Physics of CMB Anisotropies

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Cours d'hiver du LAL, Laboratoire de l'Accélérateur Linéaire
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Lecture Slides

- Available at
  - https://wwwmpa.mpa-garching.mpg.de/~komatsu/lectures--reviews.html
- Or, just find my website and follow “LECTURES & REVIEWS” link

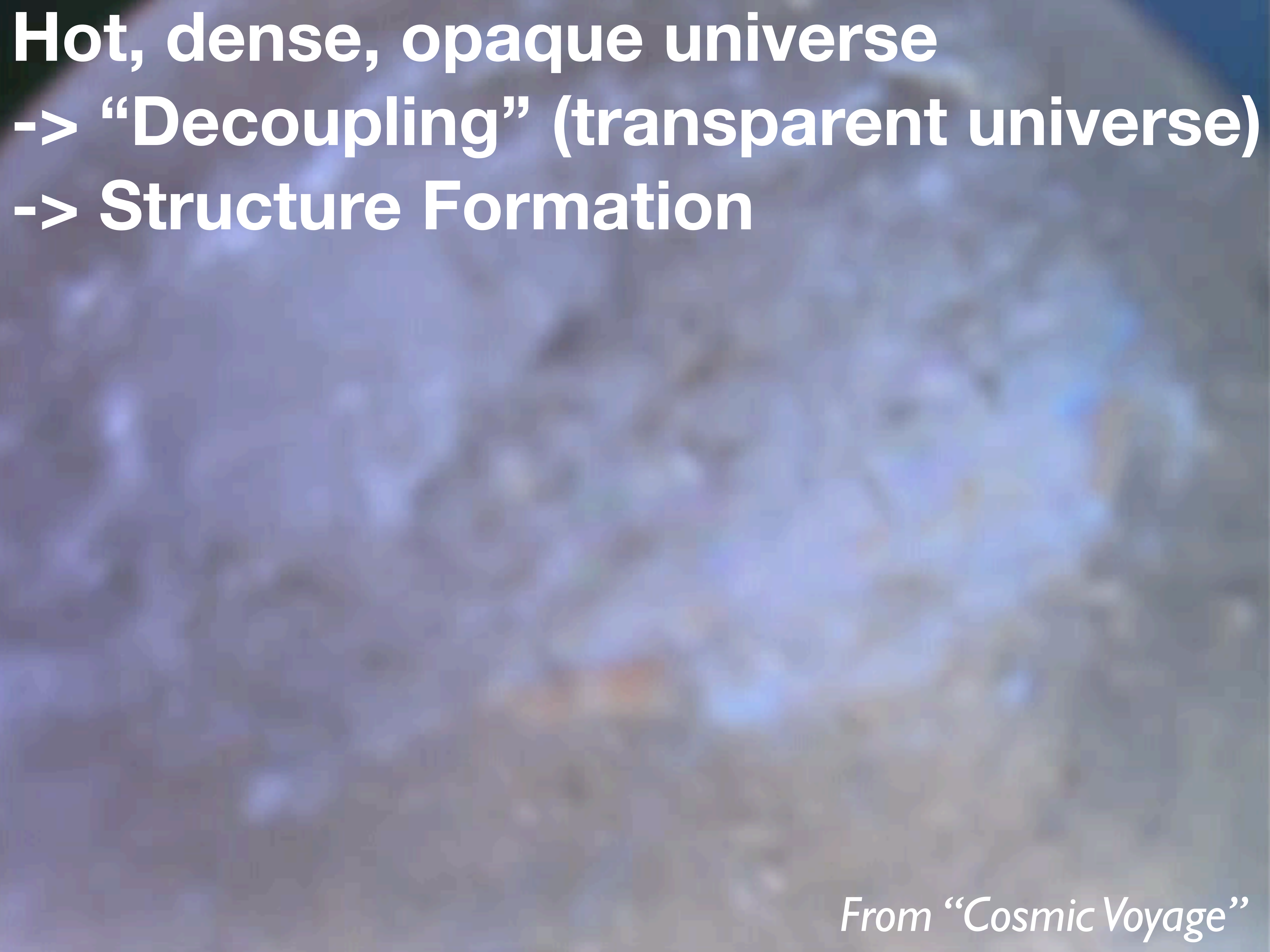
Planning: Day 1 (today)

- **Lecture 1**
  - Brief introduction of the CMB research
  - Temperature anisotropy from gravitational effects
  - Power spectrum basics

Planning: Day 2 & 3

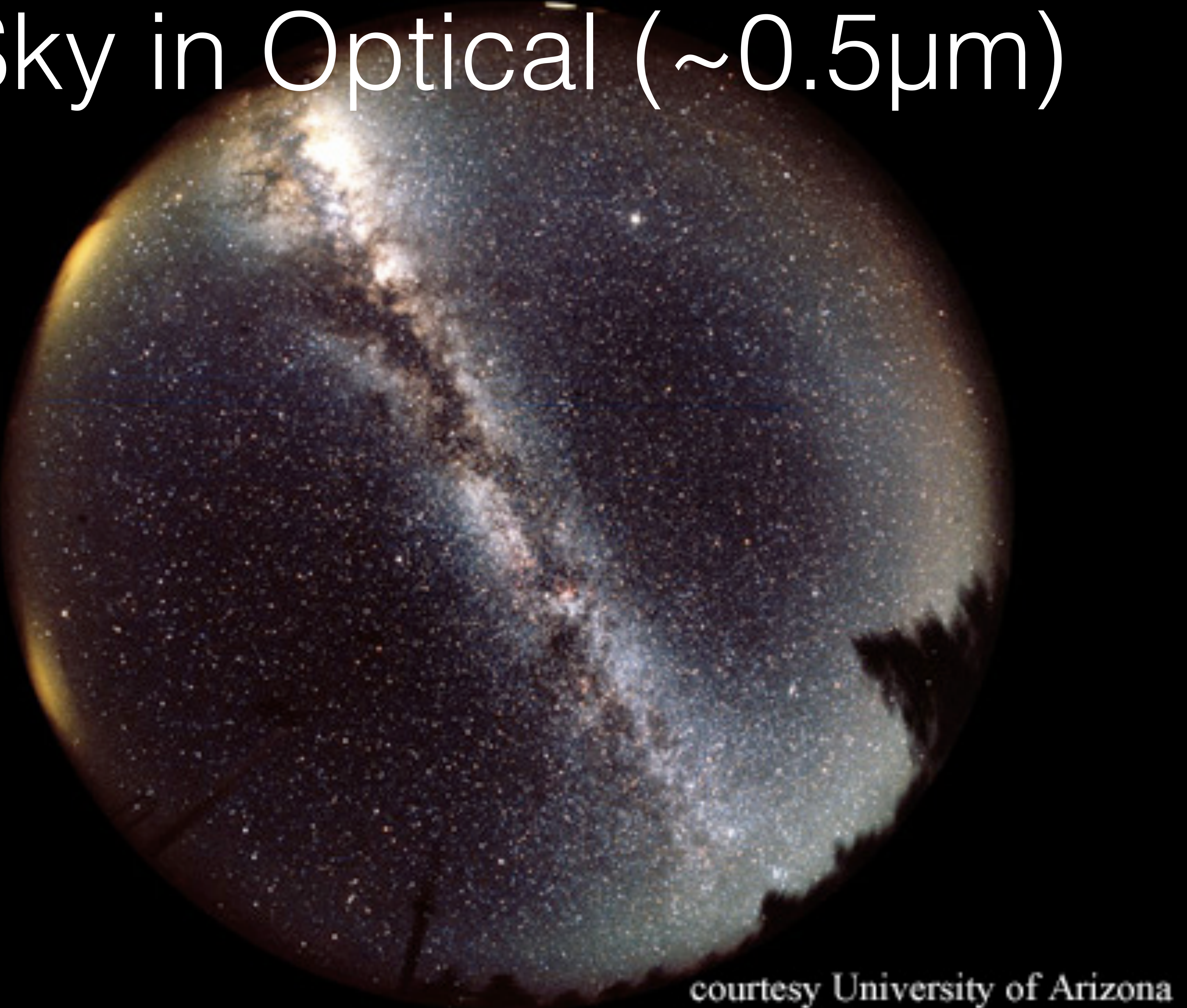
- **Lecture 2**
  - Temperature anisotropy from hydrodynamical effects (sound waves)
- **Lecture 3**
  - Cosmological parameter dependence of the temperature power spectrum
  - Polarisation of the CMB
  - Gravitational waves and their imprints on the CMB

Hot, dense, opaque universe
-> “Decoupling” (transparent universe)
-> Structure Formation

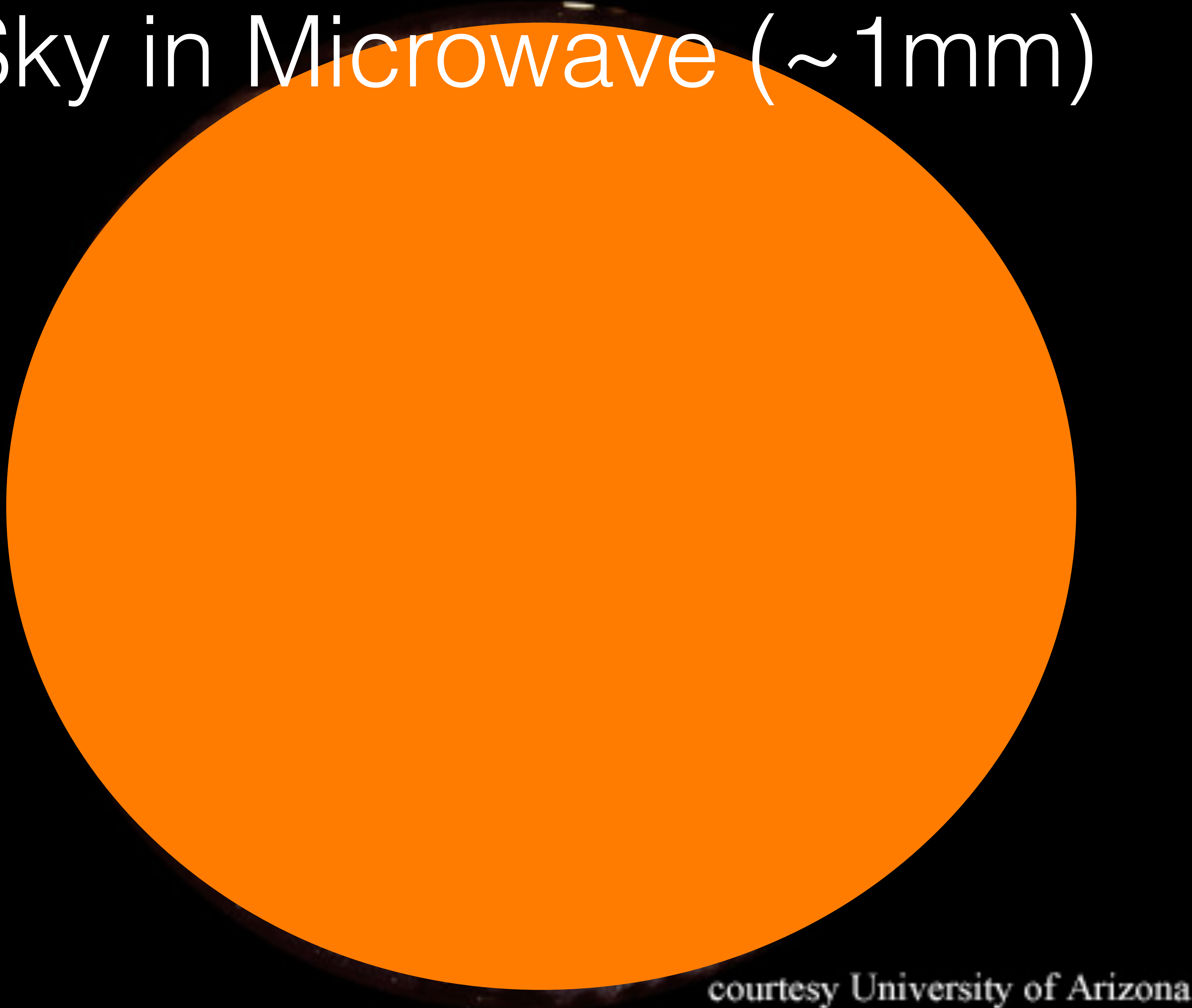


From “Cosmic Voyage”

Sky in Optical (~0.5μm)



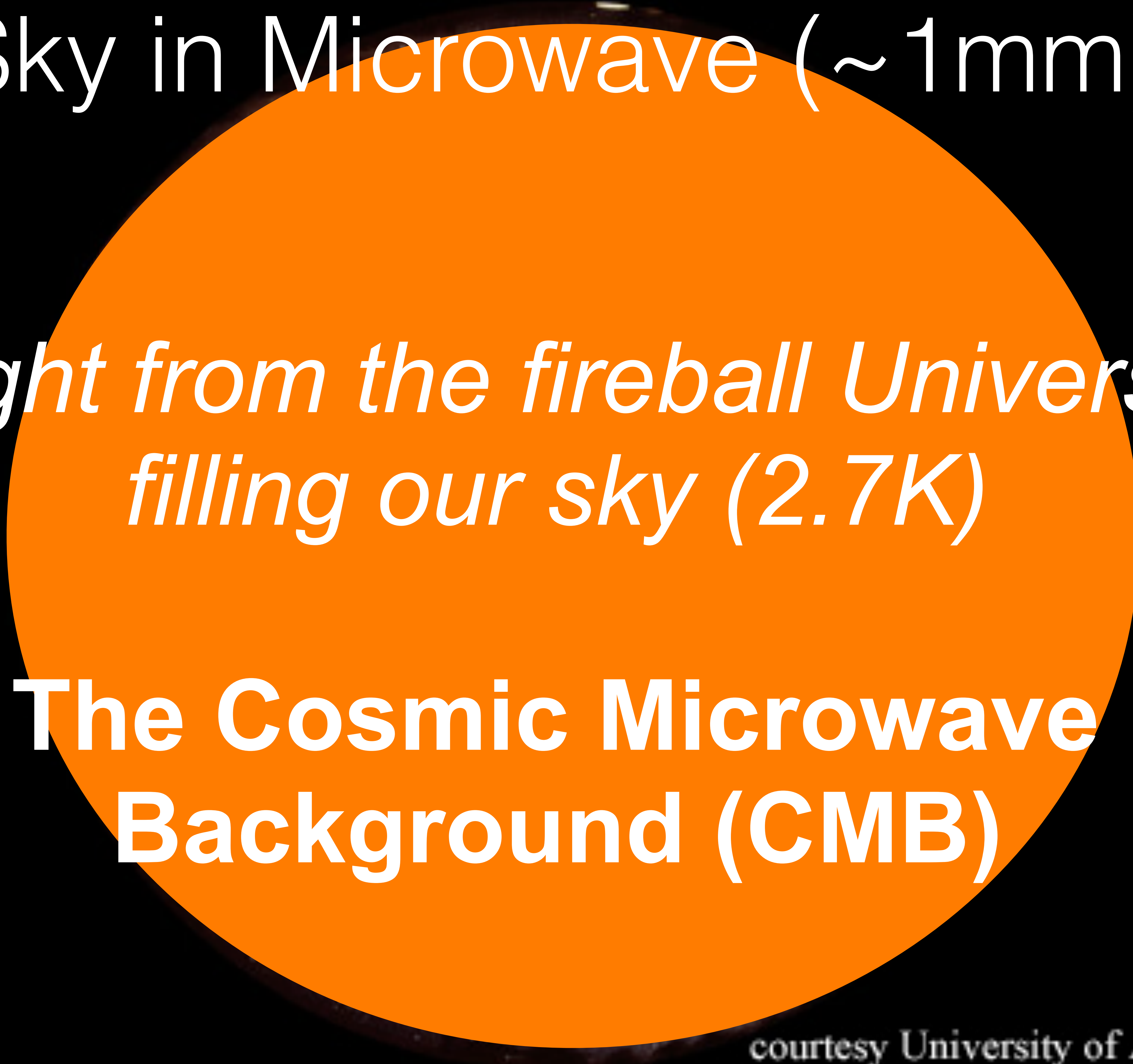
Sky in Microwave (~1mm)



Sky in Microwave (~1mm)

Light from the fireball Universe filling our sky (2.7K)

The Cosmic Microwave Background (CMB)



courtesy University of Arizona

410 photons

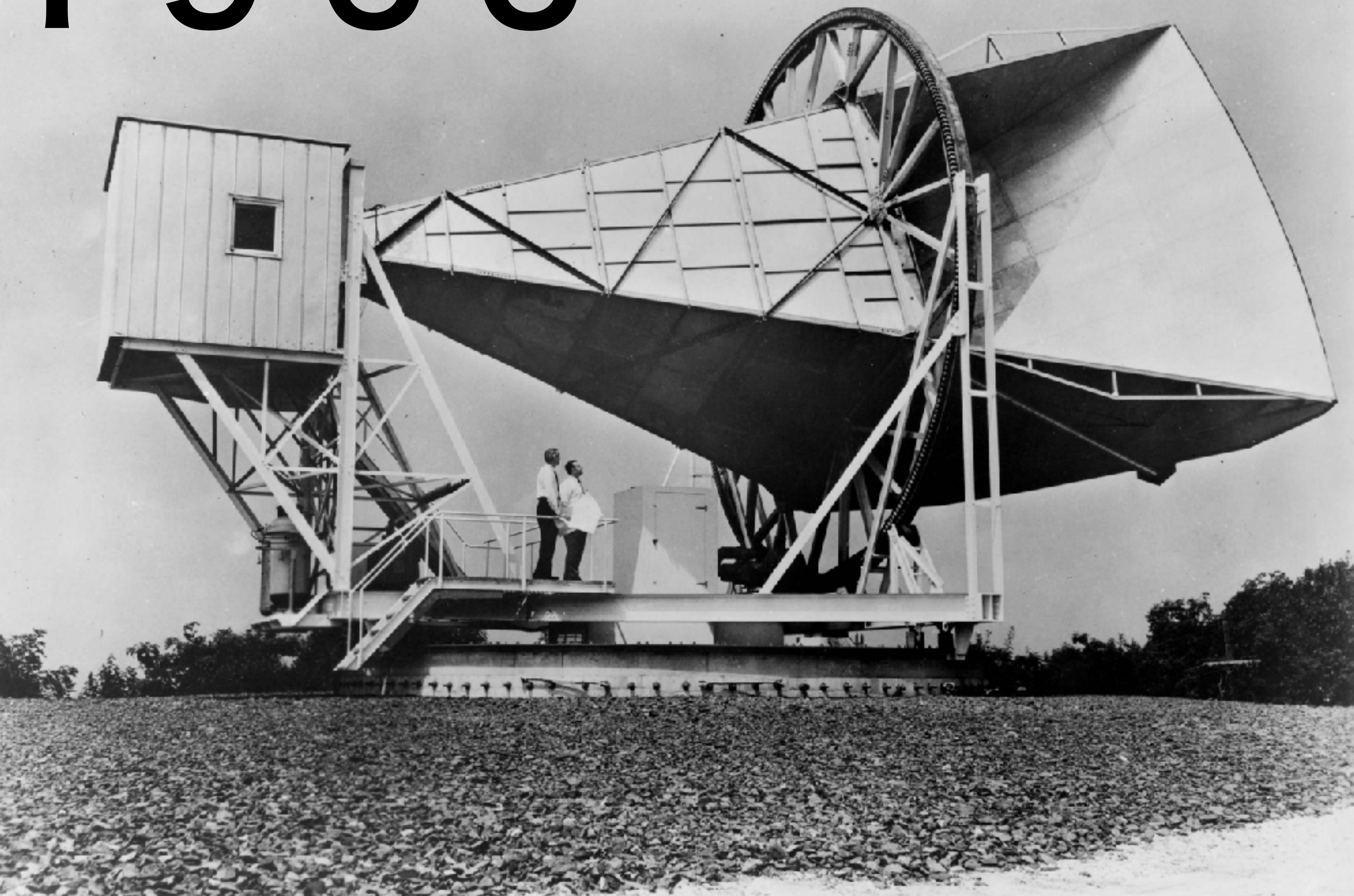
per

cubic centimeter!!

