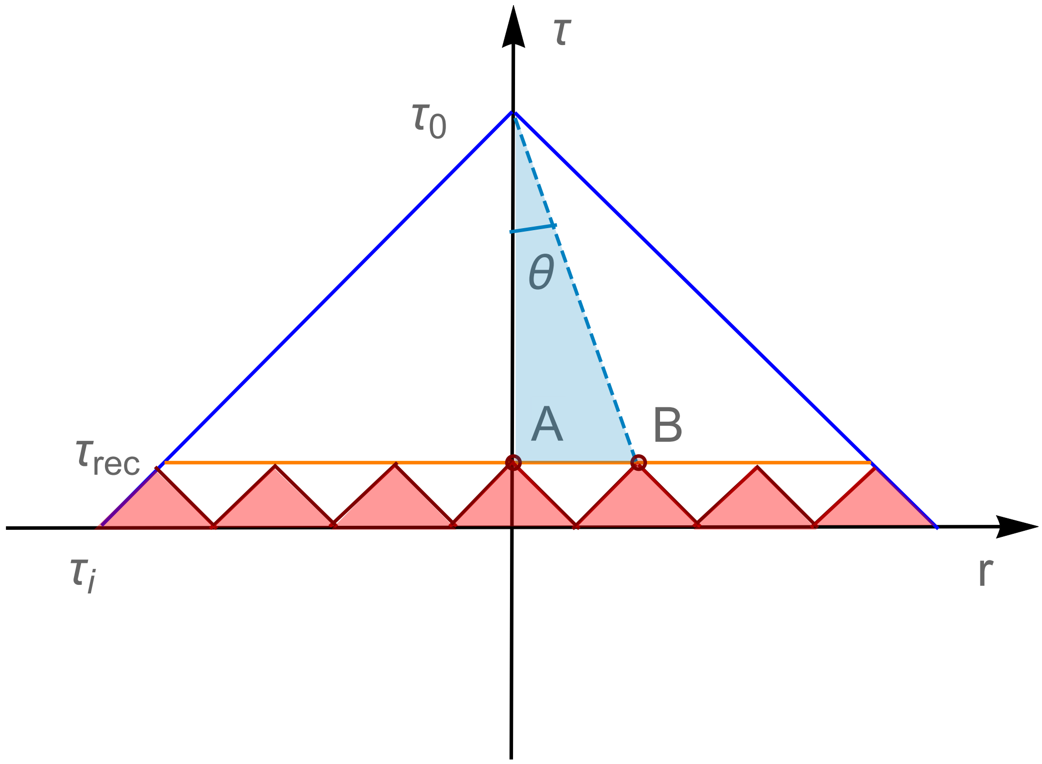

Figure 6: The (comoving) causal past of an observer today at τ_0 (redshift $z_0 = 0$), in FLRW spacetime made of ONLY ordinery matter, i.e. $1 + 3w > 0$. The orange line shows the last scattering surface, τ_{rec} (at redshift $z_{rec} \simeq 1090$). A and B are two causaly disconnected points at last scattering surface. In fact, the angle spaned by the shaded area at the last scattering surface is the comoving horizon at recombination.

In order to understand the causal structure of the cosmological spacetime, $ds^2 = 0$, we consider the radial null geodesics given as

$$r(\tau) - r(\tau_I) = \int_{\tau_I}^{\tau} d\tau' = \int_{a_I}^{a} \frac{d\ln a'}{a'H(a')}, \tag{70}$$

which is specified in terms of the comoving (particle) horizon, $(aH)^{-1}$, as

$$(aH)^{-1} = \frac{(1+3w)}{2}\tau_I \; a^{(1+3w)/2}. \tag{71}$$

Horizon problem:

Ordinary forms of matter, with positive pressure, satisfy the strong energy condition (SEC), i.e. $\rho + 3P > 0$. Thus from (71), the comoving horizon increases as the universe expands. Now, let us compute the angle spanned by the comoving horizon at recombination, θ. As we see in figure 6, θ is given as

$$\sin\theta = \frac{2(\tau_{rec} - \tau_i)}{\tau_0 - \tau_{rec}}. \tag{72}$$

On the other hand, we can read the τ integral in (68) as

$$\tau - \tau_I = -\int_{z_1}^{z_2} \frac{dz}{H(z)}, \tag{73}$$

where z is the redshift parameter

$$1 + z = \frac{1}{a(z)}, \tag{74}$$

in which we set the scale factor today to unity, $a_0 = 1$. Moreover, the Hubble parameter is

$$H(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_\gamma(1+z)^4 + \Omega_\Lambda}, \tag{75}$$

where $\Omega_m = 0.3$, $\Omega_\gamma = \frac{\Omega_m}{1+z_{eq}}$, and $\Omega_\Lambda = 1 - \Omega_m$ are the matter, radiation, and (late time) dark energy fraction ($z_{eq} = 3400$). Using (73) and (75) in (72) and solving the integral, we find

$$\theta_c \simeq 2.3^\circ. \tag{76}$$

Therefore, causal theories should have vanishing correlation functions for $\theta > \theta_c = 2.3^\circ$ and therefore CMB at the time of decoupling naively should be consisted of about 10^4 causally disconnected patches. However, we observe an almost perfectly uniform CMB temperature field across super-horizon scales at recombination. As we see in the following, cosmic inflation, an early period of accelerated expansion in which the SEC is violated, solves the horizon problem dynamically and allows our universe to arise from generic initial conditions.

3.2 Cosmic inflation

The inflation paradigm postulates a brief period (within 10^{-34} s) of quasi-exponential accelerated expansion during which the scale factor increased by over 60 e-folds. This considerable expansion is sourced by a negative pressure component in energy-momentum of the matter contents and drives the universe towards almost perfect homogeneity, isotropy, and flatness that we have observed.

Our discussion of horizon problem was based on the validity of SEC ($1 + 3w > 0$) and therefore growing Hubble sphere of the standard Big Bang cosmology. A simple solution therefore is a phase of decreasing Hubble radius in the early history of the universe,

$$\frac{d(aH)^{-1}}{dt} < 0 \quad \text{where} \quad 1 + 3w < 0. \tag{77}$$

If this lasts long enough, e.g. $\frac{a_{end}}{a_{initial}} = e^{60}$, the horizon problem can be solved.

During this phase, the spacetime is very close to a de Sitter space and the Hubble parameter, H, is almost constant. The deviation from a perfect de Sitter space is quantified in terms of two slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta = \frac{\dot{\epsilon}}{H\epsilon}, \tag{78}$$

which should be small during the slow-roll inflation. For a very nice lecture notes on differnt aspects of cosmic inflation see The Physics of Inflation by Daniel Baumann.

