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Exercise sheet 9

Exercise 9 - 1

Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous process that follows the following stochastic differential equation: $\mathrm{d}s_t/\mathrm{d}t = a\,\xi_t - b\,s_t$. Here, a and b are non-negative constants and ξ is a stationary, white random field of unit variance, i.e., $\langle \xi_t \xi_r \rangle_{(\xi)} = \delta(t-r)$. Let $S = \left\langle s s^\dagger \right\rangle_{(s)}$ be the signal covariance.

a) Assume for the moment $s_0 = 0$, $b = 0$, and $t, t' \geq 0$. Calculate $S_{tt'}$. Use S_{tt} to argue why this so-called Wiener process is a frequently used model for diffusive motion of a particle (2 points).

b) Calculate the signal power spectrum for any a and b being non-negative. Try to explain with words why the spectral normalization and the appearing characteristic frequency depend on a and b the way they do (2 points).

Hint: Transform the differential equation to Fourier space.

Exercise 9 - 2

Consider a field $\varphi \equiv \varphi_{x,t}$ with a time domain and a one-dimensional spatial domain, following the stochastic differential equation

$$\partial_t \varphi = \kappa \Delta_x \varphi + \xi \tag{1}$$

with independent Gaussian noise contribution ξ of unit variance and constant κ.

a) Calculate the auto-correlation $\langle \varphi^*_{(\omega,k)} \varphi_{(\omega',k')} \rangle$ in its full harmonic domain (temporal and spatial Fourier basis) (2 points).

b) Perform the inverse Fourier transformation in the time domain and give the expression of this auto-correlation in time - spatial frequency domain $\langle \varphi^*_{(t,k)} \varphi_{(t',k')} \rangle$ using the residue theorem. (2 points)

Exercise 9 - 3

Assume a linear measurement of some field. Assume further a log-normal model for this field and an additive Gaussian noise term, i.e.

$$d = R\mathrm{e}^s + n,\ s \hookleftarrow \mathcal{G}(s,S),\ n \hookleftarrow \mathcal{G}(n,N). \tag{2}$$

a) Derive the information Hamiltonian $H(s,d)$ for this problem.
(2 points)

b) Give a recursion relation of the type

$$m_{\mathrm{MAP}} = f(m_{\mathrm{MAP}}) \tag{3}$$

for the *maximum a posteriori* solution m_{MAP} of the signal field s.
(1 point)

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html