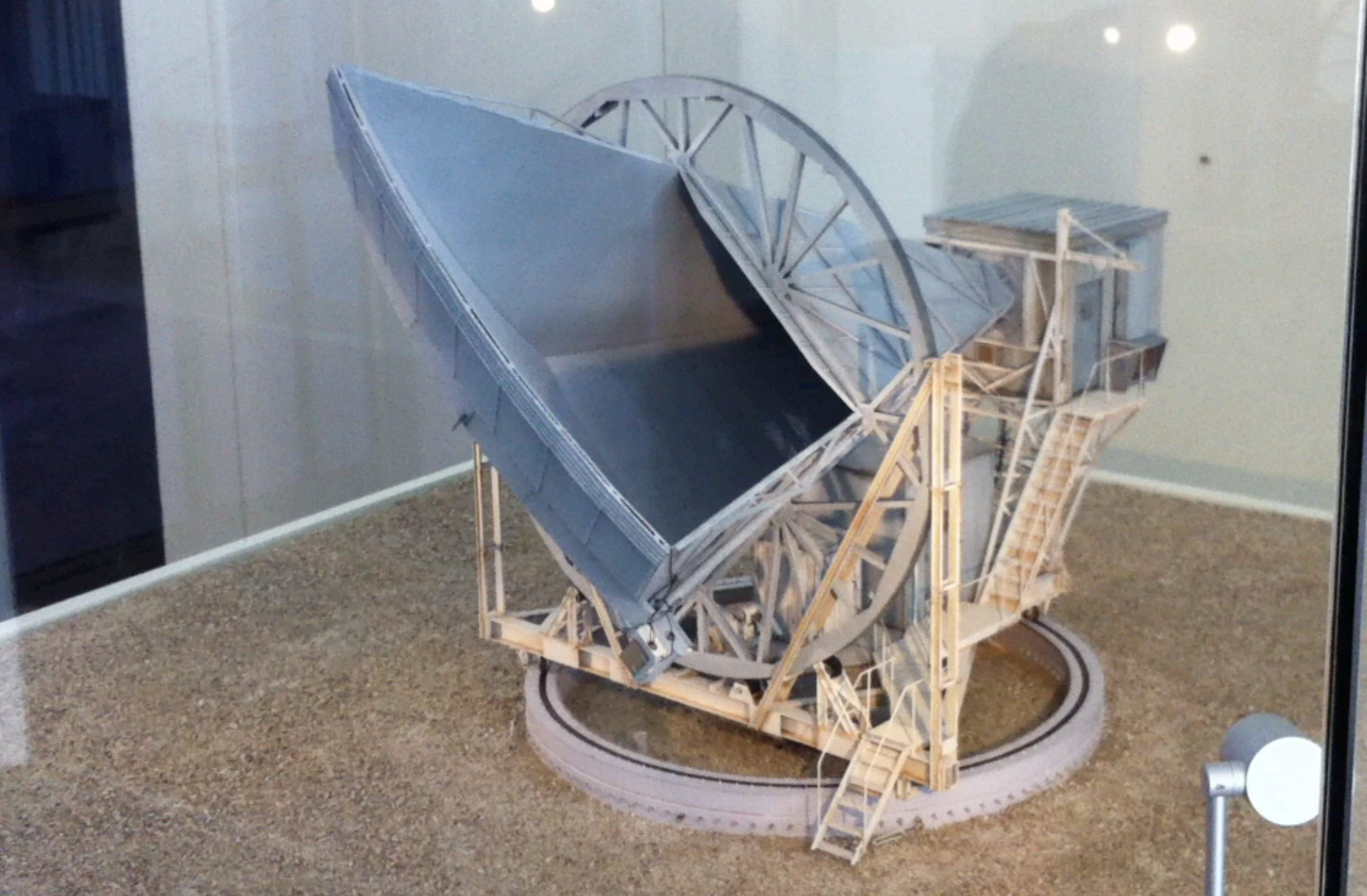
courtesy University of Arizona

All you need to do is to detect radio waves. For example, 1% of noise on the TV is from the fireball Universe

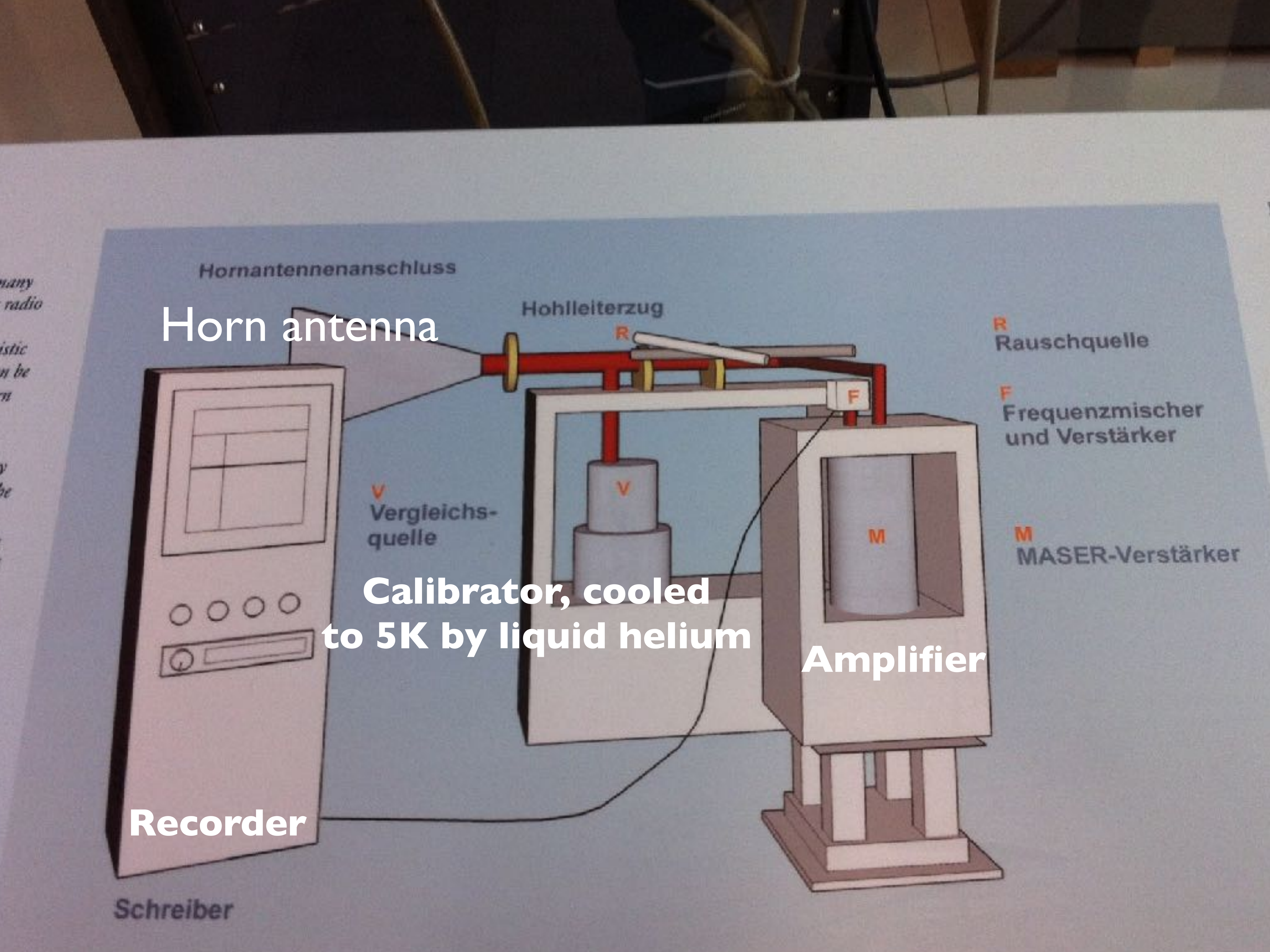
1965



1:25 model of the antenna at Bell Lab
The 3rd floor of Deutsches Museum

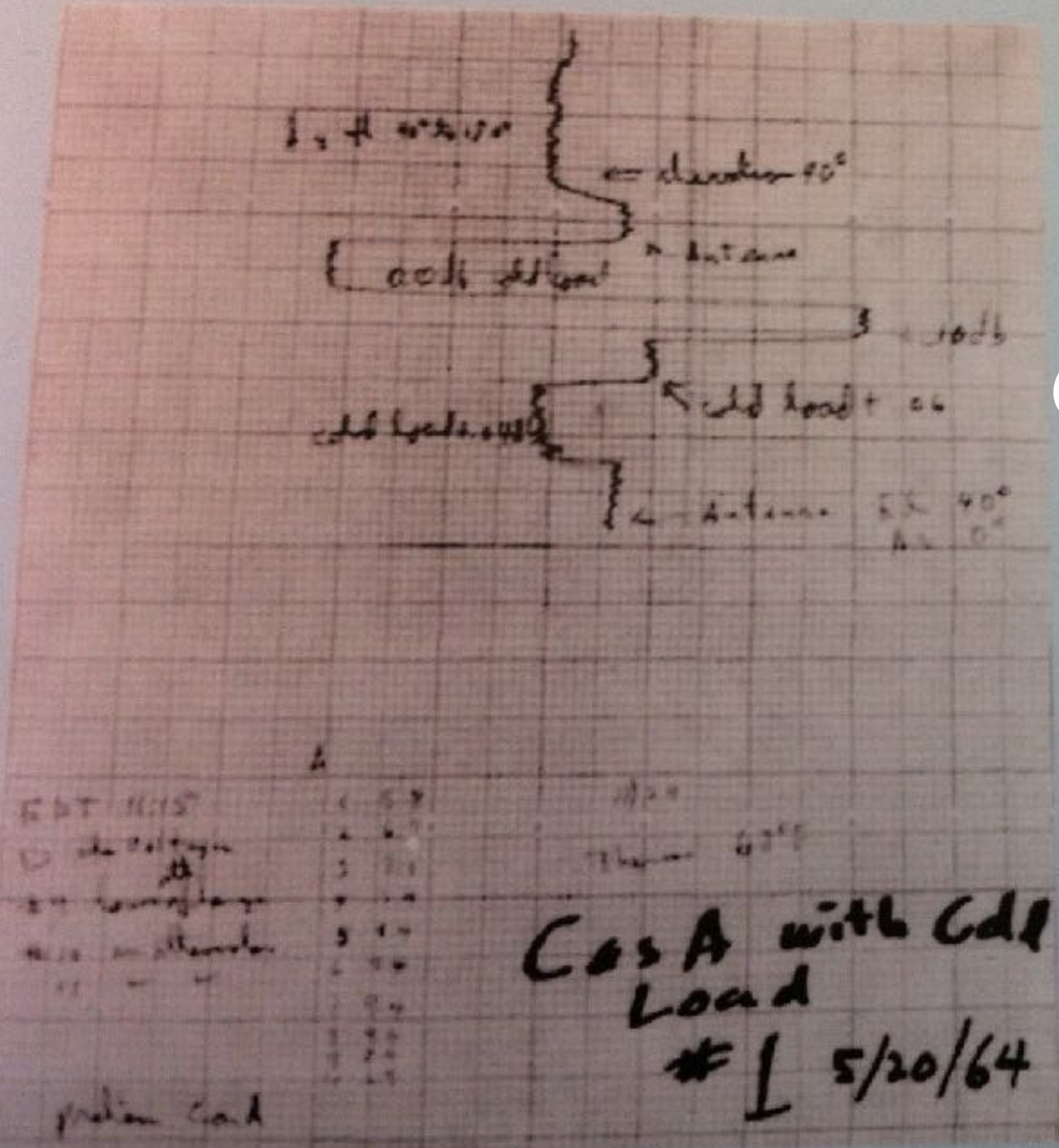


The real detector system used by Penzias & Wilson
The 3rd floor of Deutsches Museum


Hornantennenanschluss
Horn antenna
Hohlleiterzug
R
R
Rauschquelle
F
F
Frequenzmischer und Verstärker
V
Vergleichs-quelle
V
M
M
MASER-Verstärker
Calibrator, cooled to 5K by liquid helium
Amplifier
Recorder
Schreiber

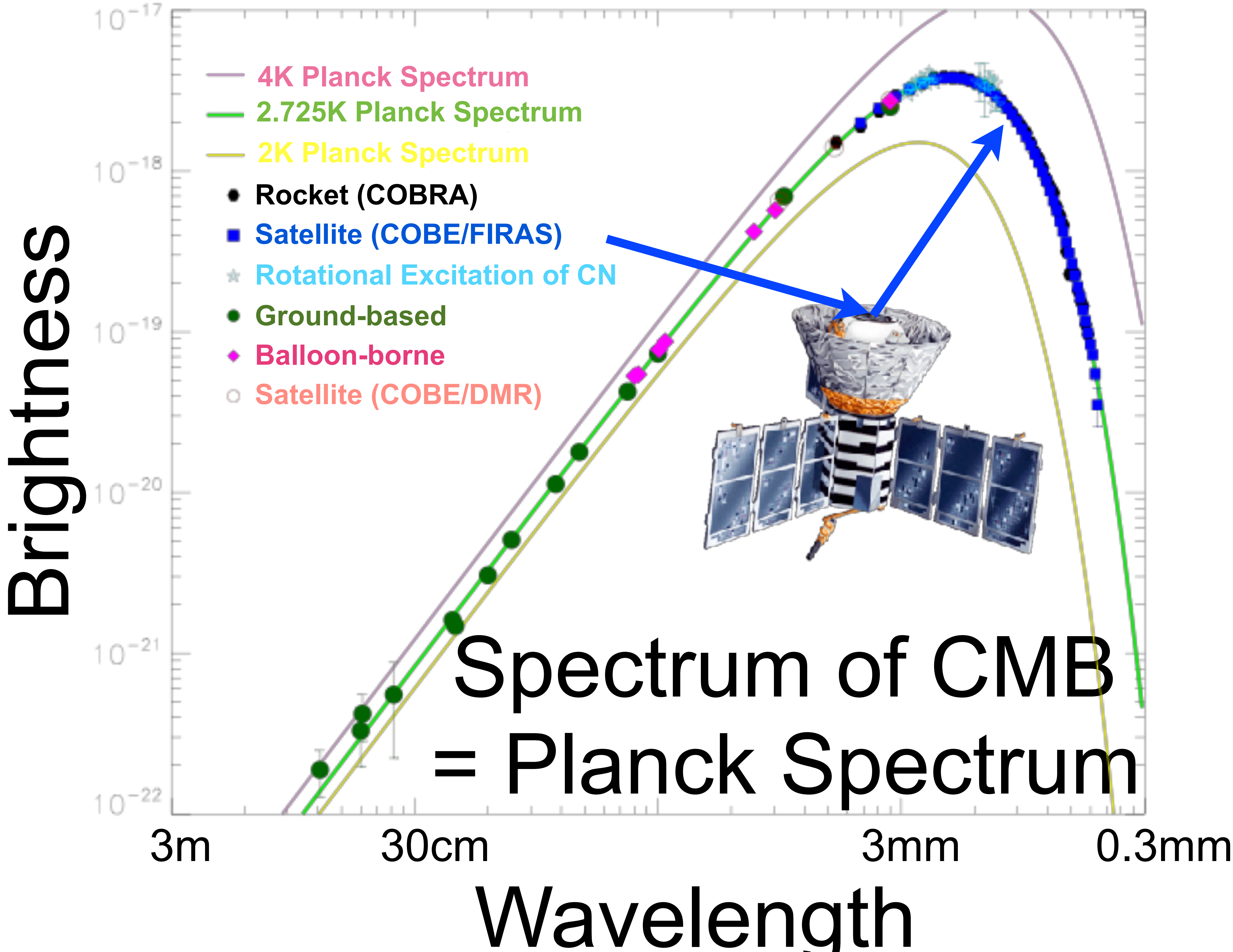
May 20, 1964 CMB Discovered

6.7–2.3–0.8–0.1 = 3.5±1.0 K



Schreiberaufzeichnung der ersten Messung des Mikrowellenhintergrundes am 20.5.1964

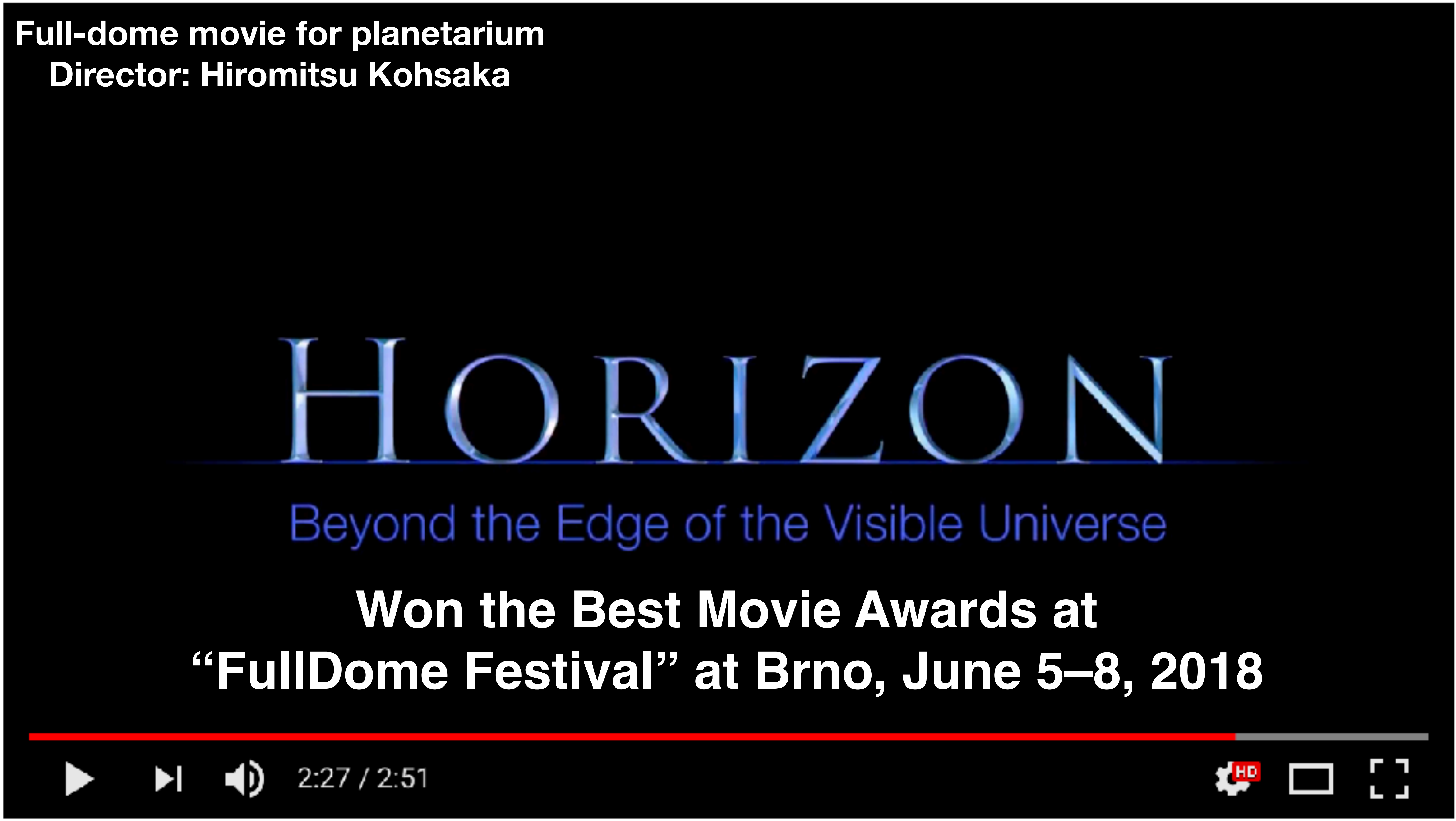
Recording of the first measurement of cosmic microwave background radiation taken on 5/20/1964.