4 Gravitational waves in cosmological eras

We perturb the metric around cosmological background and keep only the transverse traceless perturbation, i.e. gravitational waves, as

$$ds^2 = a^2\Big(- d\tau^2 + (\delta_{ij} + \gamma_{ij})dx^i dx^j\Big). \tag{79}$$

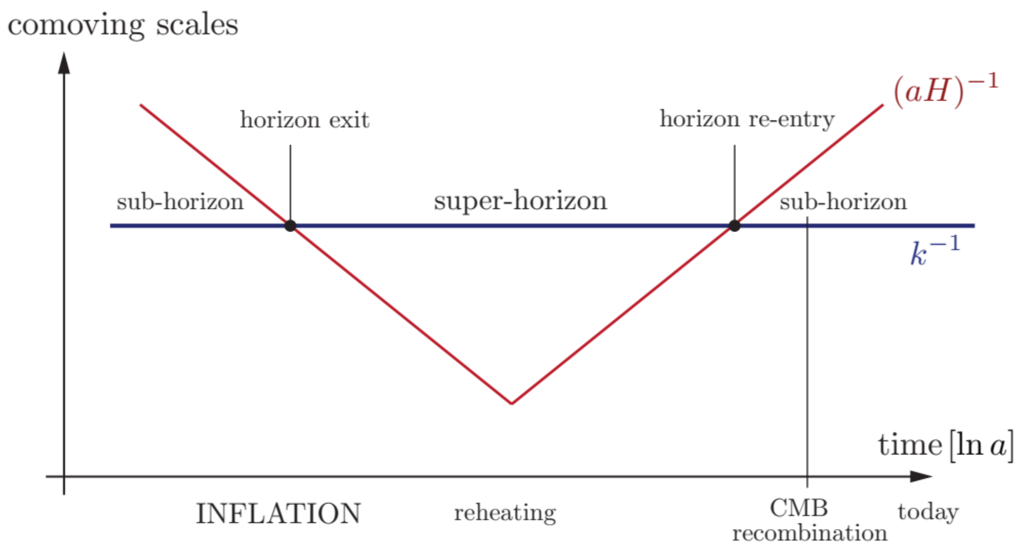

Figure 7: Evolution of conformal horizon, $(aH)^{-1}$ (red line), comparing with a given wavelength, k^{-1} (blue line), as a function of $\ln a$. Image credit: Inflation by Daniel Baumann [10].

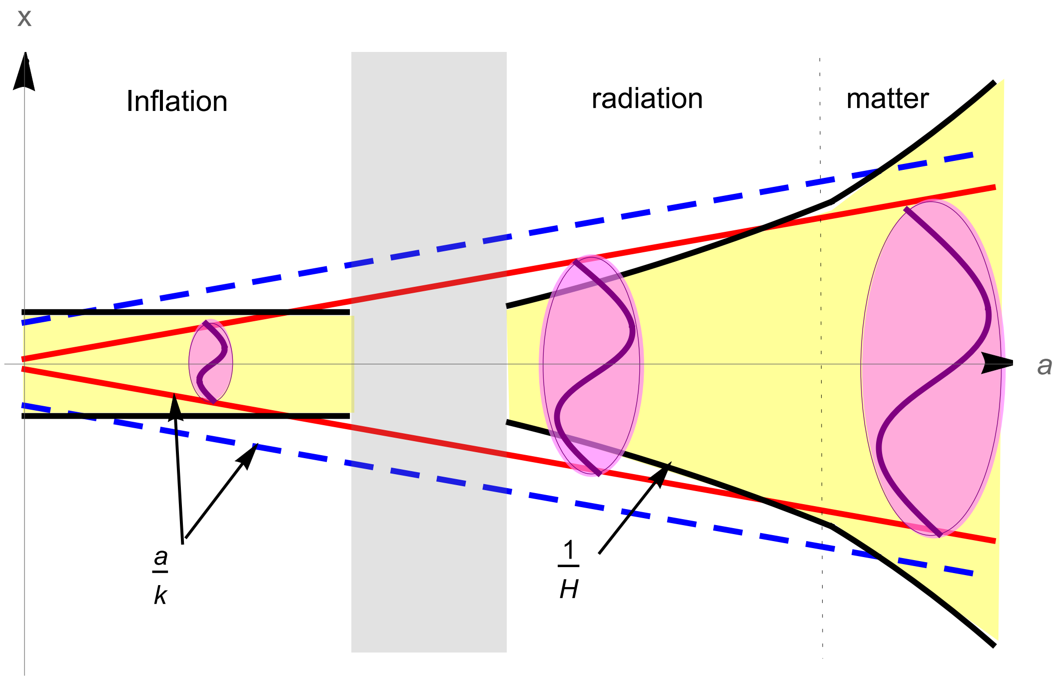

Figure 8: The Hubble radius, $\frac{1}{H}$, and the physical wavelength, $\frac{a}{k}$, as functions of the scale factor. Notice that X is the physcial coordinate, $X = ax$. The black line shows $1/H$, the red and blue lines are the physical wavelength of two modes which re-entered the cosmic Horizon during radiation domination (RD) and matter domination (MD) eras respectively. The shaded yellow region shows causally connected points, while the gray shaded one shows our ignorance about reheating.

It is more convenient to go to the Fourier space and expand the field in terms of its polarization states

$$\gamma_{ij}(\tau,\vec{k}) = \sum_{\sigma=+,\times} e^{\sigma}_{ij}(\hat{k})\gamma_\sigma(\tau,\vec{k}), \tag{80}$$

where $e^{+}_{ij}(\hat{k})$ and $e^{\times}_{ij}(\hat{k})$ are the plus and cross polarization tensors respectively. The field equation of $\gamma_\sigma(\tau,\vec{k})$ is

$$\gamma''_\sigma(\vec{k}) + 2\mathcal{H}\gamma'_\sigma(\vec{k}) + k^2\gamma_\sigma(\vec{k}) = 0, \tag{81}$$

where again a prime denotes a derivative with respect to the conformal time and $\mathcal{H} = aH$. We can extract analytically some general features of the solution by using the field redefinition

$$h_\sigma(\tau,\vec{k}) \equiv a\gamma_\sigma(\tau,\vec{k}). \tag{82}$$

The field equation takes the simple form

$$h''_\sigma(\tau,\vec{k}) + (k^2 - \frac{a''}{a})h_\sigma(\tau,\vec{k}) = 0, \tag{83}$$

where $'$ means derivative with respect to the conformal time and the effective mass term $\frac{a''}{a} = \frac{1}{2}(aH)^2(1-3w)$ is

$$\frac{a''}{a} = \begin{cases} 2\mathcal{H}^2(1-\epsilon) \simeq \frac{2}{\tau^2}(1-\epsilon) & \text{inflation} \\ 0 & \text{radiation-era}\ . \\ \frac{1}{2}\mathcal{H}^2 \simeq \frac{2}{\tau^2} & \text{matter-era} \end{cases} \tag{84}$$

When the mode functions are outside the horizon ($\frac{k}{a} > 1$), the gravitational wave is a constant while when it is well inside the horizon ($\frac{k}{a} \gg 1$), it is a simple oscillating function scales like $1/a$

$$\gamma_\sigma(\tau,\vec{k}) \propto \begin{cases} \gamma_0 & (\frac{k}{a} \ll 1) \\ \frac{1}{a}\sin(\frac{k}{a}+\alpha) & (\frac{k}{a} \gg 1) \end{cases}, \tag{85}$$

where γ_0 and α (a phase), are both given by the initial conditions. See figure 9. The energy density of the gravitational waves are given as

$$\rho_{GW} = \frac{1}{32\pi G}\sum_{\sigma=+,\times}\langle\dot{\gamma}^2_\sigma\rangle, \tag{86}$$

where the expectation value denotes average over several wavelengths. For gravitational waves inside the horizon, we have $\dot{\gamma} \propto k\cos(\frac{k}{a}+\alpha)/a^2$ which gives

$$\rho_{GW} \propto a^{-4},$$

as it is expected from any form of radiation.

