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Exercise sheet 6

Exercise 6 - 1

Assume the measurement of a signal s, which yields the data d, leads to

$$\mathcal{P}(s) = \mathcal{G}(s, S)$$
$$\mathcal{P}(s|d) = \mathcal{G}(s - m, D).$$

a) Calculate the amount of information in terms of entropy one gains via the measurement (3 points).

b) Now assume that your signal prior in the above measurement was flat. How much information in terms of entropy does one gain via the measurement? Explain the result (1 point).

Exercise 6 - 2

Assume that you are measuring a field ψ with symmetric statistics, i.e.

$$\mathcal{P}(\psi) = \mathcal{P}(-\psi) \; \forall\psi, \tag{1}$$

with a perfect instrument, i.e.

$$d = \psi. \tag{2}$$

You are interested in the power of the field, i.e.

$$s = \psi^2. \tag{3}$$

a) Calculate the signal response and the noise using the definition

$$R(s) = \langle d \rangle_{(d|s=\psi_0^2)} \tag{4}$$

of the signal response (1 point).

b) Do the same for a new data set d' that is the square of the old data set, $d' = d^2$ (1 point).

Exercise 6 - 3

Consider the potential $V(\vec{r})$ which is symmetric with respect to the radial distance $r = |\vec{r}|$,

$$V(r) = a\sqrt{r} + b. \tag{5}$$

This potential is parametrized by the unknown numbers $a, b \in \mathbb{R}$ and can be measured at strictly positive radii, i.e., $r > 0$. Furthermore, only a single data point $d \in \mathbb{R}$ can be obtained,

$$d = V(r) + n, \tag{6}$$

where the noise n is assumed to obey a Gaussian statistic $\mathcal{P}(n) = \mathcal{G}(n, N)$. The noise variance $N = N(r)$, however, depends on the measurement position,

$$N(r) = r^2 + 3. \tag{7}$$

a) Find an expression for the information entropy $S\left[\mathcal{P}(s|d)\right]$ for a Gaussian posterior

$$\mathcal{P}(s|d) = \mathcal{G}(s - m, D)\ , \tag{8}$$

with mean m and covariance D (2 points).

b) Consider the signal $s = \begin{pmatrix} a \\ b \end{pmatrix}$, for which a Gaussian prior $\mathcal{P}(s) = \mathcal{G}(s, \mathbb{1})$ can be assumed. (2 points)

- Write Eq. (6) in the form $d = R\,s + n$ and give R explicitly.
- Work out an expression for the joint probability $\mathcal{P}(d, s)$ and calculate the corresponding Hamiltonian $H(d, s) = -\log \mathcal{P}(d, s) = \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + H_0$. You may drop H_0.
- Identify the information source j and the inverse information propagator D^{-1}.

c) You verified in **a)** that information entropy $S = S(D)$ is a monotonically increasing function of $|D|$. Find the best position $\tilde{r}$ to estimate both, a and b, by minimizing $|D|$ from **b)** with respect to r (1 point).

d) Now, consider the signal $s = a$ for which b becomes a nuisance parameter. (1 point)

- Work out an expression for the joint probability $\mathcal{P}(d, a)$, and calculate the corresponding Hamiltonian $H(d, a) = -\log \mathcal{P}(d, a) = \frac{1}{2} D^{-1} a^2 - ja + H_0$. You may drop H_0.
- Identify the information source j and the information propagator D.

e) Find the best position $\tilde{r}_a$ to estimate a irrespectively of b, by minimizing D from **d)** with respect to r (1 point).

f) Guess at which radius $\tilde{r}_b$ one should measure in order to obtain the most certain estimate for the parameter b. No justification required (1 point).

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html