Brightness
4K Planck Spectrum
2.725K Planck Spectrum
2K Planck Spectrum
Rocket (COBRA)
Satellite (COBE/FIRAS)
Rotational Excitation of CN
Ground-based
Balloon-borne
Satellite (COBE/DMR)
10^{-17}
10^{-18}
10^{-19}
10^{-20}
10^{-21}
10^{-22}
3m
30cm
3mm
0.3mm
Wavelength
Spectrum of CMB
= Planck Spectrum

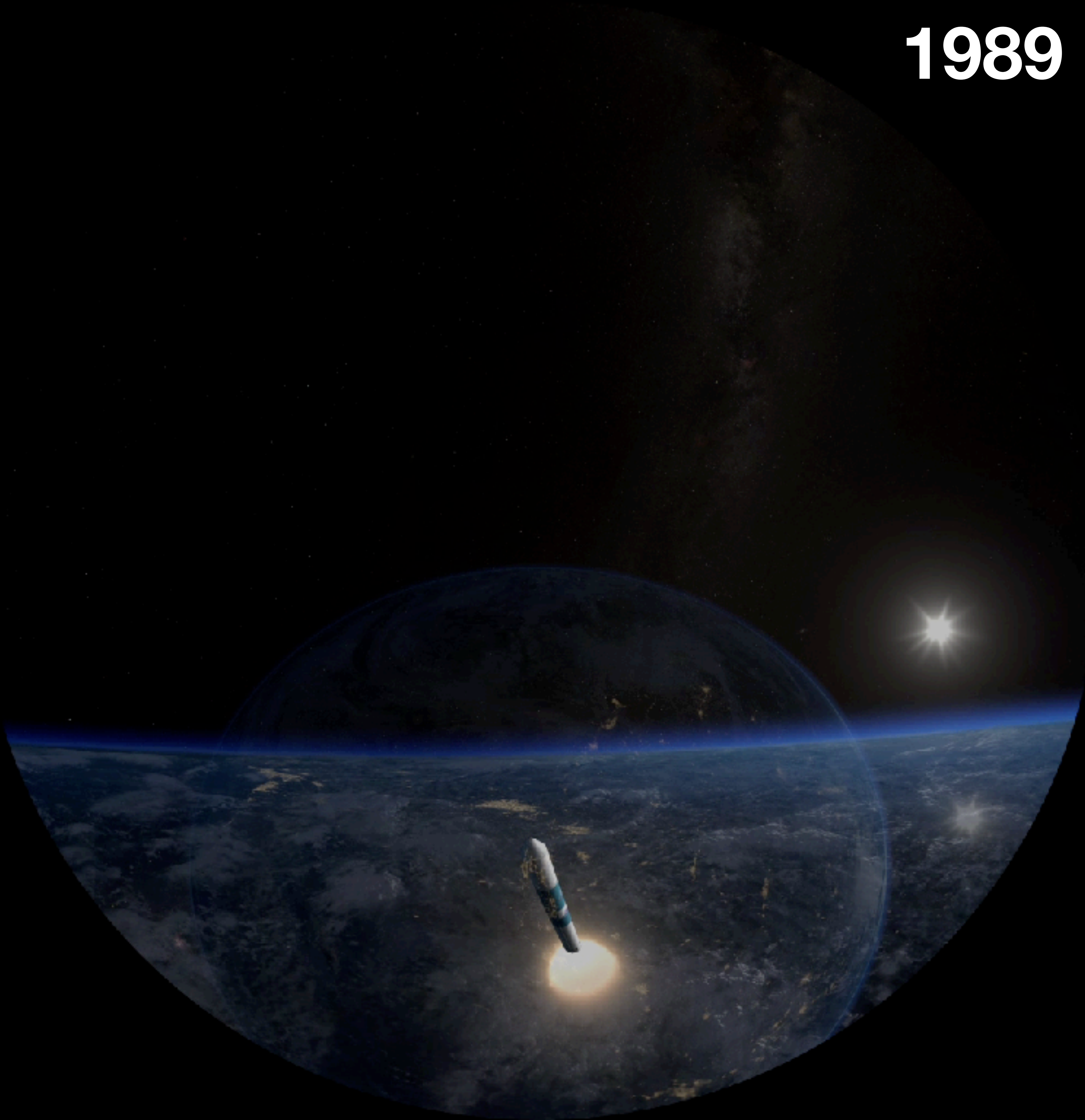
YouTube BE

horizon edge of the visible universe



HORIZON :Beyond the Edge of the Visible Universe [Trailer]

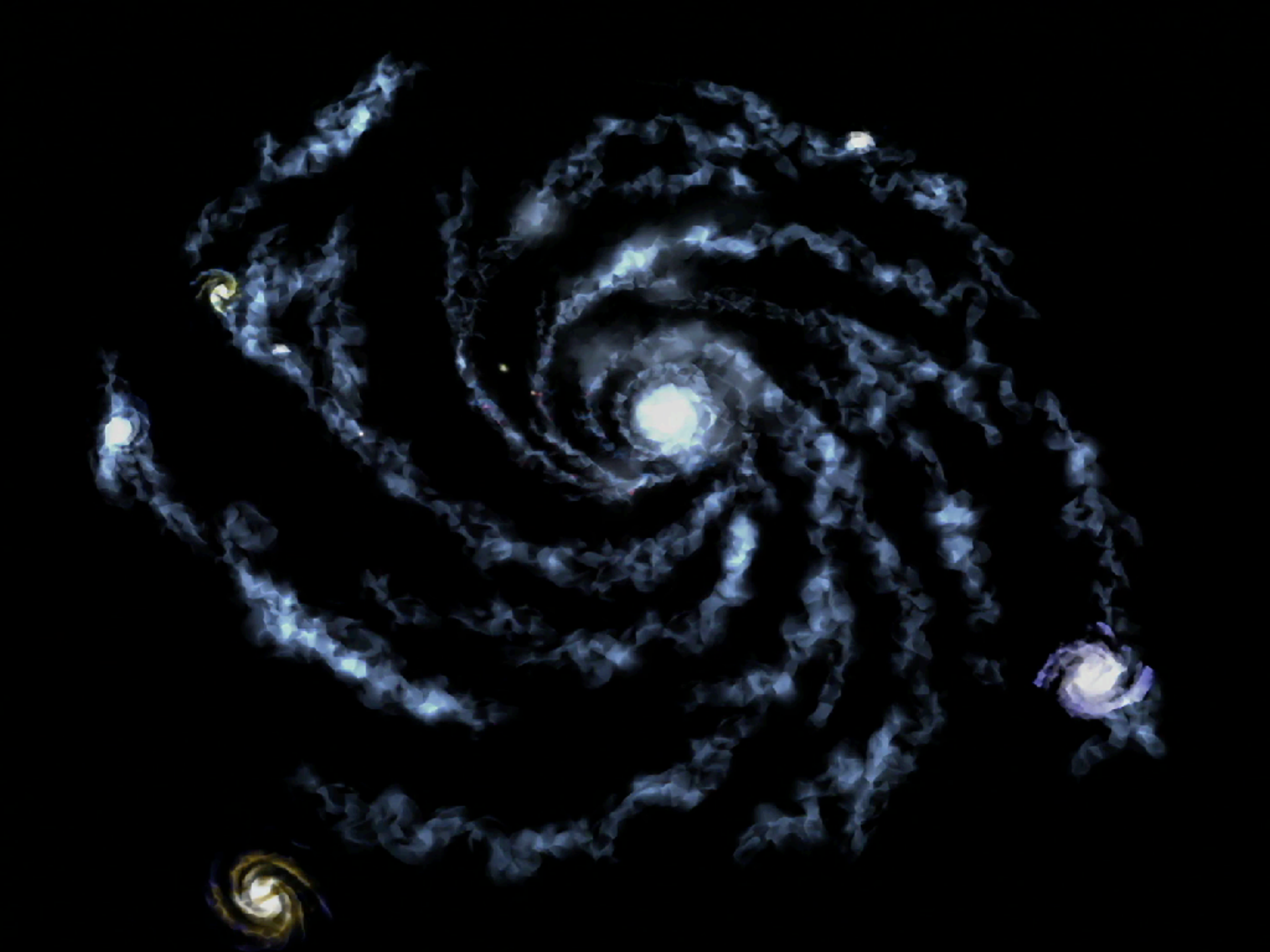
1989 COBE



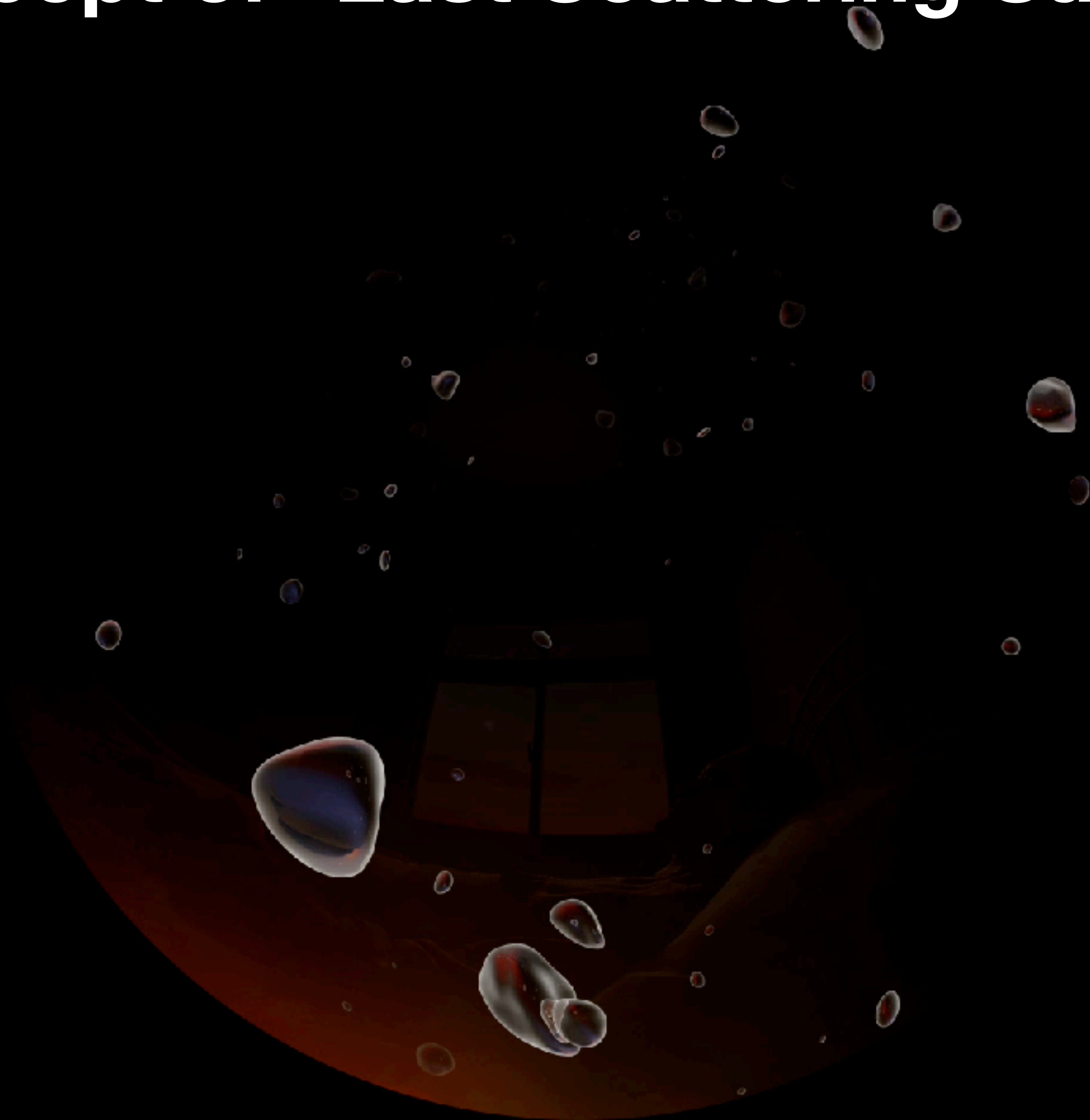
2001 WMAP

WMAP Science Team July 19, 2002

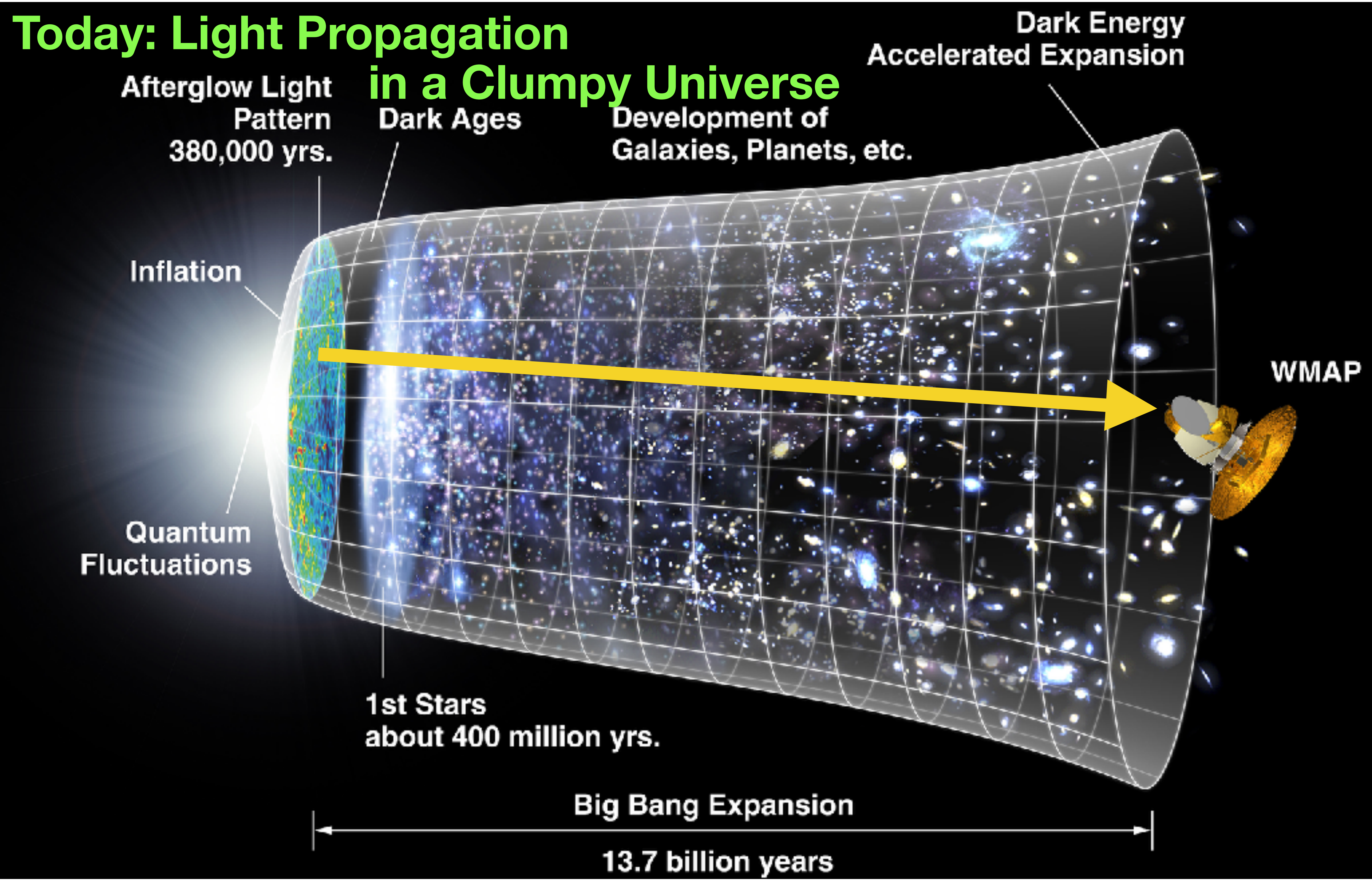
- WMAP was launched on June 30, 2001
- The WMAP mission ended after 9 years of operation