4.1 Inflation and primordial gravitational waves

Cosmic inflation generates primordial scalar perturbations that seeds all structure formation in the observable universe. More precisely, most of the inflationary models consistent with the date predict an adiabatic and almost non-Gaussian scalar perturbation, the comoving curvature perturbation ζ with a nearly scale invariant power spectrum as

$$\Delta^2_\zeta = A_s(k_*)(\frac{k}{k_*})^{n_s-1}, \tag{87}$$

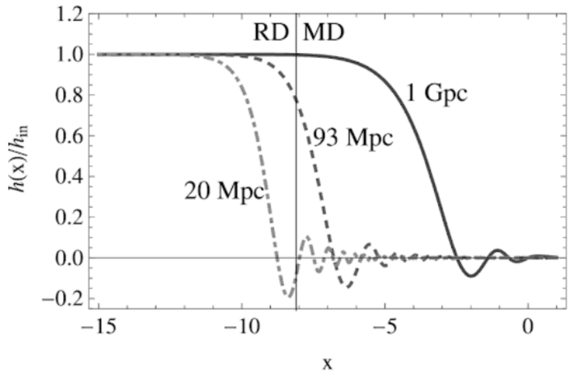

Figure 9: The time evolution of $\gamma_{+,\times}(\tau, k)$, normalized to its initial value as a fucntion of $x = \log(a)$, for three different values of the wavelength. Image credit: Gravitational Waves. Vol. 2: Astrophysics and Cosmology [11].

which specifies in terms of two numbers A_s, and the spectral tilt

$$n_s \equiv 1 + \frac{d \ln P_\zeta}{d \ln k}, \tag{88}$$

which is of the order of slow-roll parameters. Another critical prediction of inflation is the existence of a stochastic primordial GWs background (PGW) generated by tensor perturbations in the geometry of the very early universe. In this part, we discuss this mechanism. Just like CMB, this relic stochastic GWs is a random noise of GWs with no sharp, specific characters in either the time or frequency domains. However, GWs has an advantage over the CMB because while photons decoupled about 4×10^5 years after the big bang, primordial GWs could free-stream from times as early as (possibly) Planck scales.

We are interested in metric perturbations that correspond at present time to gravitational waves. The linear Einstein equation for such a metric perturbation is given by Eq. (7) where ∇_μ is the covariant derivative in FLRW metric and it is sourced by the anisotropic stress tensor of matter fields

$$\pi_{ij} = \delta T_{ij} - \frac{1}{3}\delta_{ij}\delta T^k_k. \tag{89}$$

More precisely, we have

$$\partial_0^2 \gamma_{ij} + 3H\partial_0\gamma_{ij} - a^{-2}\partial_k^2\gamma_{ij} = 8\pi G \pi^T_{ij}, \tag{90}$$

in which π^T_{ij} is the traceless and transverse part of π_{ij}. [4] Before going any further, note that the tensor sector of perturbations, h_{ij} and ${\pi^T}_{ij}$, are invariant under infinitesimal coordinate transformations. Hence they are *gauge invariant* and physical quantities.

- *Quantum origin of the perturbations:*

[4]Using the standard scalar, vector, tensor decompostion, we can decompose π^T_{ij} as

$$\pi_{ij} = \partial^2_{ij}\pi^S - \frac{1}{3}\partial^2\pi^S\delta_{ij} + 2\partial_{(i}\pi^V_{j)} + \pi^T_{ij},$$

where $\partial_i\pi^V_i = \partial_i\pi^T_{ij} = 0$ [12]. Being a perfect fluid or having irrotational flows are physical properties, thus their corresponding conditions are gauge-invariant. In other words, $\pi^S, {\pi^V}_i, {\pi^T}_{ij}$ are all invariant under infinitesimal space-time coordinate transformations.

For solving the field equation, we then need to set the initial condition. Assuming seeds of perturbations to be from the quantum fluctuations, so-called Bunch-Davies vacuum, imposes the initial value in the asymptotic past as

$$\lim_{\tau\to-\infty} h_{+,\times}(\tau,\vec{k}) = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \tag{91}$$

hence both polarization states have the same initial values. In general relativity and the absence of cosmological higher spin fields, e.g., gauge fields and fermions, to serve as a stochastic source for gravitational waves, the anisotropic stress is zero. Therefore, the resulting perturbations are originated by vacuum fluctuations

$$h_{\vec{k}}(\tau) \equiv h_+(\tau,\vec{k}) = h_\times(\tau,\vec{k}). \tag{92}$$

which are un-polarized.

4.1.1 Adiabatic perturbations

Interestingly, regardless of the cosmological era which we have, in the limit that the physical wavelength is much smaller than the Hubble rate, $k/a \ll H$, the field equation (81) has two solutions

$$h_{\vec{k},1}(\tau) = cst. \quad \text{and} \quad h_{\vec{k},2}(\tau) = \int_{\tau}^{\tau_{end}} \frac{d\tau'}{a^2(\tau')}. \tag{93}$$

which as we see, the second solution is decaying with time. Moreover, these are the solutions of a second order differential equation so in the absence of any new degree of freedom, the above are all the possible super horizon solutions. Therefore at late times and for a genetic initial condition, the gravitational waves eventually get dominated by $h_{\vec{k},1}$. The above solutions are the adiabatic modes which are the result of the Weinberg's adiabatic modes theorem [12]. Here we only use its automatic result, but later in section 5.0.1, I will get back to it and discuss this theorem. The above simple observation has an amazing physical consequence. Recalling that during inflation many Fourier modes leave the Hubble horizon which eventually during radiation or matter era return to our causal patch, the above implies that once the adiabatic mode is outside the horizon it is conserved and constant, regardless of the complicated and unknown physics inside the horizon. Thus, this powerful effect allows us to connect the distant past of our Universe to its recent past. Therefore, in the absence of the super-horizon entropy and anisotropic inertia perturbations, inflation predicts adiabatic fluctuations.

4.2 (Almost) scale invariant Power spectrum

Here by solving the field equation in (83). The general solution during inflation is expressed as a linear combination of Hankel functions

$$h_{\vec{k}}(\tau) \simeq \frac{\sqrt{\pi(k|\tau|)}}{2} e^{i(1+2\nu_R)\pi/4} \left(c_1 H^{(1)}_{\nu_R}(\tau) + c_2 H^{(2)}_{\nu_R}(\tau) \right), \tag{94}$$

where

$$\nu_R \simeq \frac{3}{2} + \epsilon. \tag{95}$$

Now, focusing on inflationary era and after imposing the usual Minkowski vacuum state for the inflationary solution in (94), we obtain $c_1 = 1$ and $c_2 = 0$ in (94). The 2-point function of gravitational waves is

$$\langle \gamma_{\vec{k}} \gamma_{\vec{k}'} \rangle = \frac{2\pi^2}{k^3} \delta^3(\vec{k} + \vec{k}') \Delta_T^2 \tag{96}$$

where Δ_T^2 is the power spectrum of the fluctuations. During inflation, the super-horizon ($k > aH$) tensor power spectrum is

$$\Delta_T^2 = \frac{2}{\pi^2} \frac{H^2}{M_{\rm Pl}^2} (\frac{k}{aH})^{n_T} \bigg|_{k=aH}, \tag{97}$$

and its deviation from exact scale invariant, tensor spectral index n_T, is

$$n_T = \frac{d \ln \Delta_T^2}{d \ln k} = -2\epsilon\,. \tag{98}$$

The power spectrum predicted by inflation is specified by the energy scale of inflation and is nearly (but not exactly) scale invariant. Another important quantity is the tensor-to-scalar ratio

$$r = \frac{\Delta_T}{\Delta_\zeta}, \tag{99}$$

which is the ratio of the power spectrum of GWs to the scalar curvature perturbations. The current upper limit on tensor fluctuations is

$$r_{0.05} < 0.07 \;\; \text{at } 95\% \text{ CL}$$

which comes from the latest joint analysis of Planck and BICEP2/Keck array measurements [13].

