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Exercise sheet 10

Exercise 10 - 1

Assume a linear measurement of some field. Assume further a log-normal model for this field and an additive Gaussian noise term, i.e.

$$d = Re^s + n, \; s \hookleftarrow \mathcal{G}(s, S), \; n \hookleftarrow \mathcal{G}(n, N). \tag{1}$$

a) Derive the information Hamiltonian $H(s, d)$ for this problem.
(2 points)

b) Give a recursion relation of the type

$$m_{\text{MAP}} = f(m_{\text{MAP}}) \tag{2}$$

for the *maximum a posteriori* solution m_{MAP} of the signal field s.
(1 point)

Exercise 10 - 2

An inference problem with an unknown prior distribution can be treated with an hierarchical Bayesian model, which estimates parameters of the prior distribution from the data itself.

You want to reconstruct the results of a random process, which produces identically distributed and independent samples s_i. You have u independent but noisy measurements of this random process which follow the likelihood

$$P(d|s) = \mathcal{G}(d - s, N), \qquad \text{with} \quad N_{ij} = \delta_{ij}\sigma^2 \tag{3}$$

You assume this process is well described by a Gaussian with zero mean,

$$P(s|p) = \mathcal{G}(s, S), \qquad \text{with} \quad S_{ij} = \delta_{ij}p, \tag{4}$$

but you have no clue about its variance p. To filter out the noise contribution to each data point d_i, an estimate of p is needed. The only information you have about p is that it is positive definite. A useful parametrization is therefore $\eta \equiv \ln(p)$. Since there is no further information about η, you assume $P(\eta) = const.$

a) Write down the joint probability distribution $P(d, s, \eta)$ and marginalize out s. Write down your solution using σ and η explicitly in order to get rid of all determinants (3 points).

Hint: You can drop all factors which do not depend on s or η, but remember that some of the normalizations are η-dependent.

b) Write down the Hamiltonian $\mathcal{H}(d, \eta) \equiv -\ln P(d, \eta) + const.$ and calculate its first derivative with respect to η (2 points).

c) Set the first derivative of the Hamiltonian to zero to derive the *maximum a posteriori* (MAP) solution for η (2 points).

Hint: It is useful to replace e^η with p and then solve for p.

d) Plug in the MAP estimator for η into the Wiener Filter formula for $\langle s \rangle_{\mathcal{P}(s|d,\eta)}$ to derive an η-independent estimator for s (1 point).

e) There is a possible problem with the resulting filter. What is it? (1 point)

Exercise 10 - 3

Consider a real-valued signal field s with a Gaussian prior,

$$\mathcal{P}(s) = \mathcal{G}(s, S), \tag{5}$$

that is observed with an instrument that exhibits an almost linear response,

$$d = R\left(s + rs^2\right) + n. \tag{6}$$

Here, R is a linear operator, $r \in \mathbb{R}$ with $|r| \ll 1$ is a small parameter that determines the strength of the nonlinearity in the instrumental response, s^2 denotes the local squaring of the signal field, i.e., $\left(s^2\right)_x = (s_x)^2$, and n is additive Gaussian noise, i.e.,

$$\mathcal{P}(n) = \mathcal{G}(n, N). \tag{7}$$

a) Consider first the case of an exactly linear response, i.e., $r = 0$. Derive the Hamiltonian

$$H(d, s) = -\log\left(\mathcal{P}(d, s)\right) \tag{8}$$

for this problem. You may drop all terms that do not depend on s (1 point) .

b) Show that the posterior probability density in the case with $r = 0$ is of Gaussian form, i.e., $\mathcal{P}(s|d) = \mathcal{G}(s - m_0, D)$, and derive expressions for its mean and covariance,

$$m_0 = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad D = \left\langle (s - m_0)(s - m_0)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{9}$$

as a function of d, S, N, and R (2 points).

c) Now consider the case with small but non-zero r. Calculate the Hamiltonian in this case and write it in the form

$$H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{\infty} \frac{1}{k!} \Lambda^{(k)}_{x_1 x_2 \cdots x_k} s_{x_1} s_{x_2} \cdots s_{x_k}, \tag{10}$$

where only the coefficients $\Lambda^{(k)}$ depend on r and we use the convention that repeated indices are integrated over. Give expressions for j, D, and all non-zero $\Lambda^{(k)}$. You do not need to calculate H_0 (3 points).

Exercise 10 - 4

This exercise is a continuation of exercise 12 − 1. Given is a Hamiltonian of the form

$$H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{4} \frac{1}{k!} \Lambda^{(k)}_{x_1 x_2 \cdots x_k} s_{x_1} s_{x_2} \cdots s_{x_k}, \tag{11}$$

where $\Lambda^{(2)} \propto r$, $\Lambda^{(3)} \propto r$ and $\Lambda^{(4)} \propto r^2$. Here r is assumed to be a very small parameter $|r| \ll 1$.

d) Write down the diagrammatic expansion of the partition function $\log\left(Z(d)\right)$ up to linear order in r. (1 point)

e) Find the diagrammatic expressions for the posterior mean and covariance,

$$m_r = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad \left\langle (s - m_r)(s - m_r)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{12}$$

up to first order in r. (1 point)

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This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html