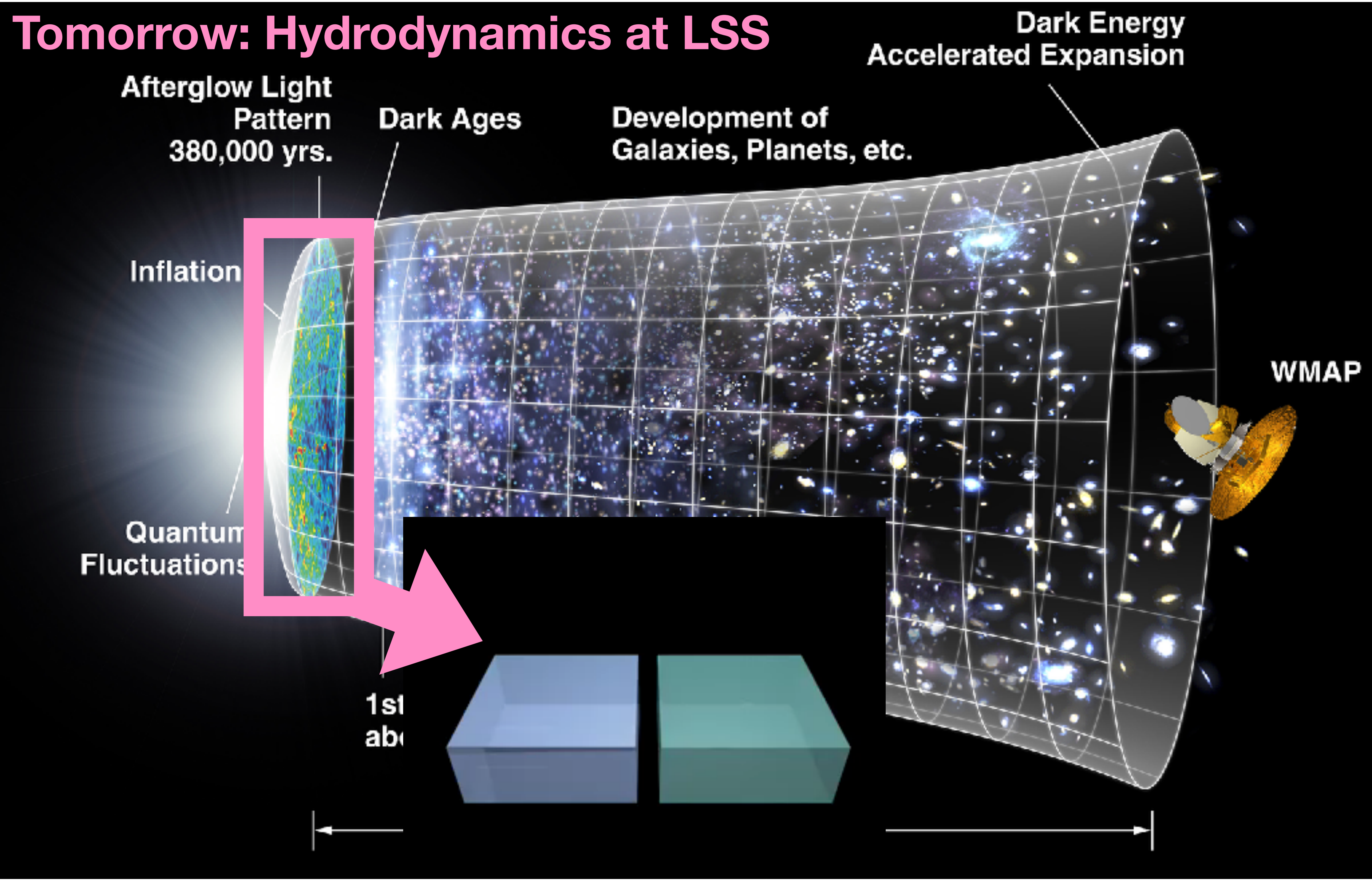
Concept of “Last Scattering Surface”



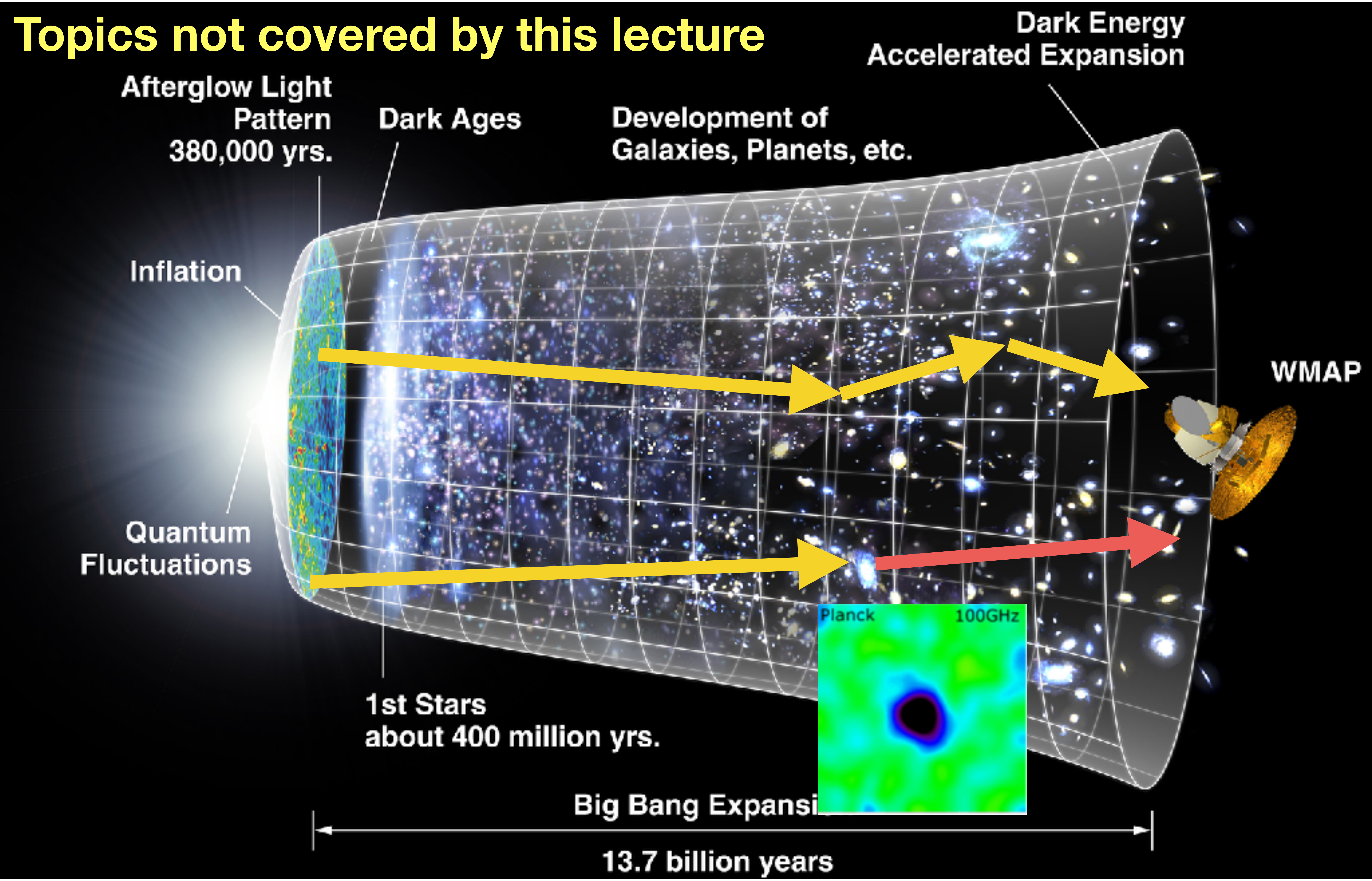
Today: Light Propagation in a Clumpy Universe



Tomorrow: Hydrodynamics at LSS

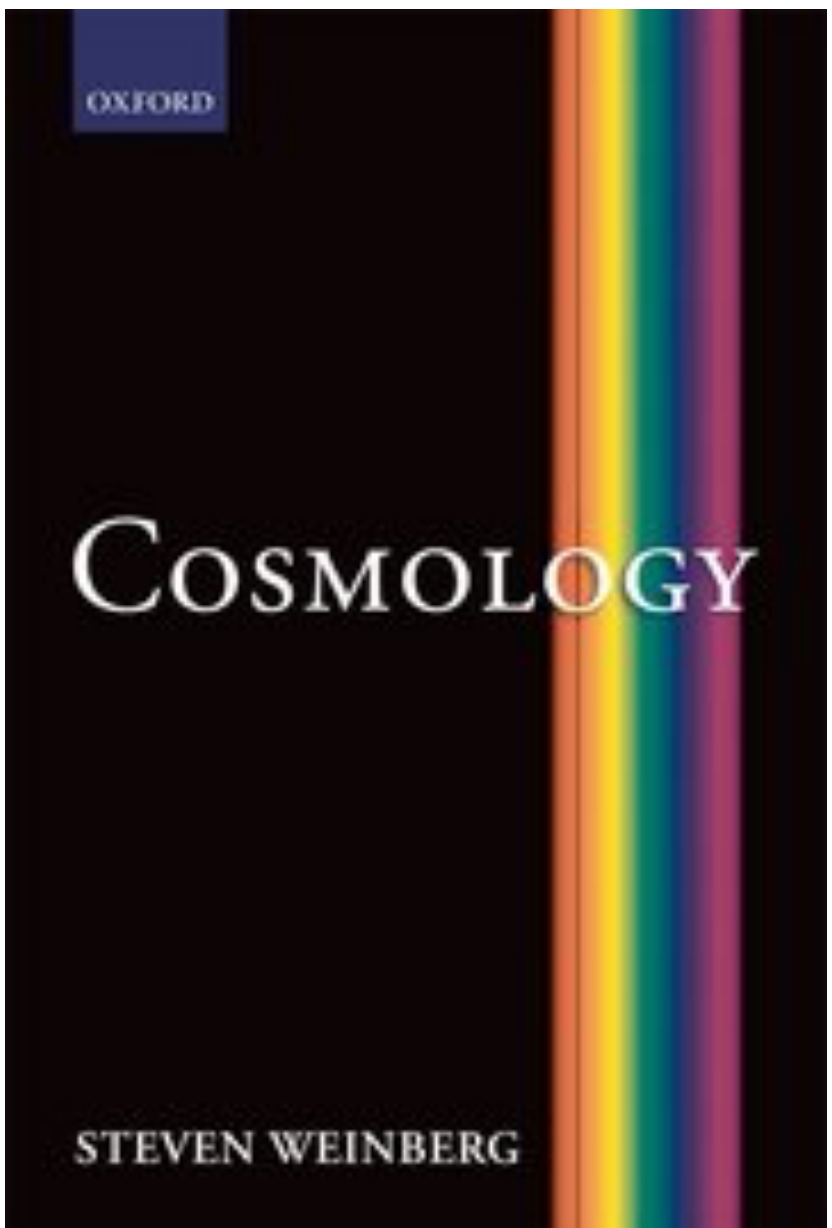


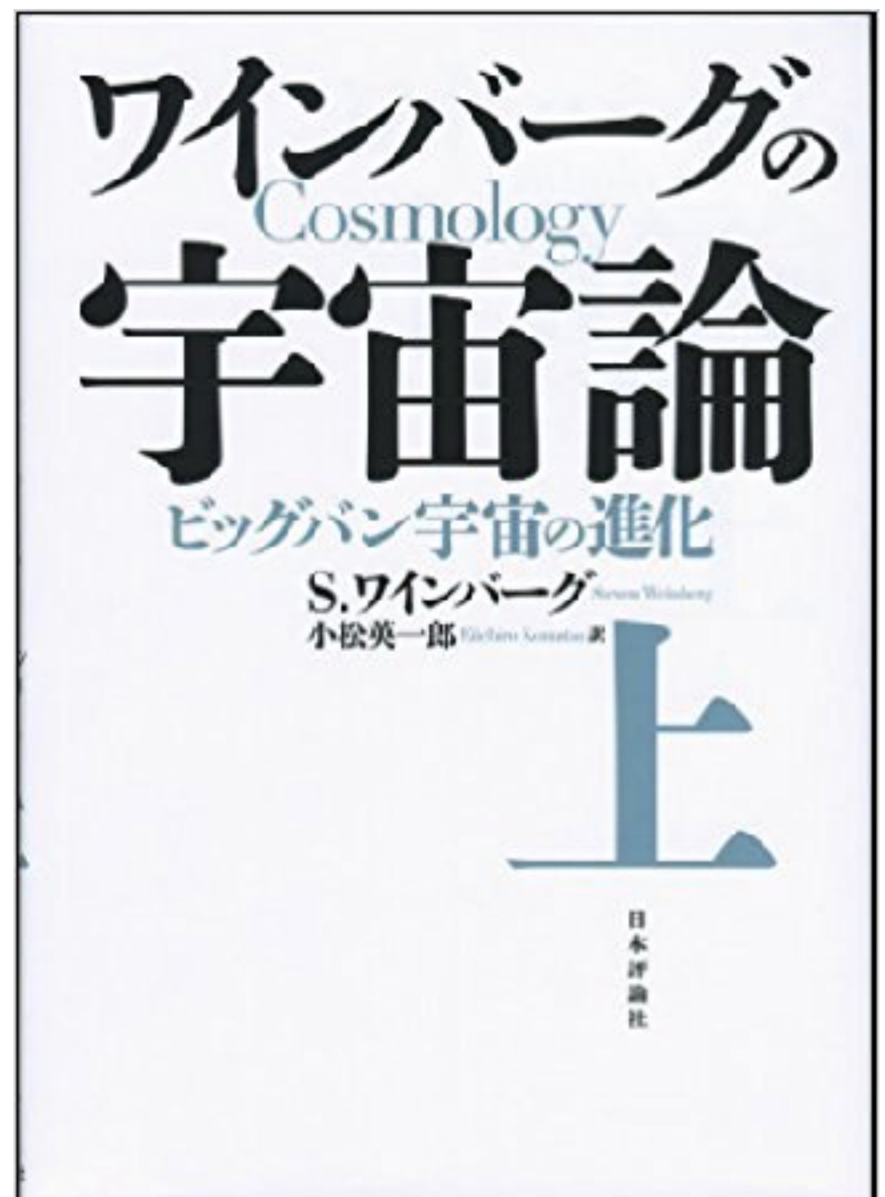
Topics not covered by this lecture



Notation

- Notation in my lectures follows that of the text book “Cosmology” by Steven Weinberg





Cosmological Parameters

- Unless stated otherwise, we shall assume a **spatially-flat Λ Cold Dark Matter** (ΛCDM) model with

$$\Omega_B h^2 = 0.022$$ **[baryon density]**

$$\Omega_M h^2 = 0.14$$ **[total mass density]**

$$\Omega_M = 0.3$$

which implies:

$$\Omega_\Lambda = 0.7, \;\; \Omega_D h^2 = 0.118, \;\; \Omega_B = 0.04714$$

$$H_0 = 100 \; h \; \mathrm{km \; s^{-1} \; Mpc^{-1}}; \;\; H_0 = 68.31 \; \mathrm{km \; s^{-1} \; Mpc^{-1}}$$

How light propagates in a clumpy universe?

- Photons gain/lose energy by **gravitational blue/redshifts**

this lecture

- Photons change their directions via **gravitational lensing**

not covered

Distance between two points in space

- Static (i.e., non-expanding) Euclidean space
  - In Cartesian coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = dx^2 + dy^2 + dz^2$$

Distance between two points in space

- Homogeneously expanding Euclidean space
  - In Cartesian **comoving** coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = a^2(t)(dx^2 + dy^2 + dz^2)$$

"scale factor"

Distance between two points in space

- Homogeneously expanding Euclidean space
  - In Cartesian **comoving** coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = a^2(t) \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \, dx^i dx^j$$

"scale factor"

δ_{ij} **=1 for i=j**

=0 otherwise

Distance between two points in space

- Inhomogeneous curved space
  - In Cartesian **comoving** coordinates $\boldsymbol{x} = (x, y, z)$

$$ds^2 = a^2 \sum_{i=1}^{3} \sum_{j=1}^{3} (\delta_{ij} + h_{ij}) dx^i dx^j$$

"metric perturbation"

-> CURVED SPACE!

Not just space…

- Einstein told us that a clock ticks slowly when gravity is strong…
- Space-time distance, ds_4, is modified by the presence of gravitational fields

$$ds_4^2 = -\exp(2\Phi)dt^2 + a^2\exp(-2\Psi)\sum_{i=1}^{3}\sum_{j=1}^{3}[\exp(D)]_{ij}dx^i dx^j$$

Φ **: Newton's gravitational potential**

Ψ **: Spatial scalar curvature perturbation**

D_{ij} **: Tensor metric perturbation [=gravitational waves]**

Tensor perturbation D_{ij}: Area-conserving deformation

- Determinant of a matrix

$$[\exp(D)]_{ij} \equiv \delta_{ij} + D_{ij} + \frac{1}{2}\sum_{k=1}^{3} D_{ik}D_{kj} + \frac{1}{6}\sum_{km} D_{ik}D_{km}D_{mj} + \cdots$$

is given by $\exp(\sum_i D_{ii})$

- **Thus, D_{ij} must be trace-less** $\sum_i D_{ii} = 0$

if it is area-conserving deformation of two points in space

Not just space…

- Einstein told us that a clock ticks slowly when gravity is strong…
- Space-time distance, ds_4, is modified by the presence of gravitational fields

$$ds_4^2 = -\exp(2\Phi)dt^2 + a^2\exp(-2\Psi)\sum_{i=1}^{3}\sum_{j=1}^{3}[\exp(D)]_{ij}dx^i dx^j$$

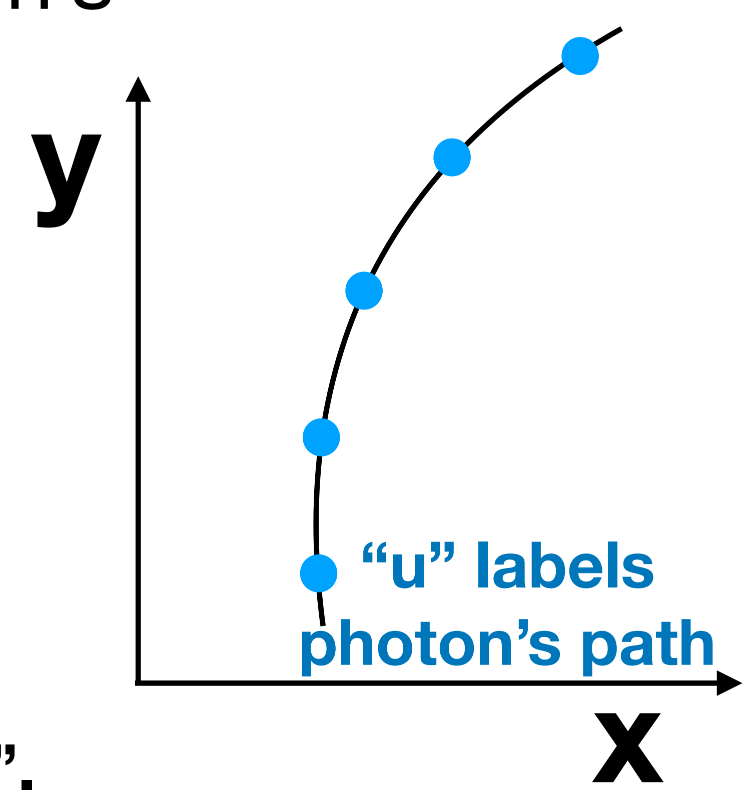
Φ **: Newton's gravitational potential**

Ψ **: Spatial scalar curvature perturbation**
is a perturbation to the determinant of spatial metric

Evolution of photon's coordinates

- Photon's path is determined such that the distance traveled by a photon between two points is minimised. This yields the equation of motion for photon's coordinates $x^\mu = (t, x^i)$

$$\frac{d^2x^\lambda}{du^2} + \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0$$

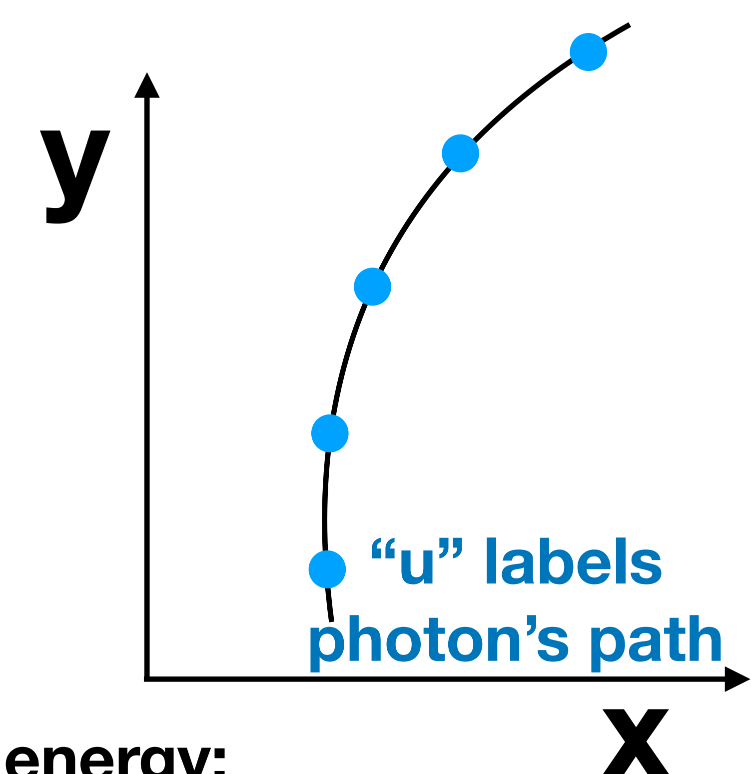


This equation is known as the "geodesic equation".
The second term is needed to keep the form of the equation unchanged under general coordinate transformation => GRAVITATIONAL EFFECTS!