4.3 Nearly non-Gaussian stochastic field

At the level of linear approximation, inflationary fluctuations have a Gaussian probability. The statistical properties of an isotropic Gaussian fields are completely determined by the 2-point function

$$\langle \varphi_{k_1} \varphi_{k_2} \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2) P_\varphi(k_1), \tag{100}$$

while any odd-point function, e.g. its bispectrum, is exactly zero

$$\langle \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \rangle = 0. \tag{101}$$

This Gaussianity is a direct consequence of i) neglecting 2nd order terms in the equation of motion as well as ii) the cosmological principle. However, both of the above are not exact conditions during inflation and cosmic fluctuations break them both. As a result, the generated perturbations are not exactly Gaussian. The deviation from Gaussianity can be formulated in terms of the bispectrum as

$$\langle \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_\varphi(k_1, k_2, k_3). \tag{102}$$

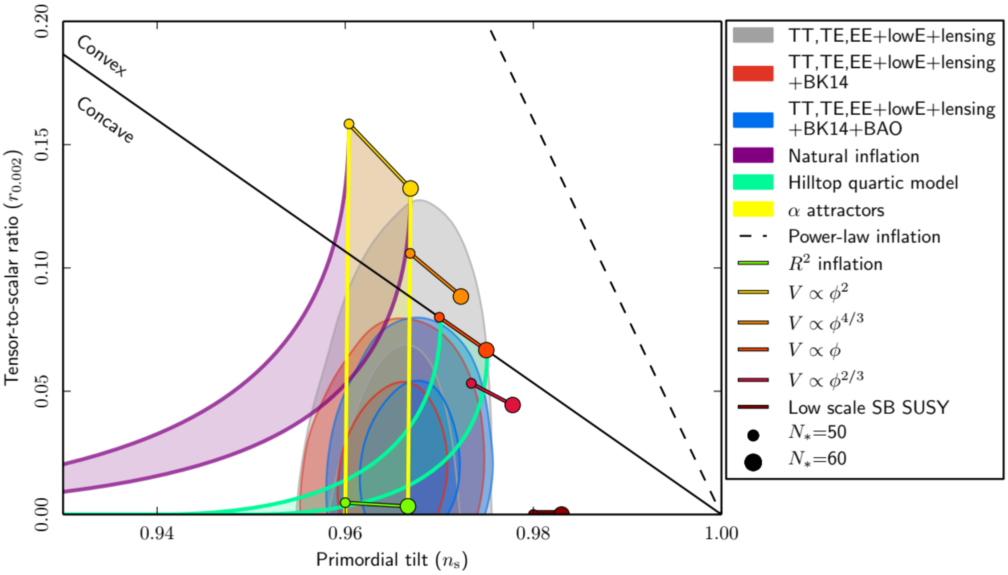

Figure 10: Marginalized joint 68 % and 95 % CL regions for n_s and r at $k = 0.002 Mpc^{-1}$ from Planck alone and in combination with BK14 or BK14 plus BAO data, compared to the theoretical predictions of selected inflationary models. Image credit: Planck 2018 results X: Constraints on inflation [14].

If the power spectrum is scale-invariant, then the shape of the Bispectrum only depends on two numbers as

$$B_\varphi(k_1, k_2, k_3) = k_1^{-6} B_\varphi(1, x_2, x_3), \tag{103}$$

where $x_2 = \frac{k_2}{k_1}$ and $x_3 = \frac{k_3}{k_1}$. In figure 12, we present the possible momentum configurations of the bispectrum. Considering k_1 to be not greater than k_2 and k_3, we have the following three possible limits. In the limit that $k_1 \ll k_2 \sim k_3$, the bispectrum is called squeezed. In case that $k_1 = k_2 = k_3$ it is called equilateral and if $k_2 = k_3$, it is folded. Depends on the details of the inflationary model, the bispectrum has a peak in different limits.

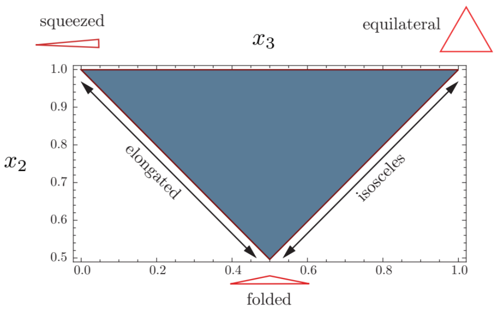

Figure 11: The shaded area shows the possible momentum configurations of the bispectrum. Image credit: Primordial Non-Gaussianity by Daniel Baumann.

Computing the gravitons 3-point function as well as the combination of 2 scalars and one graviton Bispecturms in the squeezed limit for the vacuum GWs in general relativity, one realizes that they are both slow-roll suppressed. The reason for that is the fact that the self-interaction terms are subleading by slow-roll parameters. In the squeezed limit, an adiabatic long (classical) wave gravitational wave acts as a coordinate transformation for the other short wavelength modes, either gravitons or scalars. That then leads to the Maldacena's powerful consistency relation which we review in the next section.

5 Adiabatic modes and inflationary consistency relations

Here, we briefly review Weinberg's adiabatic modes theorem and Maldacena's inflationary consistency relations. For more details on Weinberg's adiabatic modes see [12, 15] and for further information on Maldacena's inflationary consistency relations see the seminal paper [16]. The adiabatic modes and consistency relation can be extended to include the gradient expansions. Maldacena's original consistency relation has been generalized to the conformal consistency relation in [17] and to an infinite set of Ward identities in [18].

