

Exercise sheet 9

Exercise 9 - 1

Let $s : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous process that follows the following stochastic differential equation: $ds_t/dt = a \xi_t - b s_t$. Here, a and b are non-negative constants and ξ is a stationary, white random field of unit variance, i.e., $\langle \xi_t \xi_r \rangle_{(\xi)} = \delta(t - r)$. Let $S = \langle s s^\dagger \rangle_{(s)}$ be the signal covariance.

- a) Assume for the moment $s_0 = 0$, $b = 0$, and $t, t' \geq 0$. Calculate $S_{tt'}$. Use S_{tt} to argue why this so-called Wiener process is a frequently used model for diffusive motion of a particle (2 points).
- b) Calculate the signal power spectrum for any a and b being non-negative. Try to explain with words why the spectral normalization and the appearing characteristic frequency depend on a and b the way they do (2 points).

Hint: Transform the differential equation to Fourier space.

Exercise 9 - 2

Consider a field $\varphi \equiv \varphi_{x,t}$ with a time domain and a one-dimensional spatial domain, following the stochastic differential equation

$$\partial_t \varphi = \kappa \Delta_x \varphi + \xi \tag{1}$$

with independent Gaussian noise contribution ξ of unit variance and constant κ .

- a) Calculate the auto-correlation $\langle \varphi_{(\omega,k)}^* \varphi_{(\omega',k')} \rangle$ in its full harmonic domain (temporal and spatial Fourier basis) (2 points).
- b) Perform the inverse Fourier transformation in the time domain and give the expression of this auto-correlation in time - spatial frequency domain $\langle \varphi_{(t,k)}^* \varphi_{(t',k')} \rangle$ using the residue theorem. (2 points)

Exercise 9 - 3

Be s_t a stock price as a function of time t , which is believed to follow the Black-Scholes model

$$\frac{1}{s_t} \frac{ds_t}{dt} = \mu + \sigma \xi_t \tag{2}$$

with known and temporary constant drift rate μ and volatility σ . The stochastic field ξ is a white noise Gaussian random field with unit dispersion.

- a) Find a variable $x_t = x(s_t)$ which follows a drift free Wiener process ($\dot{x}_t = \sigma \xi_t$). Give its probability distribution $\mathcal{P}(x)$ (1 point).
- b) Use $\mathcal{P}(x)$ to calculate the expected (actually the average) future stock price $\bar{s}_t = \langle s_t \rangle_{(s_t | s_{t_0})}$. Why does this depend on σ^2 (2 points)?
- c) At time $t_0 = 0$ a bank has given the guarantee to a customer (for a fee, of course) that she can buy one of these stocks at a specified later time t for at most the fixed price K , independent of its actual price s_t then. Calculate the expected (average) amount of money L the bank will need to invest at “stroke” time t to fulfill its obligation from this “European call option” (2 points).

- d) Calculate now the minimal price C the bank has to ask for this call option in order not to lose any money on average. Consider that the bank has access to a risk-free interest rate r , which amplifies any amount of money C received at time $t_0 = 0$ to $C e^{rt}$ at a later time t . Find the most compact notation for this Black-Scholes option pricing formula by (i) using the property $1 - \operatorname{erf}(x) = \operatorname{erf}(-x)$ for $\operatorname{erf}(x) = \int_{-\infty}^x \mathcal{G}(x, 1)$ and (ii) expressing the current stock price through $S = \bar{s}_t e^{-rt}$, the expected future value of the stock for the bank at present time (a good brain twister, isn't it?). Compare your result to the famous Black-Scholes formula

$$C = S \operatorname{erf}(d_1) - K e^{-rt} \operatorname{erf}(d_2). \quad (3)$$

Argue why in the Black-Scholes world $S = s_{t_0}$ is assumed (2 points).

- e) [**Only for future stock traders**] Calculate the so-called "Greeks" (e.g., see Wikipedia), the derivatives of the option price C with respect to the various model parameters (μ , σ^2 , t , r , K , and S).

This exercise sheet will be discussed during the exercises.

Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,

Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

<https://wwwmpa.mpa-garching.mpg.de/~enssln/lectures/lectures.html>