*PD Dr. Torsten Enßlin*
*Vishal Johnson, Ming Huang*

# Exercise sheet 11

**Exercise 11 - 1**

**Non-Gaussianities in the Cosmic Microwave Background**

The CMB is the relic radiation from the time of (re)combination 380 000 years after the Big Bang. Its variations in temperature with direction reflect the variations in the gravitational potential $\varphi$ at this time. The simplest inflationary scenarios predict these variations to be a nearly Gaussian field. Measuring deviations from Gaussianity in this field is within the focus of contemporary research in cosmology.

The *local type* of non-Gaussianities can be modeled according to

$$\varphi = \phi + f_{\mathrm{nl}} \left( \phi^2 - \left\langle \phi^2 \right\rangle_{(\phi)} \right) = \phi + f_{\mathrm{nl}} \left( \phi^2 - \hat{\Phi} \right), \tag{1}$$

where

$$\phi \hookleftarrow \mathcal{G}(\phi, \Phi) \tag{2}$$

is an auxilliary Gaussian field and the degree of non-Gaussianity of the primordial gravitational potential $\varphi$ is quantified by the parameter $f_{\mathrm{nl}}$, which shall be assumed here to be position-independent (i.e. one number).

During cosmic recombination, the variations in the gravitational field are transformed into temperature variations in the electromagnetic radiation. This radiation has been measured in recent times by satellites, balloons, and ground based observatories. These two processes (imprinting onto temperature variations and measurement) can be combined in a linear response operator $R$. Additionaly, these measurements contain a Gaussian noise contribution, which is not correllated with the signal, i.e.

$$d = R\varphi + n, \ n \hookleftarrow \mathcal{G}(n, N). \tag{3}$$

**a)** Derive an expression for $H\left[d, \phi, f_{\mathrm{nl}}\right] = -\log\left(\mathcal{P}(d, \phi, f_{\mathrm{nl}})\right)$, assuming a flat prior for $f_{\mathrm{nl}}$.
(2 points)

Assume now uncorrelated and homogeneous noise, i.e.

$$N_{ij} = \delta_{ij}\sigma^2, \tag{4}$$

and unit response,

$$R_{ij} = \delta_{ij}. \tag{5}$$

**b)** Bring this Hamiltonian into the form

$$H\left[d, \phi | f_{\mathrm{nl}}\right] = H_0 - j^\dagger \phi + \frac{1}{2}\phi^\dagger D^{-1}\phi + \frac{1}{3!}\lambda^{(3)\dagger}\phi^3 + \frac{1}{4!}\lambda^{(4)\dagger}\phi^4 \tag{6}$$

and identify the terms $j$, $D$, $\lambda^{(3)}$, and $\lambda^{(4)}$, as well as the $f_{\mathrm{nl}}$-dependent part of $H_0$.
(3 points)

**c)** Write down the diagrammatic expansion for the logarithm of the partition function

$$\log Z_{f_{\mathrm{nl}}}(d) = \log \mathcal{P}(d|f_{\mathrm{nl}}) = \log \int \mathcal{D}\phi \ \mathcal{P}(d, \phi | f_{\mathrm{nl}}), \tag{7}$$

up to second order in $f_{\mathrm{nl}}$, i.e., use all diagrams that contain terms of lower than third order. You do not need to formulate the results algebraically, nor do you need to care about terms that are constant in $\phi$ and $f_{\mathrm{nl}}$.
(2 points)

**d)** Write down the diagrammatic version of the expectation value $\langle\phi\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}$ up to first order in $f_{\mathrm{nl}}$. Read off the corresponding algebraic formula for the expectation value in terms of $j$, $D$, $\lambda^{(3)}$, and $\lambda^{(4)}$.
(2 points)

**e)** Write down the diagrammatic version of the dispersion $\left\langle \left(\phi - \langle\phi\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}\right)\left(\phi - \langle\phi\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}\right)^\dagger \right\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}$ up to first order in $f_{\mathrm{nl}}$. Read off the corresponding algebraic formula for the dispersion in terms of $j$, $D$, $\lambda^{(3)}$, and $\lambda^{(4)}$.
(2 points)

**Wick/Isserlis' Theorem**
In the following set of exercises, we derive the Wick/Isserlis' theorem. This allows us to calculate the moments of Gaussian fields.