5.0.1 Weinberg's Adiabatic modes

We start with the construction of Weinberg's adiabatic modes in the Newtonian gauge which is essential for the derivation of the consistency relations. Fixing the gauge, uniquely specifies all the modes with finite momentum, $k \neq 0$. However, there are residual gauge symmetries for the very long-wavelength modes, which remain a symmetry of the gauge-fixed action. Starting with the flat FRW metric as the unperturbed metric

$$ds^2 = -dt^2 + a^2 d\mathbf{x}^2. \tag{104}$$

Here we focus on the implications of the diffeomorphism invariance on the nature of the very long-wavelength modes with $k/\mathcal{H} \ll 1$. Therefore, only the homogeneous diffeomorphisms are relevant. The general spatially homogeneous perturbed metric in the Newtonian gauge is

$$ds^2 = -(1+2\Phi(t))dt^2 + a^2\bigg((1-2\Psi(t))\delta_{ij} + \gamma_{ij}(t)\bigg)dx^i dx^j, \tag{105}$$

where Φ and Ψ are the Bardeen potentials and γ_{ij} is the traceless tensor perturbation, gravity wave. [5] One can decompose the general spatially homogeneous perturbed energy-momentum in the Newtonian gauge as

$$T_{00} = -\bar{\rho}g_{00} + \delta\rho(t) \quad \text{and} \quad T_{0i} = -(\bar{\rho}+\bar{P})\partial_i\delta u(t), \tag{106a}$$

$$T_{ij} = \bar{P}(t)g_{ij}(t) + a^2\bigg(\delta_{ij}\delta P(t) + \partial^2_{ij}\pi^S(t) + \pi^T_{ij}(t)\bigg), \tag{106b}$$

where a bar denotes an unperturbed quantity, $\delta\rho$, δP and δu are the perturbed density, pressure and velocity potential respectively. Moreover, π^S and π^T_{ij} are the scalar and tensor anisotropic inertia which characterize departures of $T_{\mu\nu}$ from the perfect fluid form. [6]

Under the action of diffeomorphism transformations

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \epsilon^\mu(t,\mathbf{x}), \tag{107}$$

[5] Here we neglect the vector perturbations due to their damping nature in most of the inflationary models including our axion-gauge field model.

[6] It is interesting to note that in the decomposition (106b), the effects of bulk viscosity are included in δp [15].

there is a ϵ^μ which generates spatially homogeneous transformations on the metric and preserve the Newtonian gauge [12]

$$\epsilon^0(t,\mathbf{x}) = -f(t) - \chi(\mathbf{x}), \tag{108a}$$

$$\epsilon^i(t,\mathbf{x}) = (\theta\delta_{ij} + \sigma_{ij})x^j - \partial_i\chi(\mathbf{x})\int\frac{dt}{a^2(t)}, \tag{108b}$$

where θ is a constant scalar and σ_{ij} is a constant, traceless and symmetric matrix[7], $\sigma_{ii} = 0$. Therefore, choosing the Newtonian gauge, we are still left with residual gauge symmetries for the zero wavenumber modes. These diffeomorphisms do not vanish at spatial infinity and therefore are called large gauge transformations. The scalar functions $f(t)$, $\chi(t)$ and θ act only on the scalar perturbations

$$\Phi(t) \mapsto \Phi(t) + \dot{f}(t) \quad \text{and} \quad \Psi(t) \mapsto \Psi(t) + \theta - Hf(t), \tag{109}$$

while keep the tensor perturbations untouched. On the other hand, σ_{ij} acts only on the gravitational waves as

$$\gamma_{ij}(t) \mapsto \gamma_{ij}(t) - 2\sigma_{ij}. \tag{110}$$

Therefore, if $\Phi(t)$, $\Psi(t)$ and $\gamma_{ij}(t)$ are solutions of the spatially homogeneous Einstein equations, their transformed quantities and their differences are also the spatially homogeneous solutions. In particular, in the scalar sector, we have spatially homogeneous solutions of the form

$$\Phi_{\mathrm{A}}(t) = -\dot{f}(t) \quad \text{and} \quad \Psi_{\mathrm{A}}(t) = Hf(t) - \theta, \tag{111}$$

which corresponds to a cosmic fluid given as

$$\delta\rho_{\mathrm{A}}(t) = -\dot{\bar{\rho}}f(t), \quad \delta P_{\mathrm{A}}(t) = -\dot{\bar{P}}f(t), \quad \delta u_{\mathrm{A}}(t) = f(t), \tag{112}$$

with a vanishing scalar anisotropy

$$\pi^S_{\mathrm{A}}(t) = 0. \tag{113}$$

That leads to a constant comoving curvature ($\mathcal{R} \equiv -\Psi + H\delta u$) and curvature perturbation ($\zeta \equiv -\Psi - H\delta\rho/\dot{\bar{\rho}}$)

$$\mathcal{R}_{\mathrm{A}}(t) = \zeta_{\mathrm{A}}(t) = \theta. \tag{114}$$

There is also a spatially homogeneous solution in the tensor sector as

$$\gamma^{\mathrm{A}}_{ij}(t) = 2\sigma_{ij}, \tag{115}$$

with a vanishing tensor anisotropic inertia

$$\pi^{T\mathrm{A}}_{ij}(t) = 0. \tag{116}$$

Up to now, the solutions (111) and (115) are only gauge degrees of freedoms for the $k = 0$ mode and the Weinberg's theorem relates them to the physical modes. The essential step in

[7] In general, $\epsilon^i(t,\mathbf{x})$ can have a constant term ϵ^i_0 as well as a term like $\omega_{ij}x^j$ where $\omega_{ij} = -\omega_{ji}$. Here, however, we ignored them because (due to the spatial translational and rotational symmetry of the background metric) they do not have any effects on the linear perturbed metric.

Weinberg's theorem is as follows. In case that the anisotropic inertia $\pi_{ij}(t,\mathbf{k})$ and the entropy perturbations, $\frac{\dot{\bar\rho}\delta P-\dot{\bar P}\delta\rho}{3(\bar\rho+\bar P)^2}$, vanish in the limit $k/\mathcal{H}\ll 1$, the spatially homogeneous solutions are extendible to modes with $k\neq 0$. Since solutions with non-zero wavenumbers have no residual gauge symmetry in the Newtonian gauge, these modes are physical. When solving the linearized Einstein equations, there are two scalar and two tensor physical solutions which are constant at $k/\mathcal{H}\ll 1$ (eq.s (114) and (115)), called adiabatic solutions.[8] One immediate consequence of this theorem is that these modes freeze out at horizon crossing and become indistinguishable from a redefinition of the background metric

$$\bar g_{\mu\nu}(t)+\delta g^{\rm A}_{\mu\nu}(t)=\bar{\bar g}_{\mu\nu}(\tilde x), \tag{117}$$

which implies

$$\delta g^{\rm A}_{\mu\nu}(t)=-\mathcal{L}_\epsilon\bar g_{\mu\nu}(t), \tag{118}$$

where $\mathcal{L}_\epsilon$ denotes the Lie derivative with respect to ϵ^μ.

5.0.2 Gaussianity and Maldacena's consistency relation

Now, we turn to the consistency relations which is a powerful probe of the early universe physics and holds under very general conditions, i.e., when the long-wavelength mode is adiabatic. Therefore, assuming Banch-Davis initial condition, the scalar consistency relation only holds for single clock inflationary models in which the entropy perturbation is zero. However, gravity waves consistency relations hold for more general inflationary models. More precisely, assuming Banch-Davis initial condition, only models in which $\pi^T_{ij}\neq 0$ at super-horizon scales can violate tensor consistency relations, e.g., solid inflation [19] and anisotropic inflation [20]. As shown in [21], such inflationary models violate the cosmic no-hair conjecture.