Evolution of photon's momentum

- It is more convenient to write down the geodesic equation in terms of the **photon momentum**:

$$p^\mu \equiv \frac{dx^\mu}{du}$$



then

$$\frac{dp^\lambda}{dt} + \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \Gamma^\lambda_{\mu\nu}\frac{p^\mu p^\nu}{p^0} = 0$$

Magnitude of the photon momentum is equal to the photon energy:

$$p^2 \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}p^i p^j$$

Some calculations…

$$\frac{dp^\lambda}{dt} + \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \Gamma^\lambda_{\mu\nu} \frac{p^\mu p^\nu}{p^0} = 0$$

With $ds_4^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu$ $\left(\begin{array}{l} g_{00} = -\exp(2\Phi), \quad g_{0i} = 0, \\ g_{ij} = a^2 \exp(-2\Psi)[\exp(D)]_{ij} \end{array} \right)$

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} \sum_{\rho=0}^{3} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right)$$

Scalar perturbation [valid to all orders]

$$\Gamma^0_{00} = \dot{\Phi}, \quad \Gamma^0_{0i} = \frac{\partial \Phi}{\partial x^i}, \quad \Gamma^i_{00} = \exp(2\Phi) \sum_j g^{ij} \frac{\partial \Phi}{\partial x^j},$$

$$\Gamma^i_{0j} = \left(\frac{\dot{a}}{a} - \dot{\Psi} \right) \delta^i_j, \quad \Gamma^0_{ij} = \exp(-2\Phi) \left(\frac{\dot{a}}{a} - \dot{\Psi} \right) g_{ij},$$

$$\Gamma^k_{ij} = \delta_{ij} \sum_\ell \delta^{k\ell} \frac{\partial \Psi}{\partial x^\ell} - \delta^k_i \frac{\partial \Psi}{\partial x^j} - \delta^k_j \frac{\partial \Psi}{\partial x^i},$$

Tensor perturbation [valid to 1st order in D]

$$\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j + \frac{1}{2} \sum_k \delta^{ik} \dot{D}_{kj}, \quad \Gamma^0_{ij} = \frac{\dot{a}}{a} g_{ij} + \frac{a^2}{2} \dot{D}_{ij},$$

$$\Gamma^k_{ij} = \frac{1}{2} \sum_\ell \delta^{k\ell} \left(\frac{D_{i\ell}}{\partial x^j} + \frac{D_{\ell j}}{\partial x^i} - \frac{D_{ij}}{\partial x^\ell} \right),$$

Recap

Math may be messy but the concept is transparent!

- Requiring **photons to travel between two points in space-time with the minimum path length**, we obtained the geodesic equation

- The geodesic equation contains $\Gamma^{\lambda}_{\mu\nu}$ that is required to make **the form of the equation unchanged under general coordinate transformation**

- Expressing $\Gamma^{\lambda}_{\mu\nu}$ in terms of the metric perturbations, we obtain the desired result - **the equation that describes the rate of change of the photon energy!**

$$p^2 \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} p^i p^j$$

Sachs & Wolfe (1967)

The Result

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a} + \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i - \frac{1}{2}\sum_{ij}\dot{D}_{ij}\gamma^i\gamma^j$$

γ^i is a unit vector of the direction of photon's momentum:

$$\sum_i (\gamma^i)^2 = 1$$

- Let's interpret this equation *physically*

Sachs & Wolfe (1967)

The Result

$$\frac{1}{p}\frac{dp}{dt} = \boxed{-\frac{\dot{a}}{a}} + \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i - \frac{1}{2}\sum_{ij}\dot{D}_{ij}\gamma^i\gamma^j$$

γ^i is a unit vector of the direction of photon's momentum:

$$\sum_i (\gamma^i)^2 = 1$$

- **Cosmological redshift**
  - Photon's wavelength is stretched in proportion to the scale factor, and thus the photon energy decreases as

$$p \propto a^{-1}$$

Sachs & Wolfe (1967)

The Result

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a} + \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i - \frac{1}{2}\sum_{ij}\dot{D}_{ij}\gamma^i\gamma^j$$

- **Cosmological redshift - part II**
  - The spatial metric is given by $ds^2 = a^2(t)\exp(-2\Psi)d\mathbf{x}^2$
  - Thus, locally we can define a new scale factor: $\tilde{a}(t,\mathbf{x}) = a(t)\exp(-\Psi)$
  - Then the photon momentum decreases as

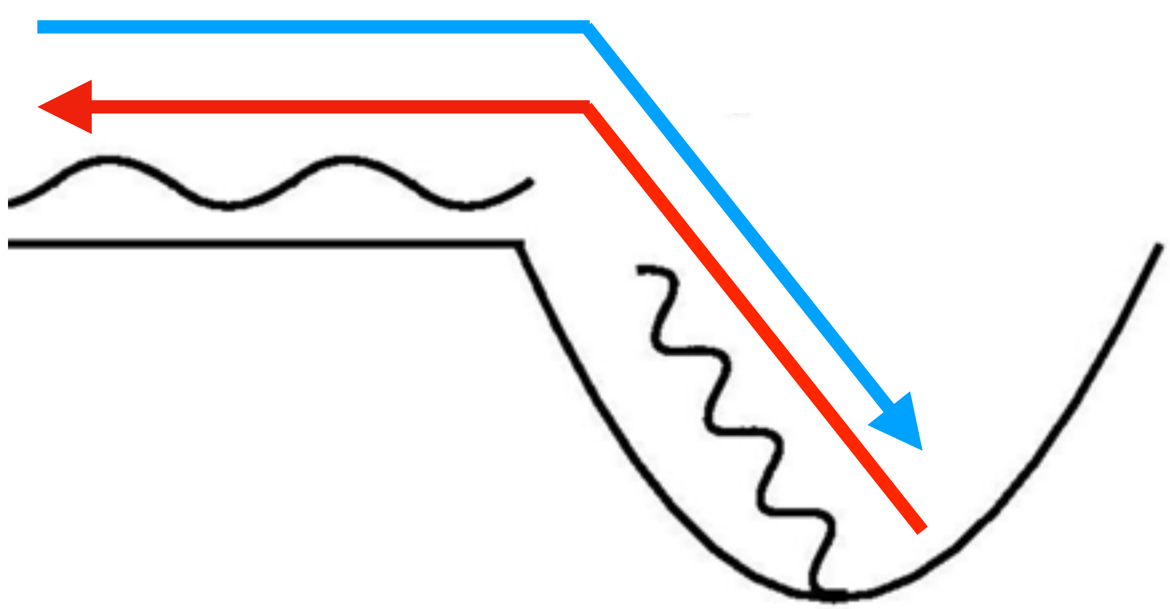
$$p \propto \tilde{a}^{-1}$$

Sachs & Wolfe (1967)

The Result

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a} + \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i - \frac{1}{2}\sum_{ij}\dot{D}_{ij}\gamma^i\gamma^j$$

- **Gravitational blue/redshift (Scalar)**



Potential well (ϕ < 0)

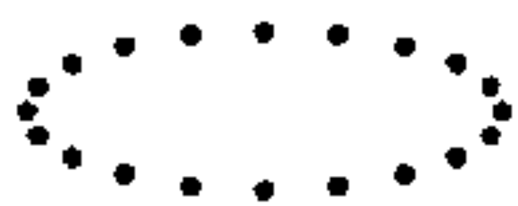
Sachs & Wolfe (1967)

The Result

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a} + \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i \boxed{- \frac{1}{2}\sum_{ij} \dot{D}_{ij}\gamma^i\gamma^j}$$

- **Gravitational blue/redshift (Tensor)**

$$D_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Sachs & Wolfe (1967)

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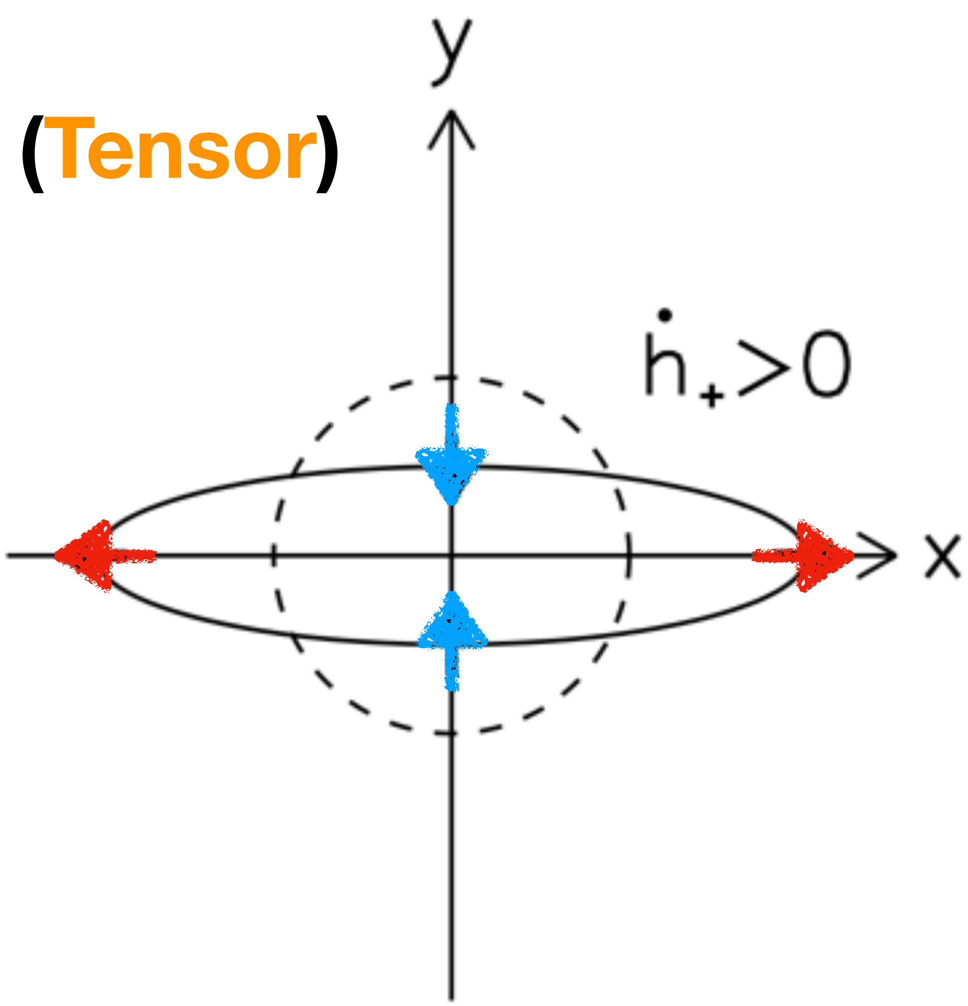


Sachs & Wolfe (1967)

Formal Solution (Scalar)

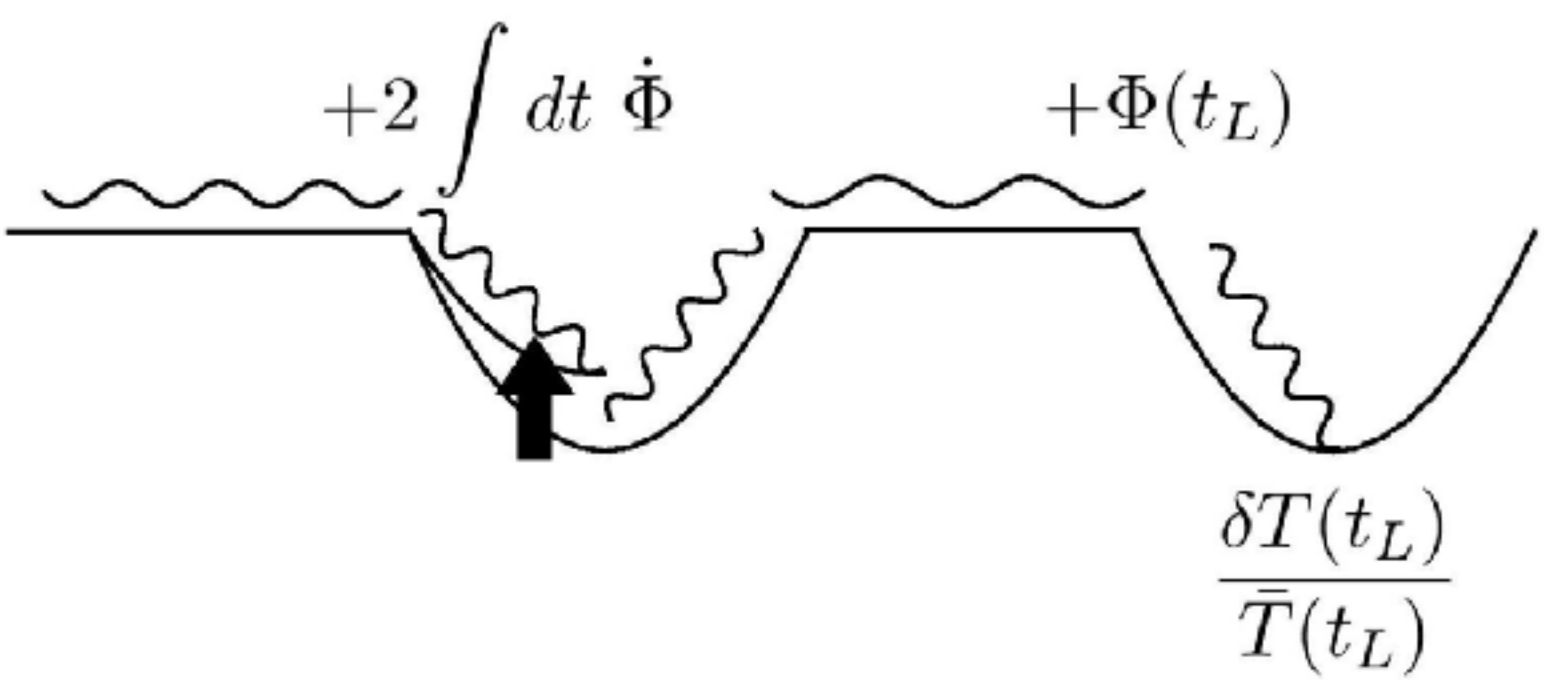
"L" for "Last scattering surface"

$$\ln(ap)(t_0) = \ln(ap)(t_L) + \Phi(t_L) - \Phi(t_0) + \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})$$

or

$$\frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i = \frac{d\Phi}{dt} - \dot{\Phi}$$

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0) + \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$



Line-of-sight direction

$$\hat{n}^i = -\gamma^i$$

Coming distance (r)

$$x^i = \hat{n}^i r$$

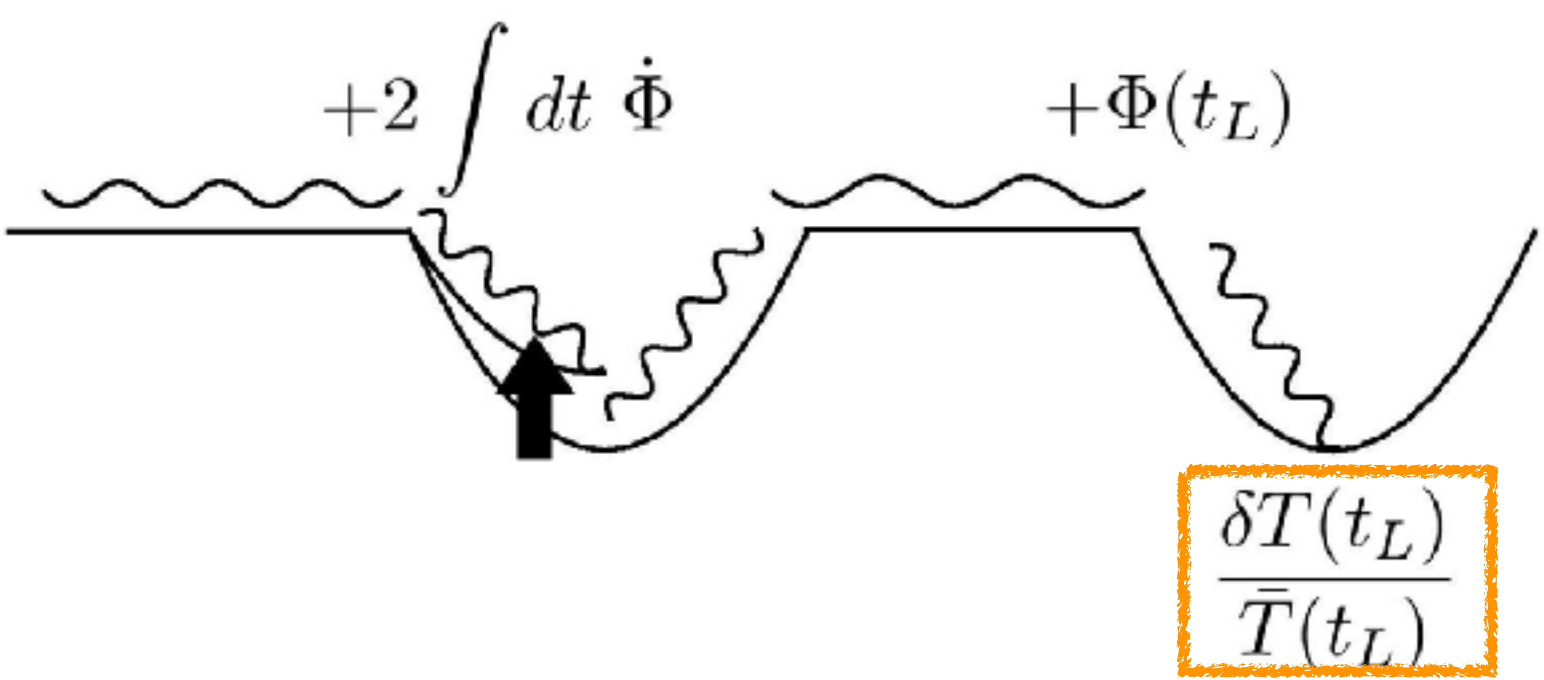
$$r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

