Intelligence and Field Theory
Summer term 2025

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Exercise sheet 11

Exercise 11 - 1

Assume that a quantity y is linearly dependent on a quantity x, i.e., $y(x) = a + bx$. Assume further that the quantity y has been measured at $m-1$ different positions $(x_i)_i,\ i = 1, \ldots, m-1$, subject to additive uncorrelated Gaussian noise, i.e.,

$$d_i = y(x_i) + n_i, \quad n \hookleftarrow \mathcal{G}(n, N), \quad N_{ij} = \delta_{ij}\sigma_i^2. \tag{1}$$

Assuming a Gaussian prior for the parameters a and b, i.e.,

$$s = \begin{pmatrix} a \\ b \end{pmatrix} \hookleftarrow \mathcal{G}(s, S), \quad S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{2}$$

a linear fit can be performed using Wiener filter theory.

You have enough money left to finance one additional measurement with uncertainty σ_m. How should you choose the position x_m for that measurement to gain optimal knowledge about the parameter a?

Hint: Use the quadratic loss function $\mathcal{L}(s, x_m) = (a - m_a)^2$ and the formalism of risk minimization. Here, m_a is the Wiener filter estimate after the m measurements (4 points).

Exercise 11 - 2

An inference problem with an unknown prior distribution can be treated with an hierarchical Bayesian model, which estimates parameters of the prior distribution from the data itself.

You want to reconstruct the results of a random process, which produces identically distributed and independent samples s_i. You have u independent but noisy measurements of this random process which follow the likelihood

$$P(d|s) = \mathcal{G}(d - s, N), \qquad \text{with} \quad N_{ij} = \delta_{ij}\sigma^2 \tag{3}$$

You assume this process is well described by a Gaussian with zero mean,

$$P(s|p) = \mathcal{G}(s, S), \qquad \text{with} \quad S_{ij} = \delta_{ij} p, \tag{4}$$

but you have no clue about its variance p. To filter out the noise contribution to each data point d_i, an estimate of p is needed. The only information you have about p is that it is positive definite. A useful parametrization is therefore $\eta \equiv \ln(p)$. Since there is no further information about η, you assume $P(\eta) = \mathit{const}$.

a) Write down the joint probability distribution $P(d, s, \eta)$ and marginalize out s. Write down your solution using σ and η explicitly in order to get rid of all determinants (3 points).

Hint: You can drop all factors which do not depend on s or η, but remember that some of the normalizations are η-dependent.

b) Write down the Hamiltonian $\mathcal{H}(d, \eta) \equiv -\ln P(d, \eta) + \mathit{const.}$ and calculate its first derivative with respect to η (2 points).

c) Set the first derivative of the Hamiltonian to zero to derive the *maximum a posteriori* (MAP) solution for η (2 points).

Hint: It is useful to replace e^η with p and then solve for p.

d) Plug in the MAP estimator for η into the Wiener Filter formula for $\langle s \rangle_{\mathcal{P}(s|d,\eta)}$ to derive an η-independent estimator for s (1 point).

e) There is a possible problem with the resulting filter. What is it? (1 point)

Exercise 11 - 3

Non-Gaussianities in the Cosmic Microwave Background

The CMB is the relic radiation from the time of (re)combination 380 000 years after the Big Bang. Its variations in temperature with direction reflect the variations in the gravitational potential φ at this time. The simplest inflationary scenarios predict these variations to be a nearly Gaussian field. Measuring deviations from Gaussianity in this field is within the focus of contemporary research in cosmology.

The *local type* of non-Gaussianities can be modeled according to

$$\varphi = \phi + f_{\mathrm{nl}} \left(\phi^2 - \left\langle \phi^2 \right\rangle_{(\phi)} \right) = \phi + f_{\mathrm{nl}} \left(\phi^2 - \hat{\Phi} \right), \tag{5}$$

where

$$\phi \hookleftarrow \mathcal{G}(\phi, \Phi) \tag{6}$$

is an auxilliary Gaussian field and the degree of non-Gaussianity of the primordial gravitational potential φ is quantified by the parameter f_{nl}, which shall be assumed here to be position-independent (i.e. one number).

During cosmic recombination, the variations in the gravitational field are transformed into temperature variations in the electromagnetic radiation. This radiation has been measured in recent times by satellites, balloons, and ground based observatories. These two processes (imprinting onto temperature variations and measurement) can be combined in a linear response operator R. Additionaly, these measurements contain a Gaussian noise contribution, which is not correllated with the signal, i.e.

$$d = R\varphi + n, \; n \hookleftarrow \mathcal{G}(n, N). \tag{7}$$

a) Derive an expression for $H[d, \phi, f_{\mathrm{nl}}] = -\log\left(\mathcal{P}(d, \phi, f_{\mathrm{nl}})\right)$, assuming a flat prior for f_{nl}. (2 points)

Assume now uncorrelated and homogeneous noise, i.e.

$$N_{ij} = \delta_{ij}\sigma^2, \tag{8}$$

and unit response,

$$R_{ij} = \delta_{ij}. \tag{9}$$

b) Bring this Hamiltonian into the form

$$H[d, \phi | f_{\mathrm{nl}}] = H_0 - j^\dagger \phi + \frac{1}{2}\phi^\dagger D^{-1}\phi + \frac{1}{3!}\lambda^{(3)\dagger}\phi^3 + \frac{1}{4!}\lambda^{(4)\dagger}\phi^4 \tag{10}$$

and identify the terms j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$, as well as the f_{nl}-dependent part of H_0. (3 points)

c) Write down the diagrammatic expansion for the logarithm of the partition function

$$\log Z_{f_{\mathrm{nl}}}(d) = \log \mathcal{P}(d|f_{\mathrm{nl}}) = \log \int \mathcal{D}\phi \; \mathcal{P}(d, \phi | f_{\mathrm{nl}}), \tag{11}$$

up to second order in f_{nl}, i.e., use all diagrams that contain terms of lower than third order. You do not need to formulate the results algebraically, nor do you need to care about terms that are constant in ϕ and f_{nl}.
(2 points)

d) Write down the diagrammatic version of the expectation value $\langle \phi \rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}$ up to first order in f_{nl}. Read off the corresponding algebraic formula for the expectation value in terms of j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$.
(2 points)

e) Write down the diagrammatic version of the dispersion $\left\langle \left(\phi - \langle \phi \rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}\right) \left(\phi - \langle \phi \rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}\right)^{\dagger} \right\rangle_{\mathcal{P}(\phi|d,f_{\mathrm{nl}})}$ up to first order in f_{nl}. Read off the corresponding algebraic formula for the dispersion in terms of j, D, $\lambda^{(3)}$, and $\lambda^{(4)}$.
(2 points)

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html