Since the tensor modes are the main focus of this work, here we present the consistency relation for the gravity waves. The scalar consistency relation is the same and one only needs to replace $\gamma^{\rm A}_{ij}$ with the (adiabatic) curvature perturbation $\zeta_{\rm A}$. The key physical point behind the consistency relations is the observation that the adiabatic long-wavelength modes can be removed by the local coordinate transformation of the background metric, i.e. (118). Hence, they act as a classical background for the short wavelength modes, which freeze out much later than the long mode. In particular, an n-point correlation function of the short modes can be written as

$$\langle\zeta(x_1)\zeta(x_2)\cdots\zeta(x_n)\rangle_{\gamma^{\rm A}_{ij}(x)}=\langle\zeta(\tilde x_1)\zeta(\tilde x_2)\cdots\zeta(\tilde x_n)\rangle, \tag{119}$$

which Taylor expanding RHS around x^i, we find the change of the short distance n-point correlation function as

$$\delta\langle\zeta(\tilde x_1)\zeta(\tilde x_2)\cdots\zeta(\tilde x_n)\rangle=\sum_{\rm I=1}^{n}\delta\vec{x}_{\rm I}.\vec\nabla_{\rm I}\langle\zeta(x_1)\zeta(x_2)\cdots\zeta(x_n)\rangle+\cdots, \tag{120}$$

where x^i and $\tilde x^i$ are related as

$$\tilde x^i=x^i+\frac{1}{2}\gamma^{\rm A}_{ij}x^j. \tag{121}$$

[8]This solution leads to equal values of $\delta\rho_x/\bar\rho_x$ for all individual elements in the cosmic fluid which explains the name adiabatic.

As a result, the (n+1)-point correlation function including the long-wavelength mode is given as

$$\langle \gamma^{\rm A}_{ij}(x)\zeta(x_1)\zeta(x_2)\cdots\zeta(x_n)\rangle \simeq \frac{1}{2}\left\langle \gamma^{\rm A}_{ij}(x)\gamma^{\rm A}_{kl}(x)\sum_{\rm I=1}^{n} x^k_{\rm I}\partial_l\langle\zeta(x_1)\zeta(x_2)\cdots\zeta(x_n)\rangle\right\rangle, \tag{122}$$

in which we only keep the dominate term that has the relevant contribution. The above equality is the consistency relation in real space. Going to the Fourier space, we can expand $\gamma_{ij}(\mathbf{q})$ as

$$\gamma_{ij}(\mathbf{q}) = \sum_{\lambda=\pm}\gamma_\lambda(\mathbf{q})\mathbf{e}_{ij}(\hat{q},\lambda), \tag{123}$$

where $\mathbf{e}_{ij}(\hat{q},\lambda)$ are the time-independent polarization tensors, $\mathbf{e}_{ij}(\hat{q},\lambda)\mathbf{e}^*{}_{ij}(\hat{q},\tilde{\lambda}) = 2\delta^{\lambda\tilde{\lambda}}$, in which $\lambda = \pm$ corresponding to the ± 2 helicity states. Moreover, the two-point function at late time is given as

$$\langle\gamma_\lambda(\mathbf{q})\gamma_{\lambda'}(\tilde{\mathbf{q}})\rangle = (2\pi)^3\delta^{(3)}(\mathbf{q}+\tilde{\mathbf{q}})P^{\rm vac}_\gamma(q)\delta^{\lambda\lambda'}, \tag{124}$$

in which

$$P^{\rm vac}_\gamma(q) = q^{-3}\left(\frac{H^2}{M^2_{\rm Pl}}\right), \tag{125}$$

is the power-spectrum. Then using the above and neglecting the gradients, we arrive at the Maldacena's consistency relation

$$\langle\gamma_\lambda(\mathbf{q})\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\cdots\zeta_{\mathbf{k}_n}\rangle' \simeq -\frac{1}{2}P^{\rm vac}_\gamma(q)\sum_{\rm I=1}^{n}\mathbf{e}_{ij}(\hat{q},\lambda)k_{{\rm I}i}\partial_{k_{{\rm I}j}}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\cdots\zeta_{\mathbf{k}_n}\rangle' \quad \text{for} \quad q\to 0, \tag{126}$$

where the prime in $\langle\cdots\rangle$ indicates that we extracted the prefactor $(2\pi)^3\delta^{(3)}(\mathbf{q}+\sum_{\rm I}^n\mathbf{k}_{\rm I})$ associated to momentum conservation. Note that the above result follows directly from the fact that adiabatic long-wavelength gravity wave is equivalent to a change of coordinate for the short wavelength mode, regardless of the super-horizon behavior of the short modes. Therefore, as far as our inflationary system generates adiabatic tensor perturbations the above consistency relation holds.

6 Gravitational waves and CMB

We now come to the effect of GWs on CMB anisotropies.

6.1 Integrated SachsWolfe effect

For that purpose we need to compute the effect of the perturbations on the photon trajectories

$$\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma}(x^\alpha(\lambda))P^\nu P^\sigma = 0, \tag{127}$$

where $P^\mu = \frac{dx^\mu}{d\lambda}$ is the photon's 4-momentum and $\Gamma^\mu_{\nu\sigma}(x^\alpha(\lambda))$ is the Christoffel symbols up to the first order of perturbations. The photon's energy measured with respect to the rest frame of the baryon-photon fluid is

$$E(x^\nu) \equiv -P^\mu(x^\nu)u_\mu(x^\nu). \tag{128}$$

Suppose that a photon emitted with energy E_E at a point, x^μ_E, and is observed at x^μ_O with E_O. Then, the temperature at x^μ_E and at x^μ_O in a rest frame of baryon-photon fluid are given as

$$\frac{E_O}{E_E} = \frac{T_O}{T_E}. \tag{129}$$

From the above and the photon's geodesics (127), the temperature perturbation today in direction $\hat{n}$ is given in terms of the perturbations at the time of the photon decoupling, τ_{dec}, as

$$\begin{aligned} \frac{\Delta T(\hat{n})}{T} &= \frac{1}{4}\delta_\gamma(\tau_{dec}, r_L\hat{n}) + \Psi(\tau_{dec}, r_L\hat{n}) + \Phi(\tau_{dec}, r_L\hat{n}) \\ &+ \int_{\tau_{dec}}^{\tau_0} d\tau \left[\Phi'(\tau, (\tau_0-\tau)\hat{n}) + \Psi'(\tau, (\tau_0-\tau)\hat{n}) - \frac{1}{2}n^i n^j h'_{ij}(\tau, (\tau_0-\tau)\hat{n})\right], \end{aligned} \tag{130}$$

where $r_L = \tau_0 - \tau_{dec}$, $\delta_\gamma(\tau_{dec}, r_L\hat{n})$ is the density perturbation in the radiation fluid and n^i is the propagation direction of the photon. The second line is the integrated SachsWolfe effect which receives contributions from the whole period from τ_{rec} to the present, τ_0. That includes the effect of a time-varying gravitational wave during the passage of the CMB photons from the last scattering to the present.

6.2 CMB polarization

Up to now, we considered the temperature fluctuations in CMB which its two-point temperature correlation function provides an important characterization of CMB anisotropies. In addition to the temperature anisotropies, there is more information to be gained by measuring CMB. More precisely, CMB photons are expected to be polarized due to the Thomson scattering by free electrons before decoupling. The polarization of CMB photons provides the cleanest and a promising method to detect primordial gravitational waves.

Figure 12: Left: A brief thermal history: nucleosynthesis, thermalization, recombination and reionization. Image credit: from Wyne Hu lecture notes [22]. CMB formed during the recombination epoch at redshift $z_{rec} \simeq 1090$, when free electrons became bound into hydrogen and helium. The universe back to full ionization again around $z \sim 10$ leaving an extended neutral period between recombination and reionization. Right: Thomson scattering of CMB radiation by a free electron. An unpolarized radiation with quadratic anisotropy becomes linearly polarized by Thomson scattering.