Sachs & Wolfe (1967)

Formal Solution (Scalar)

Initial Condition

$$\frac{\Delta T(\hat{n})}{T_0} = \boxed{\frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)}} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0) + \int_{t_L}^{t_0} dt \; (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$



Line-of-sight direction

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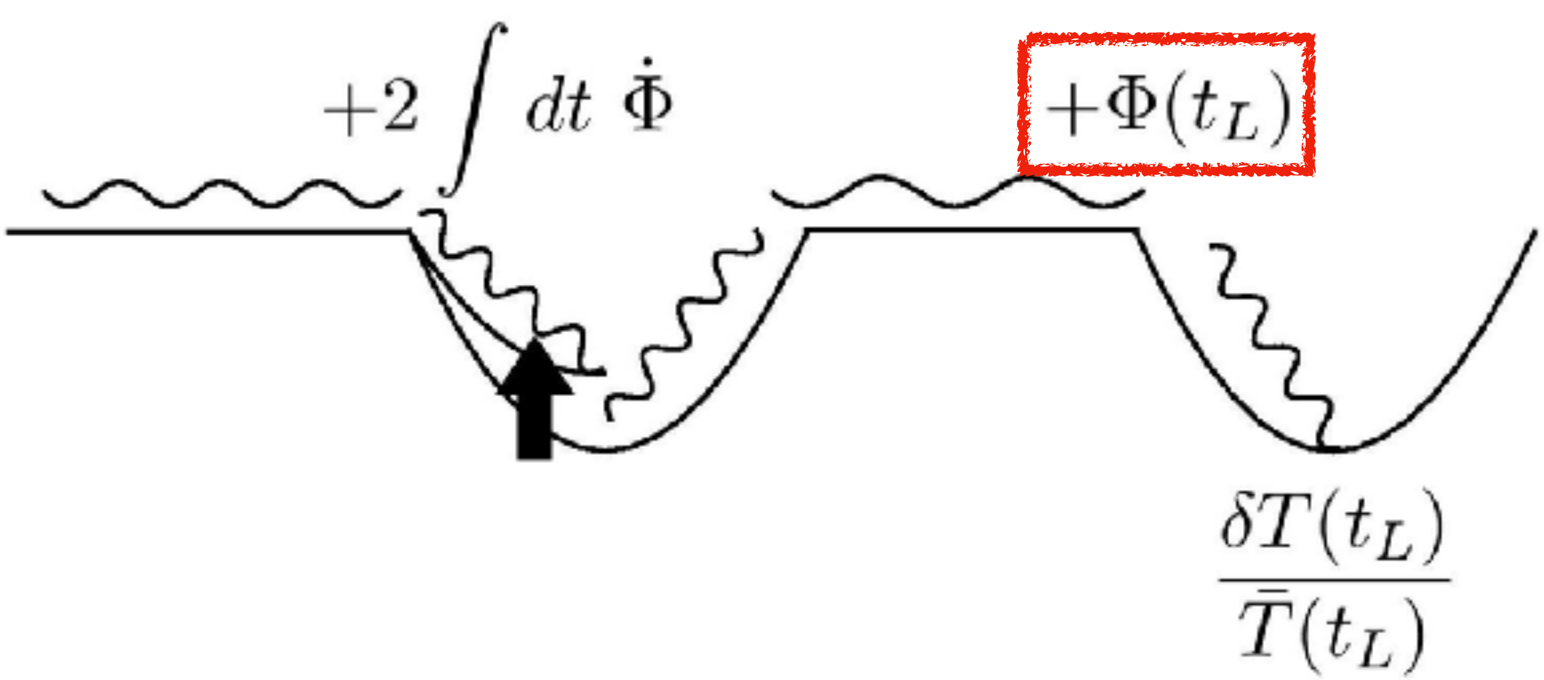
Sachs & Wolfe (1967)

Formal Solution (Scalar)

Gravitational Redshit

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} \boxed{+ \Phi(t_L, \hat{n} r_L)} - \Phi(t_0, 0)$$

$$+ \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$



Line-of-sight direction

$$\hat{n}^i = -\gamma^i$$

Comoving distance (r)

$$x^i = \hat{n}^i r$$

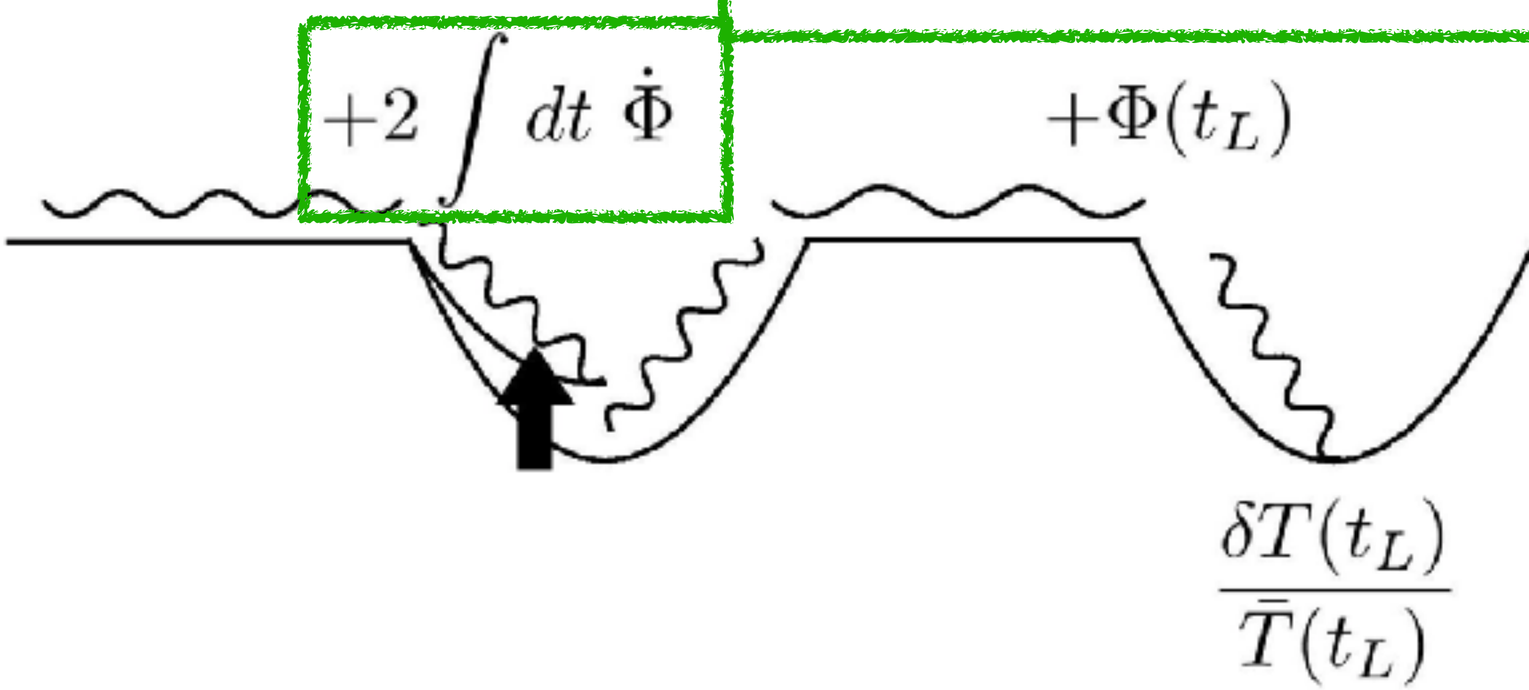
$$r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

Formal Solution (Scalar)

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0)$$

"integrated Sachs-Wolfe" (ISW) effect

$$+ \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$



Line-of-sight direction

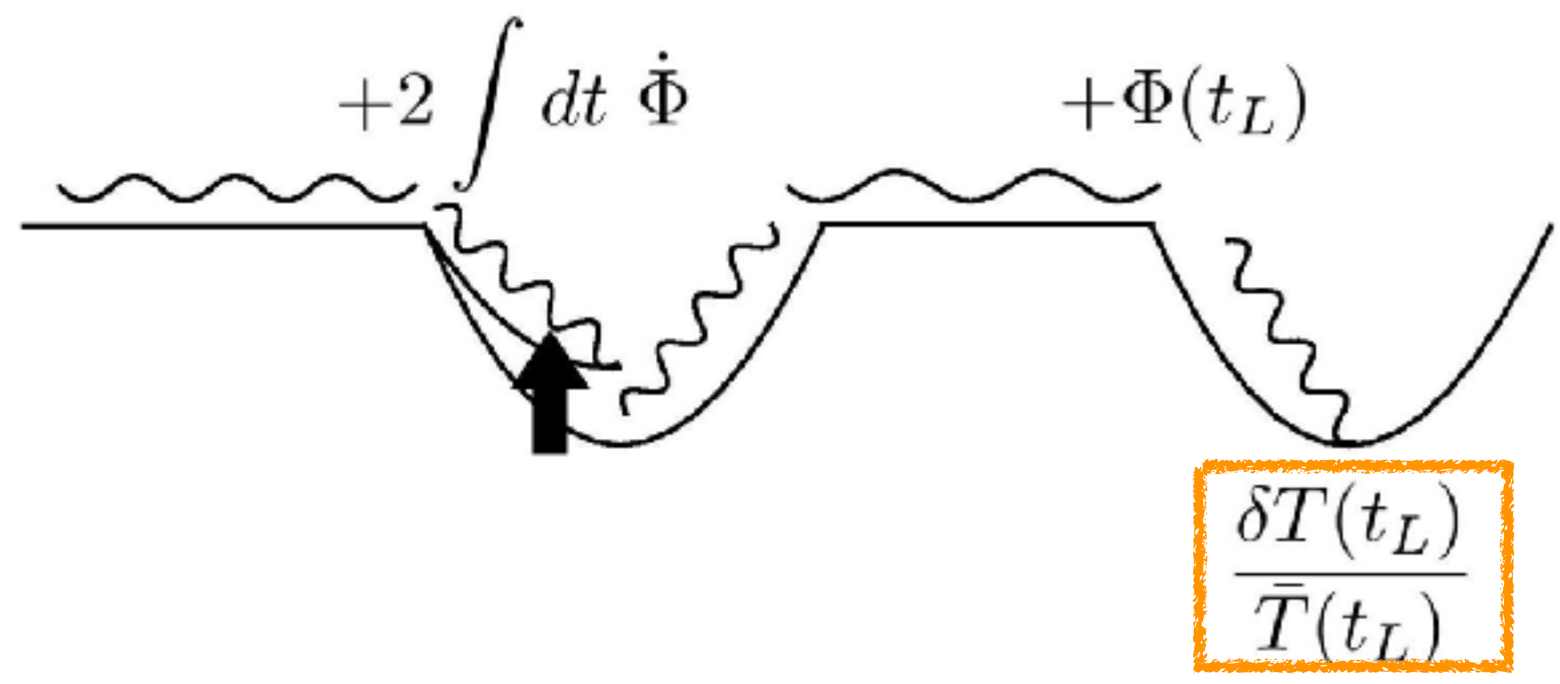
$$\hat{n}^i = -\gamma^i$$

Coming distance (r)

$$x^i = \hat{n}^i r$$

$$r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

Initial Condition



- *"Were photons hot or cold at the bottom of the potential well at the last scattering surface?"*
- This must be assumed a priori - **only the data can tell us!**

“Adiabatic” Initial Condition

- Definition: “*Ratios of the number densities of all species are equal everywhere initially*”
  - For i^{th} and j^{th} species, **$n_i(x)/n_j(x)$ = constant**
- For a quantity X(t,x), let us define the **fluctuation, *δX*,** as

$$\delta X(t, \boldsymbol{x}) \equiv X(t, \boldsymbol{x}) - \bar{X}(t)$$

- Then, the adiabatic initial condition is

$$\frac{\delta n_i(t_{\rm initial}, \mathbf{x})}{\bar{n}_i(t_{\rm initial})} = \frac{\delta n_j(t_{\rm initial}, \mathbf{x})}{\bar{n}_j(t_{\rm initial})}$$

Example: Thermal Equilibrium

- When photons and baryons were in thermal equilibrium in the past, then
  - $n_{photon} \sim T^3$ and $n_{baryon} \sim T^3$
  - That is to say, **thermal equilibrium naturally gives the adiabatic initial condition**
  - This gives

$$3\frac{\delta T(t_i, \boldsymbol{x})}{\bar{T}(t_i)} = \frac{\delta \rho_B(t_i, \boldsymbol{x})}{\bar{\rho}_B(t_i)}$$

- **"B" for "Baryons"**
- **ρ is the mass density**

Big Question

- ***How about dark matter?***

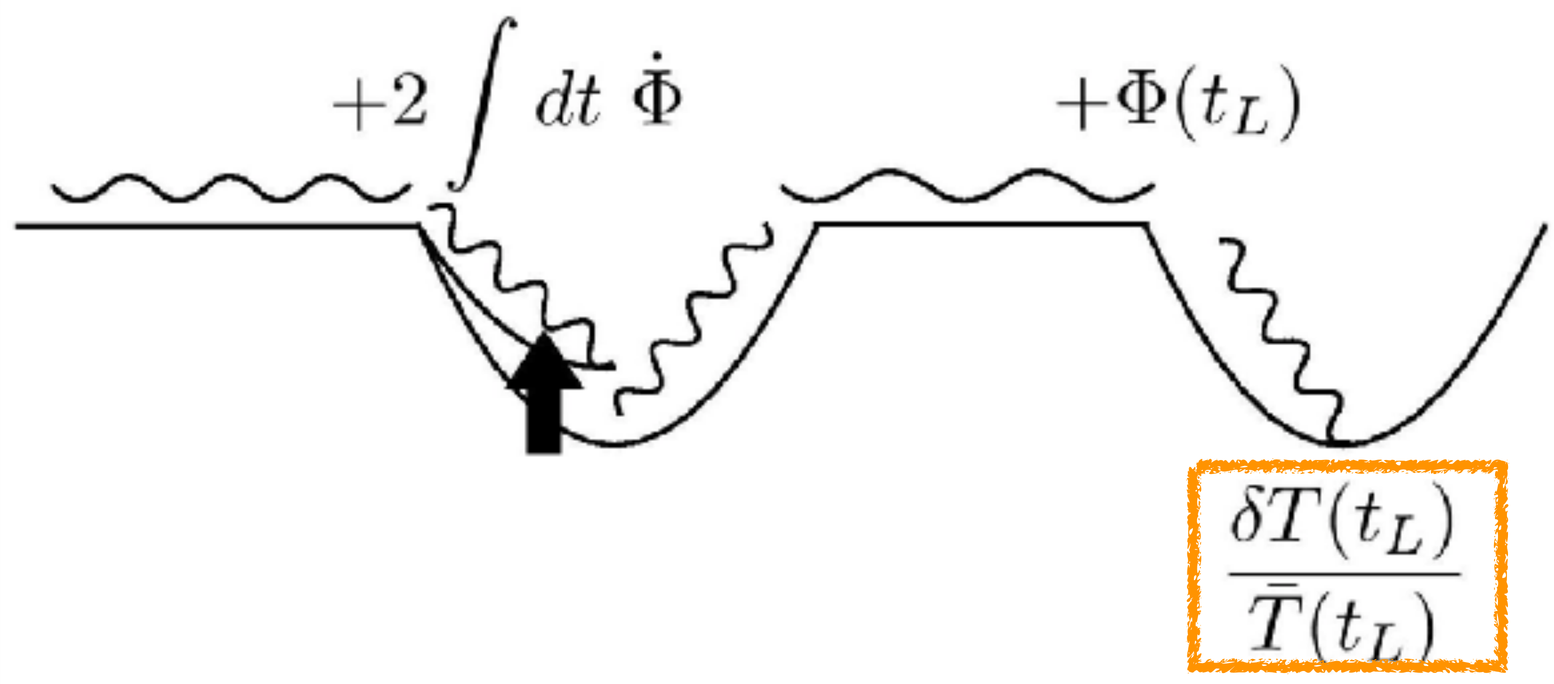
- If dark matter and photons were in thermal equilibrium in the past, then they should also obey the adiabatic initial condition

    - If not, *there is no a priori reason to expect the adiabatic initial condition!*

- The current data are consistent with the adiabatic initial condition. This means something important for the nature of dark matter!

