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Exercise sheet 7

Exercise 7 - 1

You are observing a region of the sky with a photodetector. On average a number of photons λ makes it's way from the sky into your detector per time interval.
Hint: The PDF for n photon counts per time interval is given by the Poisson distribution: $\mathcal{P}(n|\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}$

a) Calculate the expectation value and the standard deviation for the photon counts per time interval (3 Points).

b) Use the PDF to calculate the expectation value $\left\langle \frac{n!}{(n-q)!}\theta(n-q)\right\rangle_{(n|\lambda)}$ with

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases} \tag{1}$$

for any $q \in \{0, 1, \ldots, n\}$, where n denotes the photon counts per time interval (2 Points).

Exercise 7 - 2

The goal of this exercise is to evaluate the integral

$$I(x) = \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{-ikx}}{k^2 + m^2} \tag{2}$$

a) For $k \in \mathbb{C}$, the integrand can diverge. Identify the poles of the integrand and sketch them in the complex plane, sketch the integration path of $k \in (-\infty, \infty)$ as well (2 points).

b) Draw a contour line that closes the integral, connecting $k = \infty$ to $k = -\infty$. The integral of the added contour line should be 0. Illustrate why it is 0 by comparing orders in k (2 points).

<u>Hint</u>: You may assume $x > 0$.

c) The closed integral encloses one pole. Identify its order and evaluate the integral $I(x)$ using the residue theorem,

$$\int_\Gamma f = (2\pi\, i) \sum_{a \in R_f} \mathrm{ind}_\Gamma(a)\ \mathrm{Res}_a f, \tag{3}$$

where Γ is a closed contour, R_f are the poles of f, and $\mathrm{ind}_\Gamma(a)$ is the number of counter-clockwise windings of Γ around a, and the residue formula,

$$\mathrm{Res}_a f = \frac{1}{(n-1)!} \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[(z-a)^n f(z)\right], \tag{4}$$

where n is the order of the pole a (2 points).

<u>Hint</u>: A clockwise winding is a negative counter-clockwise winding.

Exercise 7 - 3

A signal $s : \mathbb{R}^u \to \mathbb{R}$ with Gaussian statistics and known covariance $S = \left\langle s s^\dagger \right\rangle_{(s)}$ is measured via $d_x = s_x + n_x$. The noise follows Gaussian statistics and is homogeneous except for a slight enhancement in an area Ω, i.e., $N_{xy} = \langle n_x n_y \rangle_{(n)} = \delta(x-y)\,(1 + \epsilon\,\Theta_\Omega(x))\,\sigma^2$. Here, $\Theta_\Omega(x) = 1$ for $x \in \Omega$ and $\Theta_\Omega(x) = 0$ for $x \notin \Omega$. Consider the Wiener filter for this inference problem.

a) Calculate perturbatively to first order in ϵ the effect of the noise inhomogeneity on the real-space structure of the propagator (3 points).

Hint: The following relation for the differentiation of a regular matrix A depending on a parameter p might be useful:

$$\begin{aligned}
&\frac{\partial}{\partial p}\left(A^{-1}A\right) = 0 \\
\Leftrightarrow\ & 0 = \frac{\partial A^{-1}}{\partial p}A + A^{-1}\frac{\partial A}{\partial p} \\
\Leftrightarrow\ & \frac{\partial A^{-1}}{\partial p} = -A^{-1}\frac{\partial A}{\partial p}A^{-1}
\end{aligned}$$

b) Calculate N in its Fourier representation for general Ω and for $\Omega = [-L, L]$ in the one-dimensional case (3 points).

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html