The polarization of an electromagnetic wave is described in terms of the Stokes parameters. Consider a monochromatic electromagnetic plane wave that propagates along the direction $\hat{x}_3$, can be decomposed as

$$\vec{E} = e^{-i(wt-\vec{k}.\vec{x})}\begin{pmatrix} E_1 e^{i\theta_1} \\ E_2 e^{i\theta_2} \\ 0 \end{pmatrix}. \tag{131}$$

We can describe the above radiation field with the 4 Stocks parameters

$$I \equiv \langle E_1^2\rangle + \langle E_2^2\rangle, \tag{132}$$
$$Q \equiv \langle E_1^2\rangle - \langle E_2^2\rangle, \tag{133}$$
$$U \equiv 2\langle E_1 E_2 \cos(\theta_1 - \theta_2)\rangle, \tag{134}$$
$$V \equiv 2\langle E_1 E_2 \sin(\theta_1 - \theta_2)\rangle, \tag{135}$$

where only 3 of the above are independent, i.e.

$$I^2 = Q^2 + U^2 + V^2.$$

See figure 13 for the polarization corresponding to each of the Stokes parameters.

- The quantity I is total intensity of the wave, here the temperature anisotropy.
- Q describes the difference between the linear polarization in $\hat{e}_1$ and $\hat{e}_2$ directions.
- The U and V parameters give information on the phases. In particular, expanding the wave in the ± 1 helicity polarization states, $\hat{e}_\pm = (\hat{e}_1 \pm i\hat{e}_2)/\sqrt{2}$, we have

$$V = \langle E_+\rangle^2 - \langle E_-\rangle^2, \tag{136}$$

  which represents the difference between the positive and negative helicity intensities. The U can be written as

$$U = 2\langle E_+ E_- \sin(\theta_+ - \theta_-)\rangle. \tag{137}$$

- Note that the mechanisms that generates CMB polarization produces only linear polarizations, and no circular one. Therefore, in the absence of any parity violation interaction during inflation, we can set $V = 0$.
- Finally, the angle of polarization is

$$\tan\alpha = \frac{E_2}{E_1}. \tag{138}$$

Under a rotation of angle φ around the $\hat{e}_3$ as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{139}$$

the intensity, I, and the helical (circular) polarization, V, remain invariant. However, the linear polarizations, Q, and U, transforms like spin-2 fields

$$\big(Q'(\hat{e}_3) \pm iU'(\hat{e}_3)\big) = e^{\pm 2i\varphi}\big(Q(\hat{e}_3) \pm iU(\hat{e}_3)\big). \tag{140}$$

Therefore, I and V can be expanded in terms of scalar (spin-0) spherical harmonics, just as the temperature is expanded as

$$\Theta(\hat{n}) = \sum_{l,m} a_{lm}\ Y_{lm}(\hat{n}), \tag{141}$$

while $Q \pm iU$ can be expanded in terms of spin-weighted (spin-2) spherical harmonics as

$$Q(\hat{n}) \pm iU(\hat{n}) = \sum_{l,m} \tilde{a}^{\pm}_{lm}\ {}_{\pm 2}Y_{lm}(\hat{n}). \tag{142}$$

For any arbitrary $\hat{n}$ direction, a function ${}_sf(\hat{n})$ defined on the sphere is called to be of spin s if under a rotation of angle φ around the $\hat{n}$, it transforms as

$${}_sf'(\hat{n}) = e^{-is\varphi}\, {}_sf(\hat{n}).$$

Any function of spin-s on the sphere can be expanded in terms of the spin-s weighted spherical harmonics, ${}_sY_{lm}(\hat{n})$. The spin-weighted spherical harmonics are related to the scalar (spin-0) spherical harmonics as

$$\begin{aligned} {}_sY_{lm}(\hat{n}) &\equiv \sqrt{\tfrac{(l-s)!}{(l+s)!}}(\partial^+)^s Y_{lm}(\hat{n}) && \text{for} \quad 0 \le s \le l && (143) \\ {}_sY_{lm}(\hat{n}) &\equiv \sqrt{\tfrac{(l+s)!}{(l-s)!}}(-1)^s(\partial^-)^{-s} Y_{lm}(\hat{n}) && \text{for} \quad -l \le s \le 0, && (144) \end{aligned}$$

where $\partial^{\pm}$ are the spin raising and lowering operators given as

$$\partial^{\pm}\ {}_sf(\hat{n}) = -\sin^{\pm s}\theta \left(\partial_\theta \pm \frac{i}{\sin\theta}\partial_\phi\right)\left[\sin^{\mp s}\theta\ {}_sf(\hat{n})\right]. \tag{145}$$

Using (143) and (144), we can express the polarization as

$$(\partial^{\pm})^2\big(Q(\hat{n}) \pm iU(\hat{n})\big) = \sqrt{\frac{(l+2)!}{(l-2)!}} \sum_{l,m} \tilde{a}^{\pm}_{lm}\ Y_{lm}(\hat{n}). \tag{146}$$

Since the Stokes parameters, $Q \pm iU$, are not invariant under rotation, it is more convenient to expand them as

$$\big(Q(\hat{n}) \pm iU(\hat{n})\big) = \sum_{l,m} \left(a^E_{lm} \pm i a^B_{lm}\right)\ {}_{\pm 2}Y_{lm}(\hat{n}). \tag{147}$$

Therefore, one can define two scalar (spin-0) fields instead of the spin-2 fields, $Q \pm iU$, as

$$E(\hat{n}) = \sum_{l,m} a^E_{lm} Y_{lm}(\hat{n}) \quad \text{and} \quad B(\hat{n}) = \sum_{l,m} a^B_{lm} Y_{lm}(\hat{n}). \tag{148}$$

See left panel of figure 14.

- The scalar quantities E and B, completely determine the linear polarization fields.
- The E-mode is curl-free and even under the action of parity. Its polarization vectors are radial around cold sports (under dense) and tangential around hot spots (over dense).
- The B-mode is the divergence-free field and odd under parity. Its polarization vectors have vorticity around the under and over dense areas.

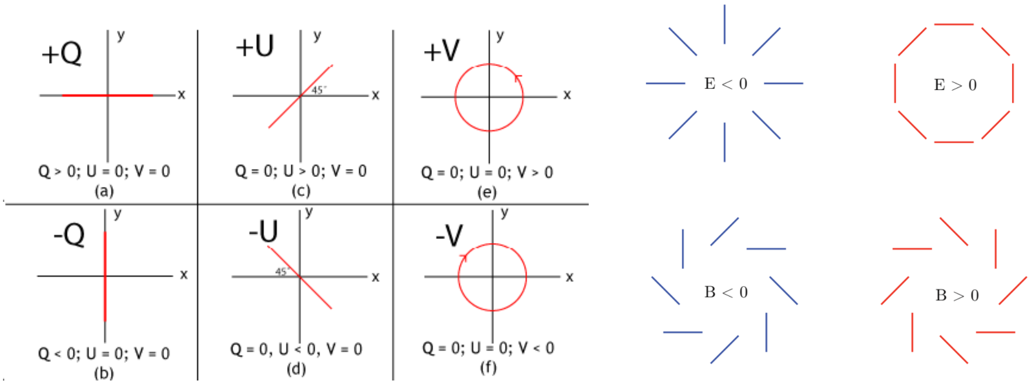

Figure 13: Left: The polarization corresponding to $U = V = 0$ (a and b), $Q = V = 0$ (c and d), and $Q = U = 0$ (e and f). Right: E-mode and B-mode patterns of polarization. The polarization vectors around a cold spot and hot spot are shown in blue and red respectively.