We shall assume the adiabatic initial condition throughout the lectures

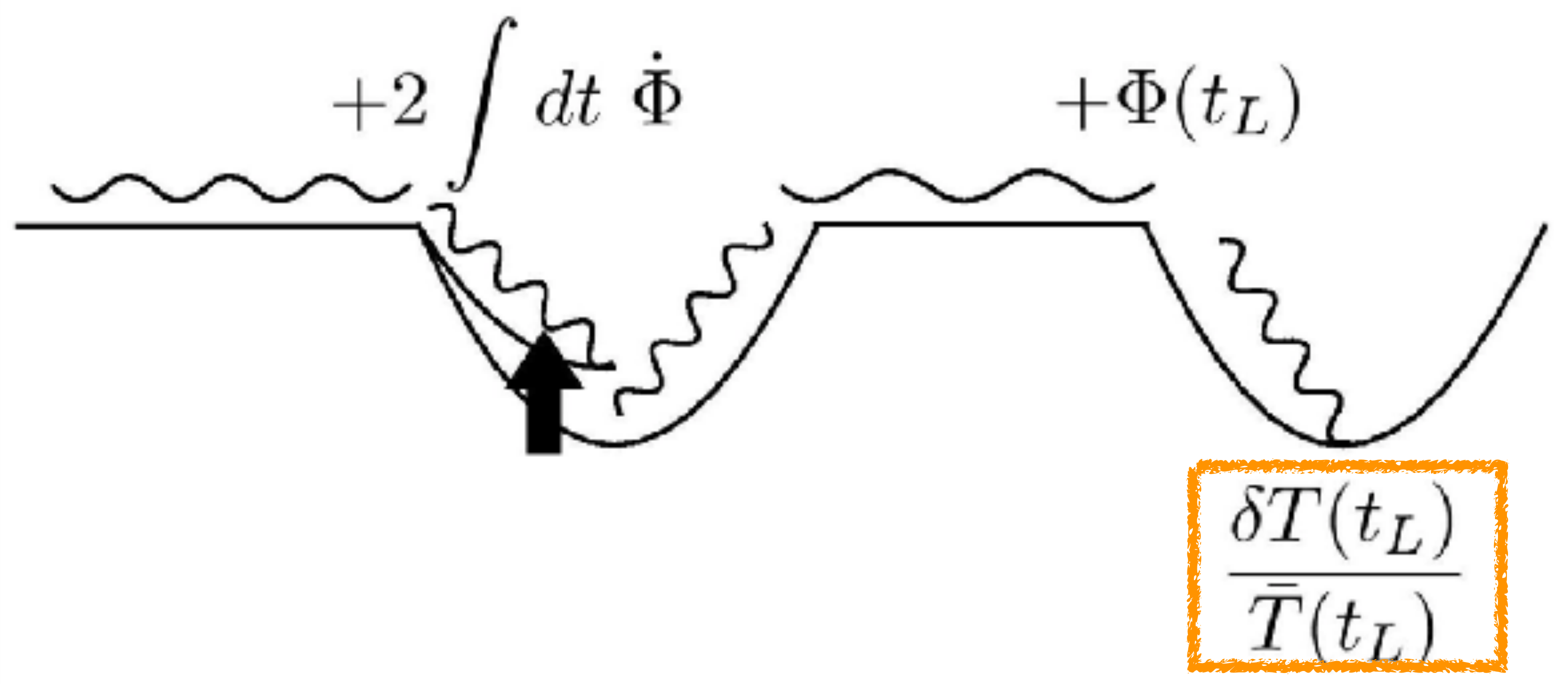
Adiabatic Solution



- At the last scattering surface, the temperature fluctuation is given by the matter density fluctuation as

$$\frac{\delta T(t_L, \mathbf{x})}{\bar{T}(t_L)} = \frac{1}{3}\frac{\delta\rho_M(t_L, \mathbf{x})}{\bar{\rho}_M(t_L)}$$

Adiabatic Solution

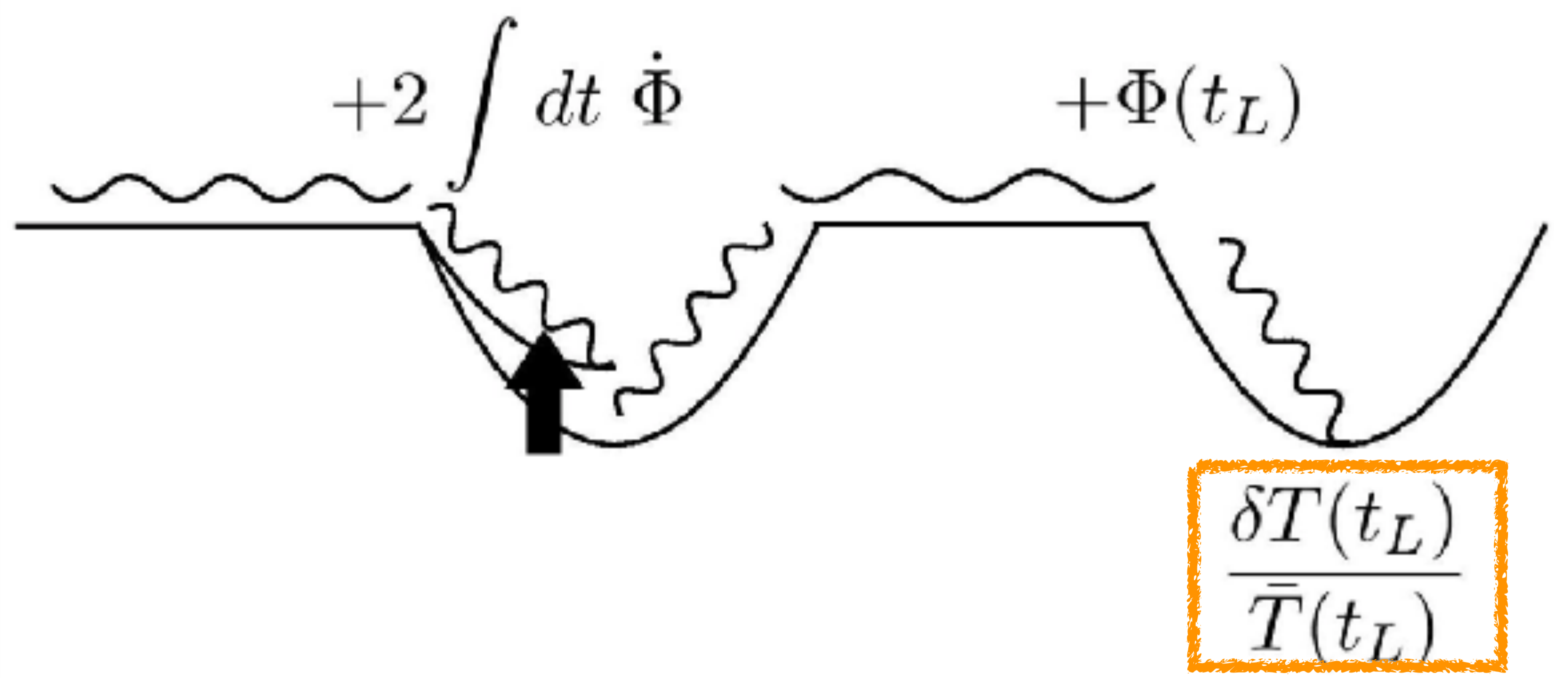


- On large scales, the matter density fluctuation during the matter-dominated era is given by $\delta\rho_M/\bar{\rho}_M = -2\Phi$; thus,

$$\frac{\delta T(t_L,\mathbf{x})}{\bar{T}(t_L)} = \frac{1}{3}\frac{\delta\rho_M(t_L,\mathbf{x})}{\bar{\rho}_M(t_L)} = -\frac{2}{3}\Phi(t_L,\mathbf{x})$$

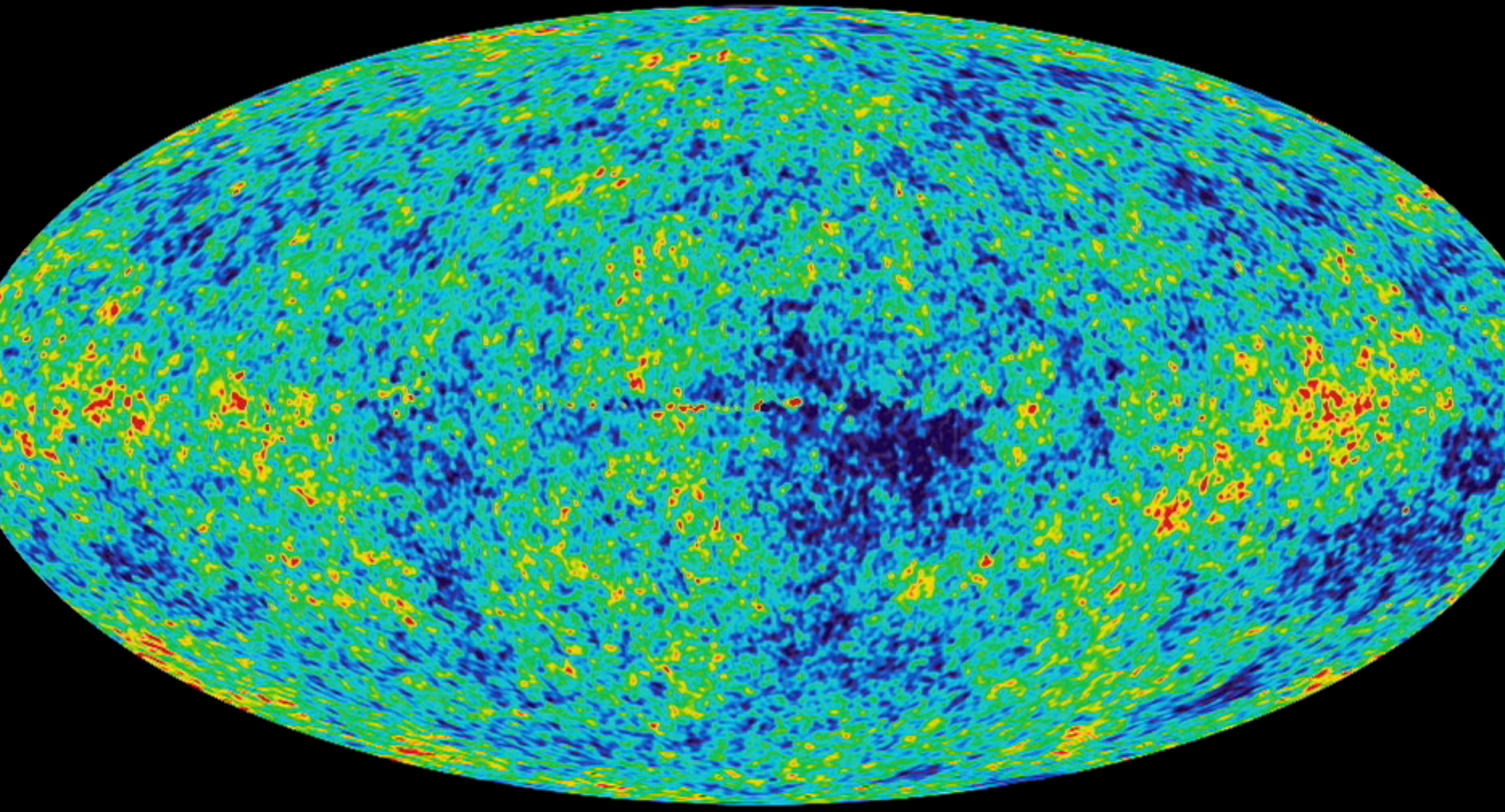
Hot **at the bottom of the potential well, but…**

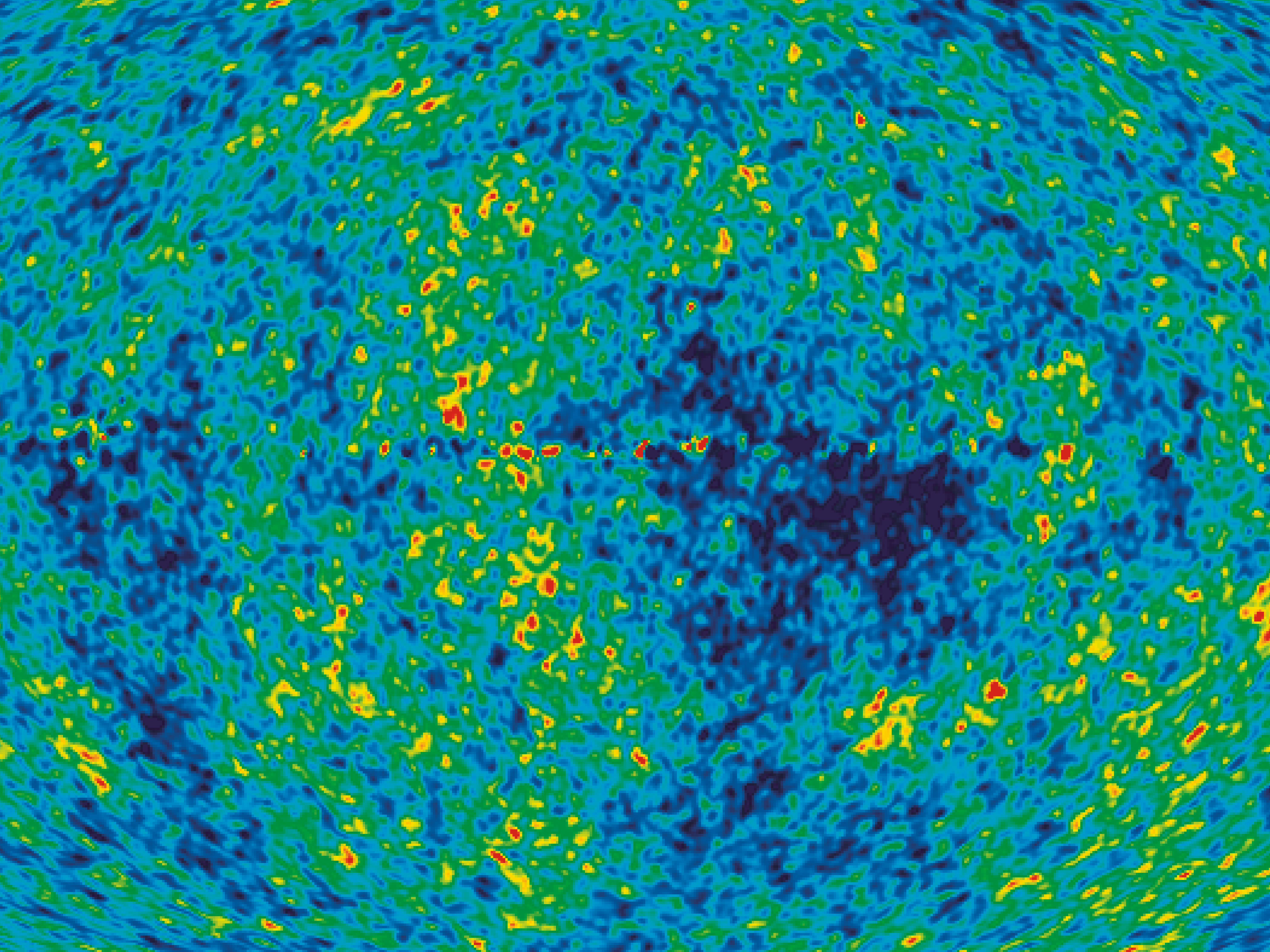
Over-density = Cold spot



- Therefore: $$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{r}_L)$$

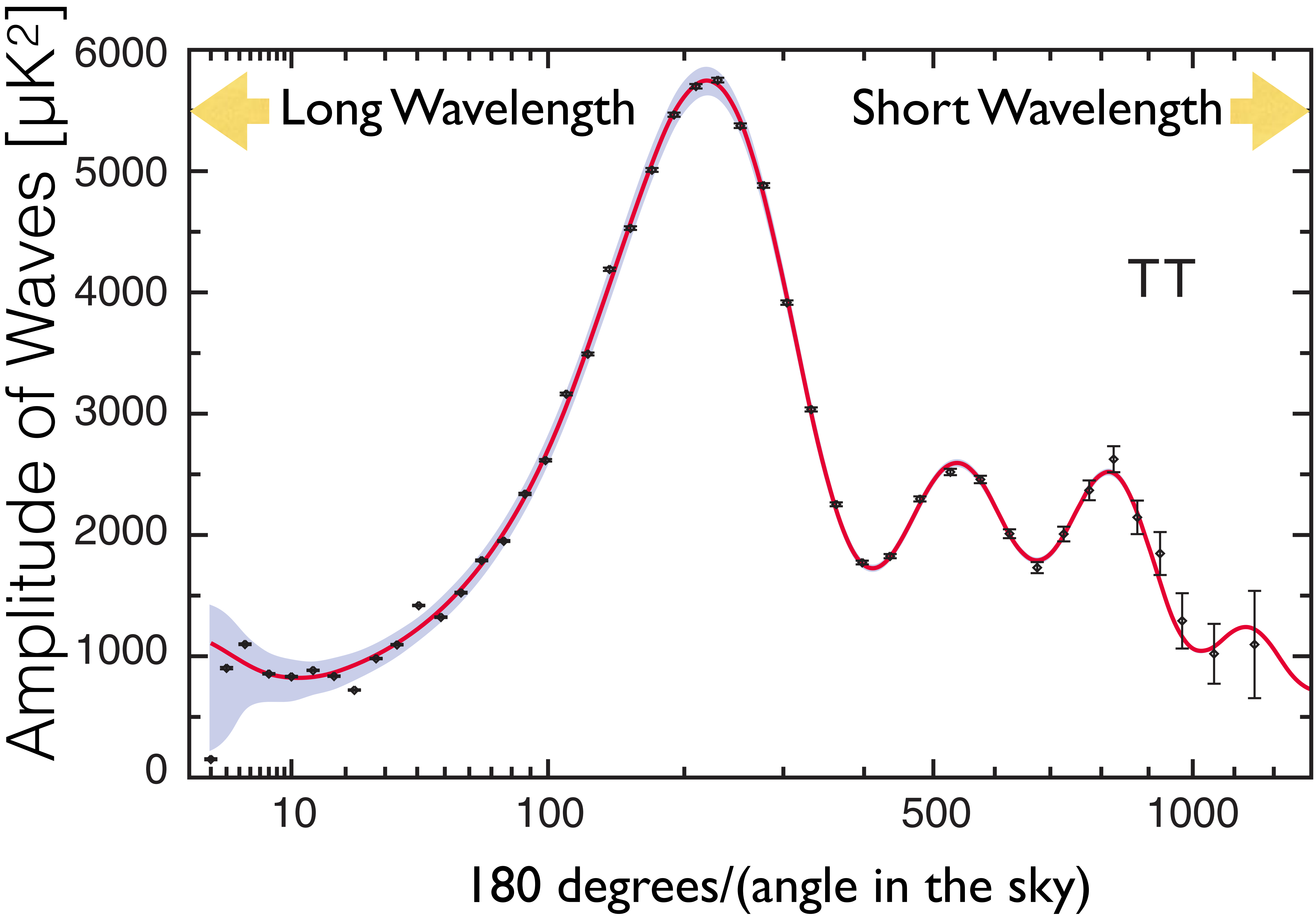
This is negative in an over-density region!

