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Exercise sheet 1

Exercise 1 - 1

Happy Birthday!
A number of persons, k, meet. Assume that the probability of a person to have their birthday is the same for every day of the year. Assume further that the number of days per year is always 365.

a) How high is the probability that the birthday of at least q of these people is on the first of January (2 Points)?

b) How high is the probability of at least two persons in the room having their birthday on the same day (1 Point)?

c) For which k is this probability larger than 50% (1 Point)?

Exercise 1 - 2

Any Boolean circuit can be constructed using the NOT, OR, and AND gates. In this question we explore the family of universal gates.

Universality of NOT, OR, and AND: Any Boolean circuit with inputs $I_1 \ldots I_n$ and outputs $O_1 \ldots O_m$ can be constructed out of single output circuits with output O_i (henceforth called O).

We describe a basis of Boolean circuits out of which any generic Boolean circuit can be constructed: each element of the basis is such that one of the outputs is 1 whereas all the others are 0.

For example, a basis of one input gates is:

$$\overline{I_1} = \begin{array}{c|c} I_1 & O \\ \hline 0 & 1 \\ 1 & 0 \end{array} \quad \text{and} \quad I_1 = \begin{array}{c|c} I_1 & O \\ \hline 0 & 0 \\ 1 & 1 \end{array} . \tag{1}$$

a) Complete the table for the two input basis gates using AND and NOT as logical connectors:

$$\overline{I_1}\,\overline{I_2} = \begin{array}{cc|c} I_1 & I_2 & O \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{array} \qquad \overline{I_1}\,I_2 = \begin{array}{cc|c} I_1 & I_2 & O \\ \hline 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{array} . \tag{2}$$

(1 point)

b) Write down the truth table for "$I_1 \Leftrightarrow I_2$". Express this statement in terms of the above basis using OR to connect the truth tables. (2 points)

c) Consider the three input logical gate:

$$\begin{array}{ccc|c} I_1 & I_2 & I_3 & O \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array} , \tag{3}$$

write an expression for this gate using the symbols I_1, I_2, and I_3 and using the AND, OR, and NOT gates. Can this argument be generalised to arbitrary n-input logic gates? (3 points)

Universal Logic Gates: We now demonstrate the possible universal binary (two-input) logic gates. A universal gate is one out of which all logical circuits can be constructed — in effect, the AND, OR, and NOT gates have to be constructed out of the universal gates. Note that with unary (single-input) gates one can never construct an OR or an AND gate.

d) Construct the AND gate out of OR and NOT gates. Construct the OR gate out of AND and NOT gates. (2 points)

e) Show that the NAND ($\uparrow$), which is AND followed by NOT, and NOR ($\downarrow$), which is OR followed by NOT, gates are universal. (2 points)

f) Are there other universal gates? List them. (3 points)

Exercise 1 - 3

A sports game in which two players play for a point in each round, the game is won by the first player who leads by two points. Player A always has the probability θ to win a round. What are his chances?

a) State the three probabilities (as a function of θ) that after two rounds player has A won, has lost, or has to continue the game. Call them in the following w, l, and c, respectively (1 points).

b) State now the three probabilities (as a function of w, l, and c) that after (at most) four rounds player A has won, has lost, or that the game still continues (1 points).

c) State now the three probabilities that after (at most) an infinite numbers of rounds player A has won, lost, or that the game still continues (2 points).

d) Now you hear that player A has won the game. What is the probability $P(n|W)$ (as a function of θ) that the game ended with round n? If you were not able to solve c) use the following:

$$P(W, n \leq \infty) = \frac{\theta^2}{1 - 2\theta(1-\theta)} \qquad P(L, n \leq \infty) = \frac{(1-\theta)^2}{1 - 2\theta(1-\theta)} \qquad P(C, n \leq \infty) = 0$$

(3 points)

Exercise 1 - 4

Weather in Markovia
You are traveling to the beautiful country of Markovia. Your travel guide tells you that the weather w_i in Markovia on a particular day i is sunny, $w_i = \text{s}$, for 80% of all days or it is cloudy, $w_i = \text{c}$, for 20% of all days. There are no other weather conditions in Markovia and the weather changes only during nights. The probability for a weather change is 10% if it is sunny,

$$P(w_{i+1} = \text{c}|w_i = \text{s}) = 0.1, \tag{4}$$

and 40% if it is cloudy,

$$P(w_{i+1} = \text{s}|w_i = \text{c}) = 0.4, \tag{5}$$

irrespective of what it has been on earlier days, $P(w_{i+1}|w_i,\, w_{i-1},\, w_{i-2},\, \ldots) = P(w_{i+1}|w_i)$.

a) You arrive on a sunny day, $w_i = \text{s}$, in Markovia. Calculate the probability that it was cloudy there the day before, $P(w_{i-1} = \text{c}|w_i = \text{s})$ Hint: Use Bayes-Theorem. (2 points).

b) What is the total probability for a weather change $P(w_{i+1} \neq w_i)$ in Markovia during an arbitrary night (1 point)?

c) The Markovian weather forecast for some day i predicts a sunshine probability of

$$p_i = P(w_i = \mathrm{s}\,|\mathrm{forecast}). \tag{6}$$

What is the sunshine probability there for the following day,

$$p_{i+1} = P(w_{i+1} = \mathrm{s}\,|\mathrm{forecast})? \tag{7}$$

(1 point)

d) Verify or correct the travel guide's statement on the frequency of 80% sunny and 20% cloudy days in Markovia (1 point).

Hint: Your result of question **c)** might be useful for this.

e) Implement an algorithm to simulate the weather in Markovia and verify your results numerically (optional).

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html