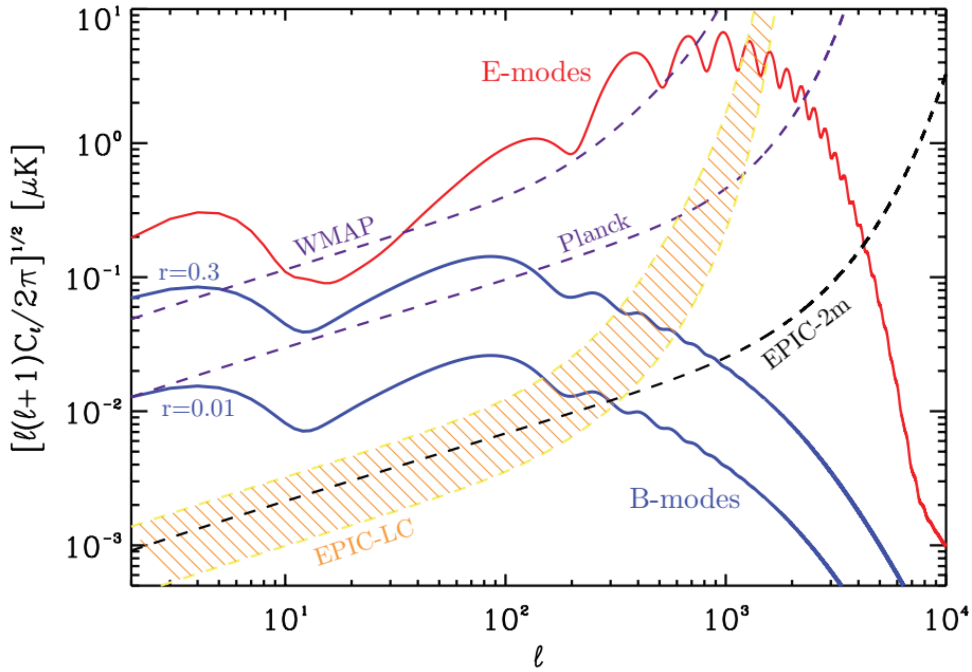

Figure 14: E- and (delensed) B-mode power spectra for a tensor-to-scalar ratios $r = 0.3$, and for $r = 0.01$. Shown are also the experimental sensitivities for WMAP, Planck and two different realizations of a future CMB satellite (CMBPol) (EPIC-LC and EPIC-2m).

- The angular power spectra are defined as

$$C_l^{XY} \equiv \frac{1}{2l+1} \sum_m \langle a_{lm}^X a_{lm}^Y \rangle \quad \text{where} \quad X, Y = T, E, B. \tag{149}$$

  The autocorrelations of E- and B-modes, denoted by EE and BB, are presented in the right panel of figure 14.

- Scalar perturbations create only E-modes and no B-modes. However, vector perturbations create mainly B-modes.

- Tensor perturbations create both E-modes and B-modes.

Primordial gravitational waves and B-modes have not yet been detected. Recalling that scalars do not produce B-modes while tensors do, a detection of primordial B-modes is a smoking gun of primordial gravitational waves, and therefore of inflation.

References

[1] N. T. Bishop and L. Rezzolla, *Extraction of Gravitational Waves in Numerical Relativity*, *Living Rev. Rel.* **19** (2016) 2, [`arXiv:1606.02532`].

[2] L. Bieri, D. Garfinkle, and N. Yunes, *Gravitational Waves and Their Mathematics*, `arXiv:1710.03272`.

[3] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, `arXiv:1703.05448`.

[4] R. M. Wald, *General Relativity.* Chicago Univ. Pr., Chicago, USA, 1984.

[5] H. Bondi, *Plane gravitational waves in general relativity*, *Nature* **179** (1957) 1072–1073.

[6] H. Bondi, F. A. E. Pirani, and I. Robinson, *Gravitational waves in general relativity. 3. Exact plane waves*, *Proc. Roy. Soc. Lond.* **A251** (1959) 519–533.

[7] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond.* **A269** (1962) 21–52.

[8] R. M. Wald, *Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant*, *Phys. Rev.* **D28** (1983) 2118–2120.

[9] L. Bieri and P. T. Chruciel, *Future-complete null hypersurfaces, interior gluings, and the Trautman-Bondi mass*, `arXiv:1612.04359`.

[10] D. Baumann, *Inflation*, in *Physics of the large and the small, TASI 09, proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder, Colorado, USA, 1-26 June 2009*, pp. 523–686, 2011. `arXiv:0907.5424`.

[11] M. Maggiore, *Gravitational Waves. Vol. 2: Astrophysics and Cosmology.* Oxford University Press, 2018.

[12] S. Weinberg, *Cosmology.* Oxford University Press, New York, USA, 2008.

[13] **BICEP2, Keck Array** Collaboration, P. A. R. Ade et al., *Improved Constraints on Cosmology and Foregrounds from BICEP2 and Keck Array Cosmic Microwave Background Data with Inclusion of 95 GHz Band*, *Phys. Rev. Lett.* **116** (2016) 031302, [`arXiv:1510.09217`].

[14] **Planck** Collaboration, Y. Akrami et al., *Planck 2018 results. X. Constraints on inflation*, `arXiv:1807.06211`.

[15] S. Weinberg, *Adiabatic modes in cosmology*, *Phys. Rev.* **D67** (2003) 123504, [`astro-ph/0302326`].

[16] J. M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013, [`astro-ph/0210603`].

[17] P. Creminelli, J. Noreña, and M. Simonović, *Conformal consistency relations for single-field inflation*, *JCAP* **1207** (2012) 052, [`arXiv:1203.4595`].

[18] K. Hinterbichler, L. Hui, and J. Khoury, *An Infinite Set of Ward Identities for Adiabatic Modes in Cosmology*, *JCAP* **1401** (2014) 039, [arXiv:1304.5527].

[19] S. Endlich, B. Horn, A. Nicolis, and J. Wang, *Squeezed limit of the solid inflation three-point function*, *Phys. Rev.* **D90** (2014), no. 6 063506, [arXiv:1307.8114].

[20] N. Bartolo, S. Matarrese, M. Peloso, and A. Ricciardone, *Anisotropic power spectrum and bispectrum in the $f(\phi)F^2$ mechanism*, *Phys. Rev.* **D87** (2013), no. 2 023504, [arXiv:1210.3257].

[21] A. Maleknejad and M. M. Sheikh-Jabbari, *Revisiting Cosmic No-Hair Theorem for Inflationary Settings*, *Phys. Rev.* **D85** (2012) 123508, [arXiv:1203.0219].

[22] W. Hu, *Lecture Notes on CMB Theory: From Nucleosynthesis to Recombination*, arXiv:0802.3688.