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Exercise sheet 7

Exercise 7 - 1

You have conducted a measurement of a quantity at n positions $\{x_i\}_i$, yielding n data points $\{(x_i, d_i)\}_i$. Now you want to fit some function to these data points. To this end, you write the function as a linear combination of m basis functions $\{f_j(x)\}_j$, i.e.,

$$f(x) = \sum_{j=1}^{m} s_j f_j(x). \tag{1}$$

If, for example, you were to fit a second order polynomial, you could choose the monomials as basis functions, i.e., $f(x) = s_2 x^2 + s_1 x + s_0$.
The fitting process now comes down to determining the coefficients $\{s_j\}_j$, allowing for some Gaussian and independent measurement error, i.e.,

$$d_i = \sum_{j=1}^{m} s_j f_j(x_i) + n_i. \tag{2}$$

Assume that you do not know anything about the coefficients a priori, i.e., $S^{-1} \equiv 0$, where $S_{ik} = \langle s_i s_k \rangle_{\mathcal{P}(s)}$.

a) Write down the response matrix for this problem (1 point).

b) For a given set of m basis functions, how many data points n are at least necessary for the calculation of the posterior mean of the coefficients (2 points)?

c) Now let's make a linear fit. Assuming $N_{ik} = \langle n_i n_k \rangle_{\mathcal{P}(n)} = \eta^{-1} \delta_{ik}$, choose two basis functions and work out the explicit formula for the posterior mean of the two coefficients (3 points).

Exercise 7 - 2

A signal $s : \mathbb{R}^u \to \mathbb{R}$ with Gaussian statistics and known covariance $S = \left\langle s s^\dagger \right\rangle_{(s)}$ is measured via $d_x = s_x + n_x$. The noise follows Gaussian statistics and is homogeneous except for a slight enhancement in an area Ω, i.e., $N_{xy} = \left\langle n_x n_y \right\rangle_{(n)} = \delta(x-y)\left(1 + \epsilon\, \Theta_\Omega(x)\right) \sigma^2$. Here, $\Theta_\Omega(x) = 1$ for $x \in \Omega$ and $\Theta_\Omega(x) = 0$ for $x \notin \Omega$. Consider the Wiener filter for this inference problem.

a) Calculate perturbatively to first order in ϵ the effect of the noise inhomogeneity on the real-space structure of the propagator (3 points).

Hint: The following relation for the differentiation of a regular matrix A depending on a parameter p might be useful:

$$\begin{aligned} &\frac{\partial}{\partial p}\left(A^{-1} A\right) = 0 \\ \Leftrightarrow\ &0 = \frac{\partial A^{-1}}{\partial p} A + A^{-1} \frac{\partial A}{\partial p} \\ \Leftrightarrow\ &\frac{\partial A^{-1}}{\partial p} = -A^{-1} \frac{\partial A}{\partial p} A^{-1} \end{aligned}$$

b) Calculate N in its Fourier representation for general Ω and for $\Omega = [-L, L]$ in the one-dimensional case (3 points).

Exercise 7 - 3

Given a field $s : \mathcal{S}^2 \to \mathbb{C}$ on the two-dimensional sphere, assume that it is statistically homogeneous and isotropic, i.e., $S(\hat{n}, \hat{n}') = \langle s(\hat{n}) s^*(\hat{n}') \rangle = S(\hat{n} \cdot \hat{n}')$, where $\hat{n}$ and $\hat{n}'$ are unit vectors that give directions or, equivalently, points on $\mathcal{S}^2$. Prove that the covariance matrix S is diagonal in the basis given by the spherical harmonic functions and its entries are independent of m, i.e.,

$$S_{(\ell m)(\ell' m')} := \langle s_{\ell m} s^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell. \tag{3}$$

(3 points)
Hint: Use the following properties of the spherical harmonic functions $Y_{\ell m}$ and the Legendre polynomials P_ℓ:

$$s(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} s_{\ell m} Y_{\ell m}(\hat{n}), \qquad s_{\ell m} = \int_{\mathcal{S}^2} \mathrm{d}\Omega \; s(\hat{n}) Y^*_{\ell m}(\hat{n}) \tag{4}$$

$$\int_{\mathcal{S}^2} \mathrm{d}\Omega \; Y^*_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}) = \delta_{\ell \ell'} \delta_{m m'} \tag{5}$$

$$P_\ell(\hat{n} \cdot \hat{n}') = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{n}) Y_{\ell m}(\hat{n}') \tag{6}$$

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html