**Exercise 11 - 2**

**a)** Consider a Gaussian distributed random field $s \hookleftarrow \mathcal{G}(s,S)$ and a modification of it:

$$\tilde{\mathcal{G}}(s,S,J) = \frac{1}{\sqrt{|2\pi S|}} \exp\left(-\frac{1}{2}s^\dagger S^{-1} s + J^\dagger s\right). \tag{8}$$

Use the partition function $\mathcal{Z}(J) = \int \mathcal{D}s\, \tilde{\mathcal{G}}(s,S,J)$ to derive the following equality:

$$\frac{\delta}{\delta J_{x_n}} \cdots \frac{\delta \mathcal{Z}(J)}{\delta J_{x_1}}\bigg|_{J=0} = \langle s^{x_1} \dots s^{x_n}\rangle_{\mathcal{G}(s,S)}. \tag{9}$$

In this exercise, $\langle \quad \rangle = \langle \quad \rangle_{\mathcal{G}(s,S)}$ unless otherwise specified. (2 points)

**b)** Complete the square in the exponent for $\tilde{\mathcal{G}}(s,S,J)$ to derive an expression for $\mathcal{Z}(J)$ which only depends on $J$ and the covariance $S$. (2 points)

**c)** Using the above expressions derive $\langle s^{x_1}\rangle$ and $\langle s^{x_1}s^{x_2}\rangle$. (2 points)

**d)** Use the expansion of the exponential function:

$$e^x = 1 + x + {}^1\!/_{2!}x^2 \dots \tag{10}$$

and equation 9 to show that for odd $n$, $\frac{\delta}{\delta J_{x_n}} \cdots \frac{\delta \mathcal{Z}(J)}{\delta J_{x_1}}\Big|_{J=0} = \langle s^{x_1} \dots s^{x_n}\rangle_{\mathcal{G}(s,S)} = 0$. (2 points)

**Exercise 11 - 3**

**Combinatorial Interlude**

**a)** Consider a set $s$ of $n = r \times m$ objects. In how many ways can the $n$ objects be partitioned into $m$ sets of $r$ objects each? Denoting the set of all possible partitions as $\mathcal{P}^s_r$, we seek to find $|\mathcal{P}^s_r|$
(3 points)

For example, dividing the four objects $o_4 = \{o_1, o_2, o_3, o_4\}$ into two sets of 2 would have the partition:

$$\begin{aligned}\mathcal{P}^{o_4}_2 = \quad & \{\{\{o_1,o_2\},\{o_3,o_4\}\},\\ & \{\{o_1,o_3\},\{o_2,o_4\}\},\\ & \{\{o_1,o_4\},\{o_2,o_3\}\}\}\end{aligned} \tag{11}$$

**b)** For the set of objects $o_6 = \{o_1, o_2, o_3, o_4, o_5, o_6\}$, find the partition $\mathcal{P}_2^6$. (2 points)

**Exercise 11 - 4**

**a)** Combining the insights from the previous exercises, show that

$$\langle s^{x_1} \dots s^{x_n} \rangle = \begin{cases} 0 & n \text{ odd} \\ \sum_{p \in \mathcal{P}_2^{s^n}} \left( \prod_{\{s^{x_i}, s^{x_j}\} \in p} \langle s^{x_i} s^{x_j} \rangle \right) & n \text{ even} \end{cases} \tag{12}$$

where $s^n = \{s^{x_1}, \dots s^{x_n}\}$ is the set of points at which the correlation function is evaluated. (4 points)

**b)** Calculate $\langle s^{x_1} s^{x_2} s^{x_3} s^{x_4} \rangle$. (2 points)

*This exercise sheet will be discussed during the exercises.*
*Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,*
*Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,*

*https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html*