Exercise sheet 12

Baker-Campbell-Hausdorff formula
Consider the operators X and Y. Their commutator is represented by:

$$\mathrm{ad}_X Y = [X, Y] = XY - YX. \tag{1}$$

Exercise 12 - 1

Our first aim is to calculate $e^X Y e^{-X}$.

a) Write XY as a combination of YX and $\mathrm{ad}_X Y$. (1 point)

We define $\mathrm{ad}_X^{k+1} Y = \mathrm{ad}_X(\mathrm{ad}_X^k Y) = [X, \mathrm{ad}_X^k Y]$ with the base case $\mathrm{ad}_X^0 Y = Y$. Thus, $[X, [X, Y]]$ can be written as $\mathrm{ad}_X(\mathrm{ad}_X Y) = \mathrm{ad}_X^2 Y$ and so on.

We now use induction to prove that

$$X^n Y = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{ad}_X^k Y) X^{n-k} \quad \forall n. \tag{2}$$

The base case is $X^0 Y = (\mathrm{ad}_X^0 Y) X^0 = Y$, which holds trivially. The case $n = 1$ is part (a).

b) Assume now that

$$X^n Y = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{ad}_X^k Y) X^{n-k} \tag{3}$$

and use this to show that $X^{n+1} Y = \sum_{k=0}^{n+1} \binom{n+1}{k} (\mathrm{ad}_X^k Y) X^{n+1-k}$. (3 points)

Thus, we have inductively proved equation 2

We define $e^{\mathrm{ad}_X} = \sum_{n=0}^{\infty} {}^1\!/\!{}_{n!}\, \mathrm{ad}_X^n$.

c) Use the expansion of e^X to show that $e^X Y = (e^{\mathrm{ad}_X} Y) e^X$. (3 point)
Hint: Use the fact that $\sum_{n=0}^{\infty} \sum_{k=0}^{n} = \sum_{k=0}^{\infty} \sum_{j=n-k=0}^{\infty}$.

It is thus seen that $e^X Y e^{-X} = e^{\mathrm{ad}_X} Y$.

d) Use the fact that $e^X Y = (e^{\mathrm{ad}_X} Y) e^X$ to show that

$$e^X e^Y = e^{e^{\mathrm{ad}_X}(Y)} e^X. \tag{4}$$

(2 points)
Hint: First consider $e^X Y^2$ and then $e^X Y^n$.

Note that $e^{e^{\mathrm{ad}_X}(Y)} e^X \neq e^{e^{\mathrm{ad}_X}} (Y) e^X$! Think about the difference between the two.

Exercise 12 - 2

We now calculate the expectation for particular operators.

a) Consider operators X and Y such that their commutator $\mathrm{ad}_X Y = [X, Y]$ commutes with X. Calculate the expression in equation 4. (2 points)

Consider now $X = \hat{a}^x = D^{xz}\, {}^{d}\!/\!{}_{dm^z}$ and $Y = (\hat{a}^\dagger)^y = m^y$ from the lecture.

b) Calculate the commutator $[\hat{a}^x, (\hat{a}^\dagger)^y]$ and show that this commutes with $\hat{a}^x$. (1 point)

c) Move $e^{\hat{a}^x}$ through $e^{(\hat{a}^\dagger)^y}$ in $e^{\hat{a}^x}e^{(\hat{a}^\dagger)^y}$ using the results above. (1 points)

d) Use the results from the above exercises and the lecture notes to calculate $\langle e^{s^x}e^{s^y}e^{s^z}\rangle = e^{\hat{\Phi}^x}e^{\hat{\Phi}^y}e^{\hat{\Phi}^z}1$. Here, $\hat{\Phi}^x = \hat{a}^x + (\hat{a}^\dagger)^x$. (2 points)

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html