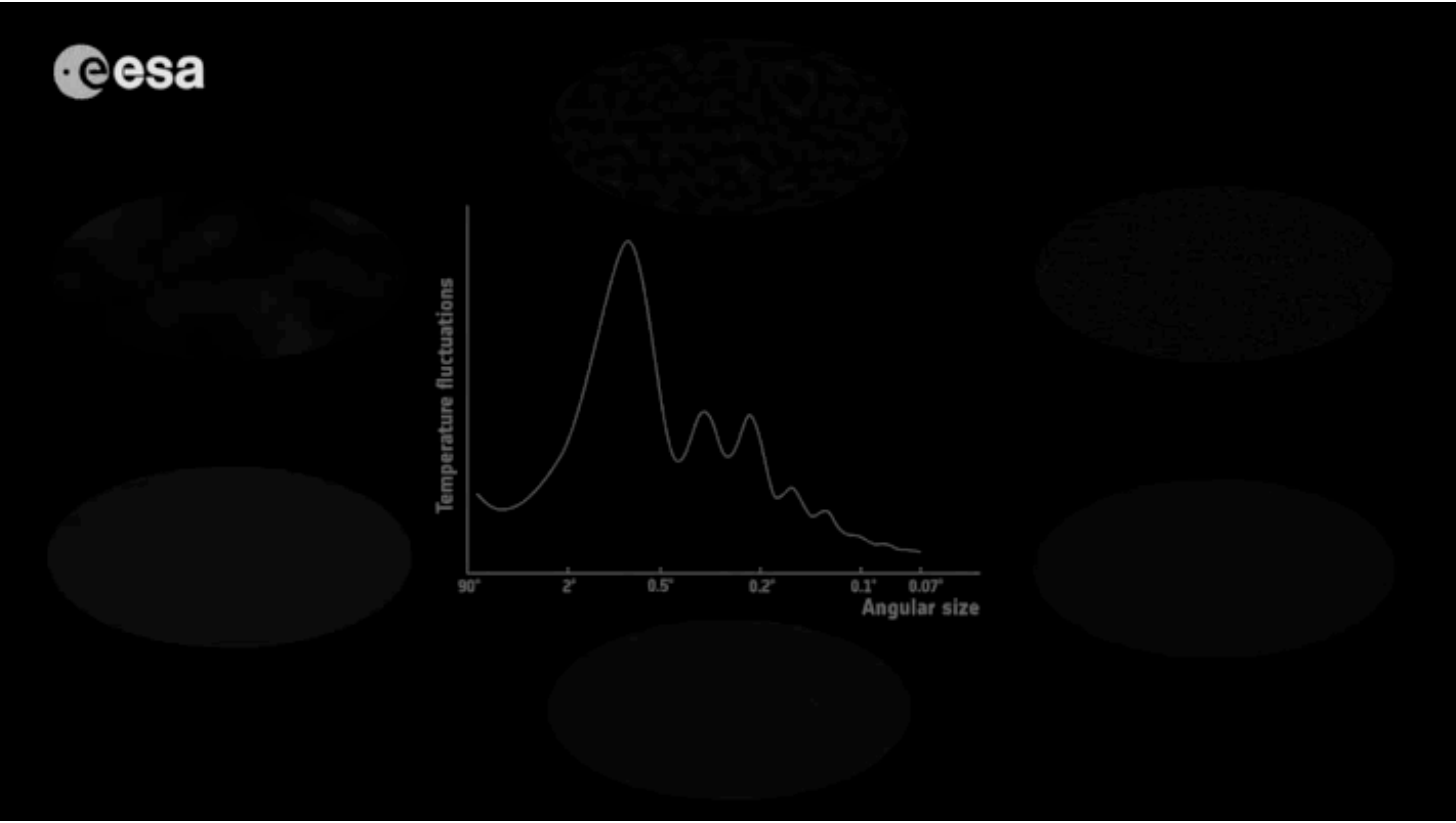


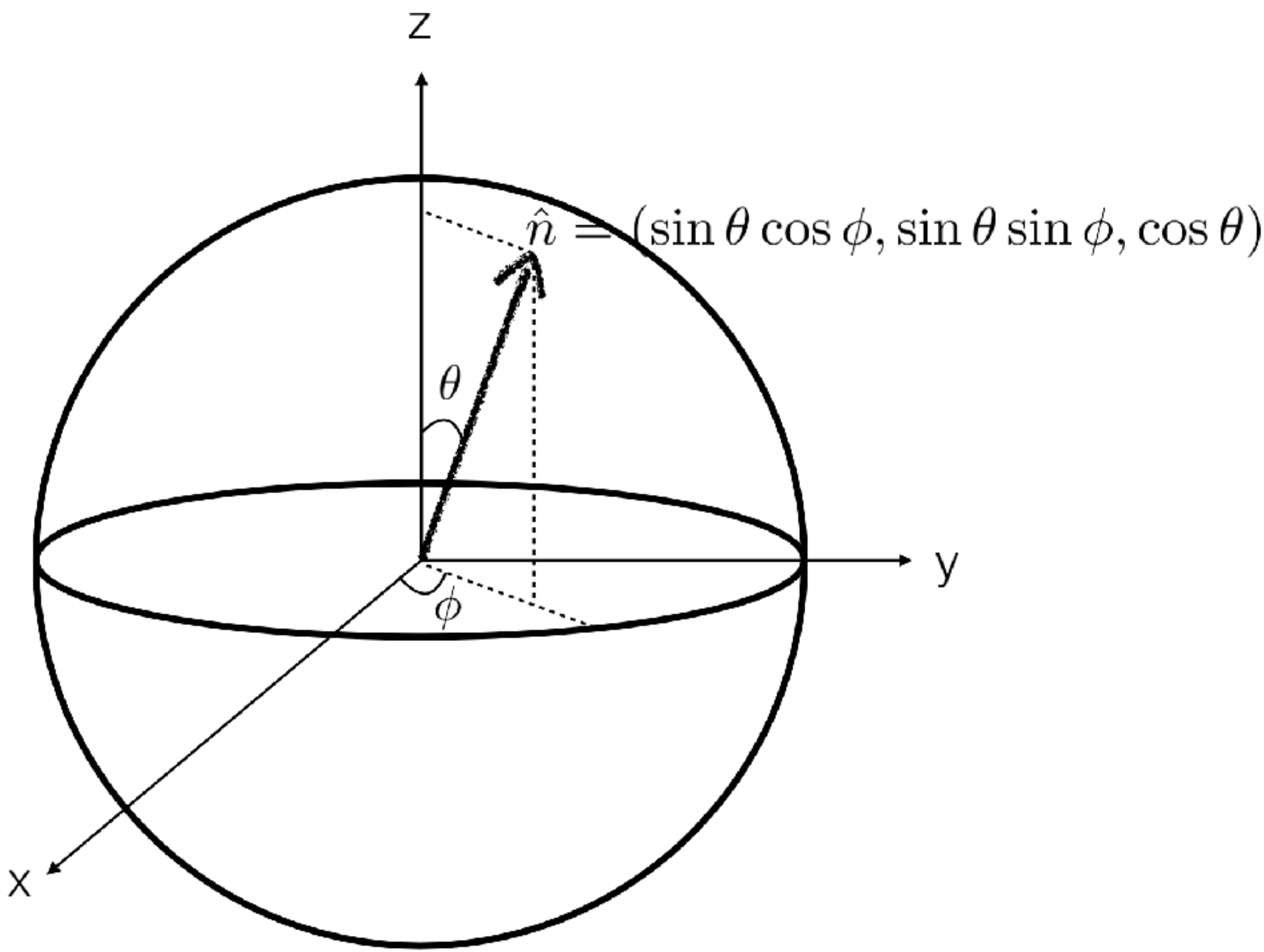


Data Analysis

- Decompose temperature fluctuations in the sky into a set of waves with various wavelengths
- Make a diagram showing the strength of each wavelength


Long Wavelength
Short Wavelength
TT
Amplitude of Waves [μK²]
6000
5000
4000
3000
2000
1000
0
10
100
500
1000
180 degrees/(angle in the sky)

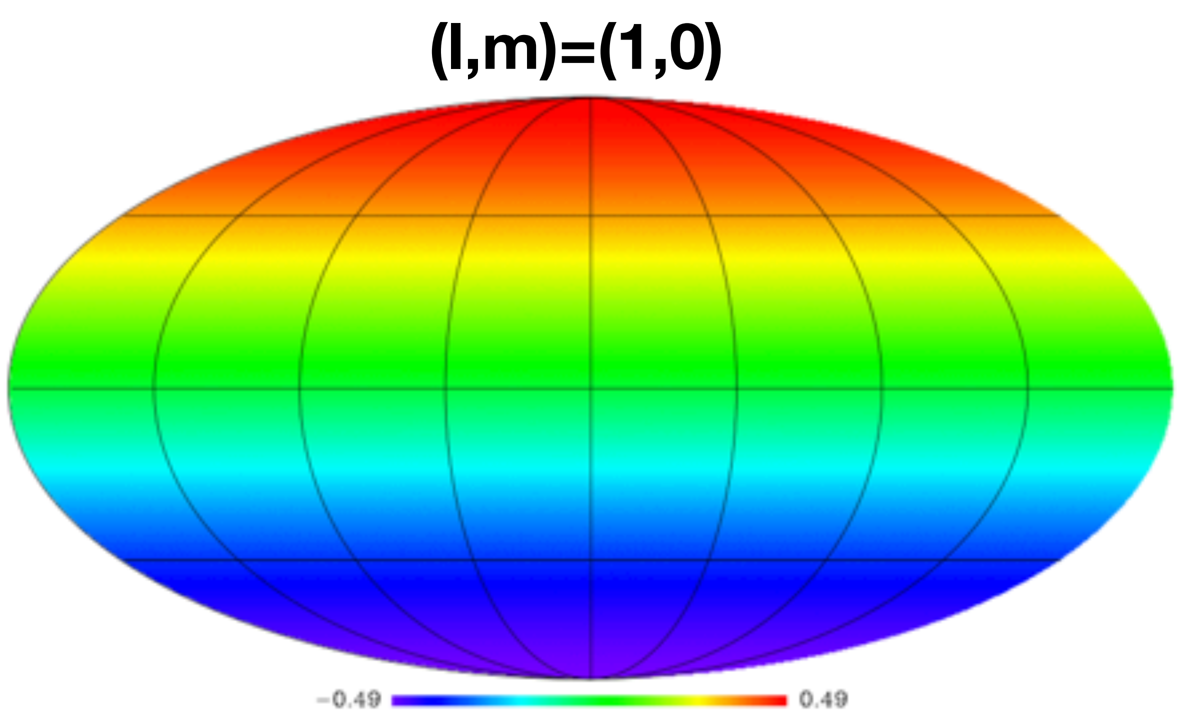

Temperature fluctuations
90°
2°
0.5°
0.2°
0.1°
0.07°
Angular size

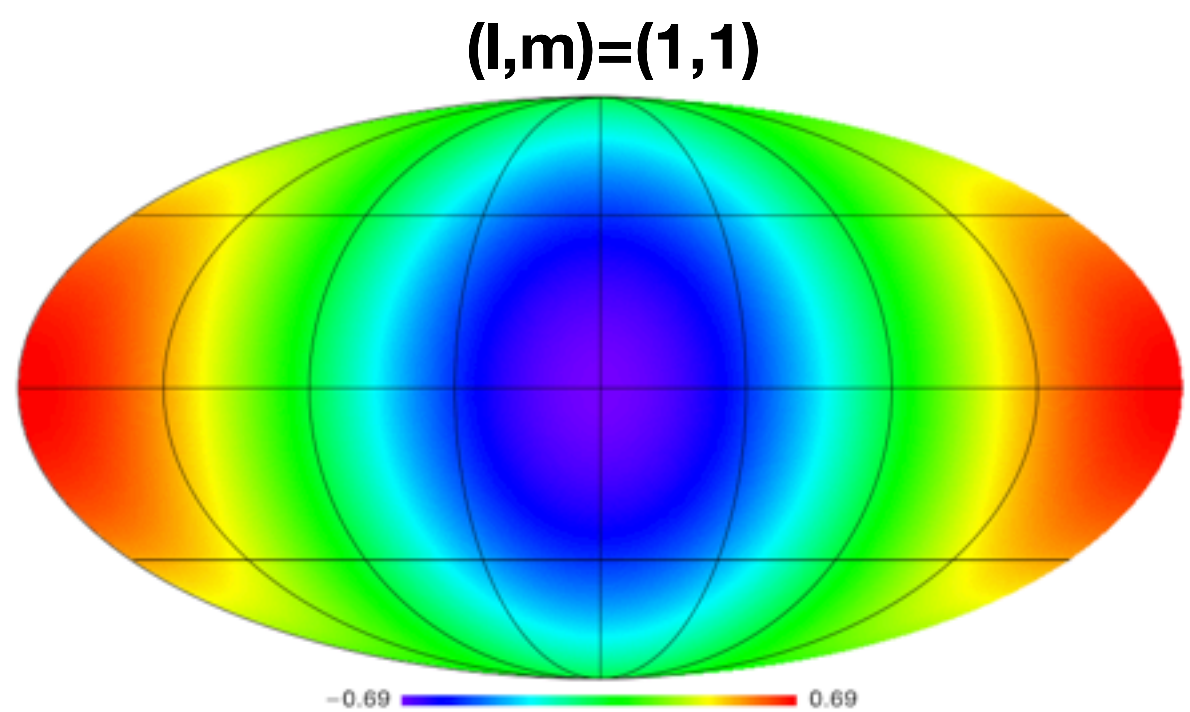

z
$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
θ
y
ϕ
x

Spherical Harmonic Transform

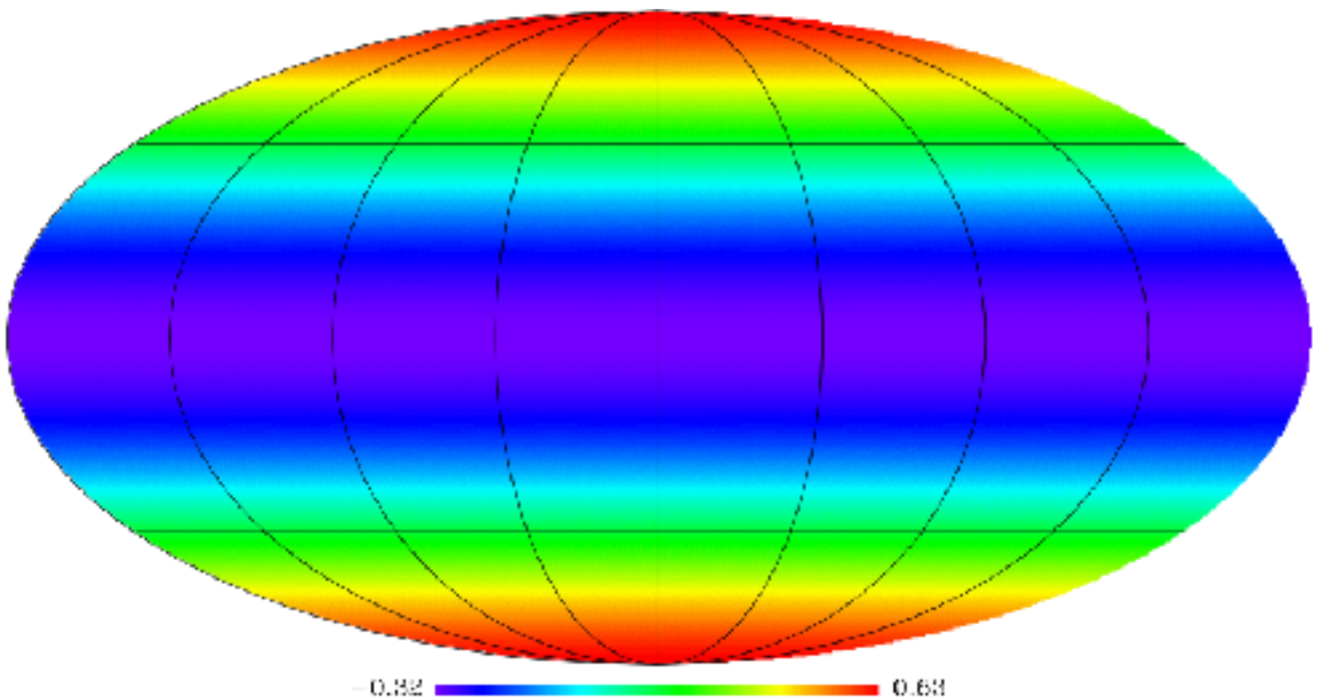
$$\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n})$$

- Values of a_{lm} depend on coordinates, but the squared amplitude, $\sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^*$, does not depend on coordinates

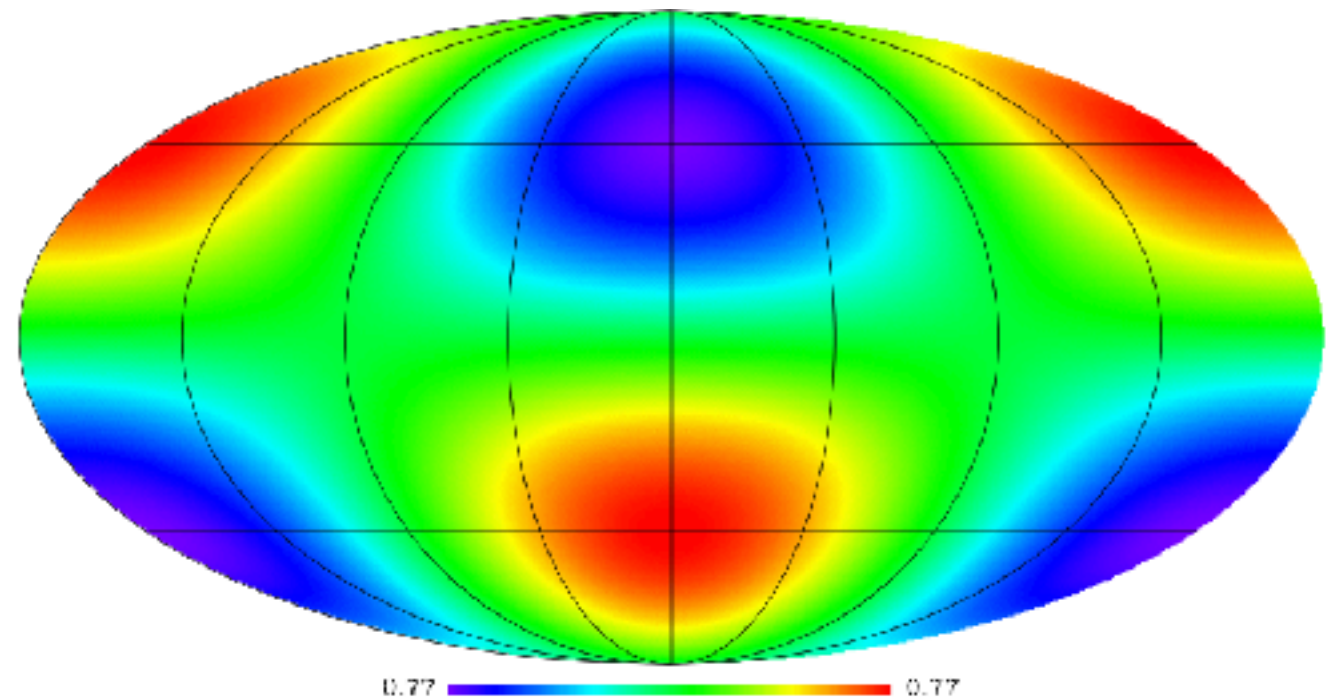




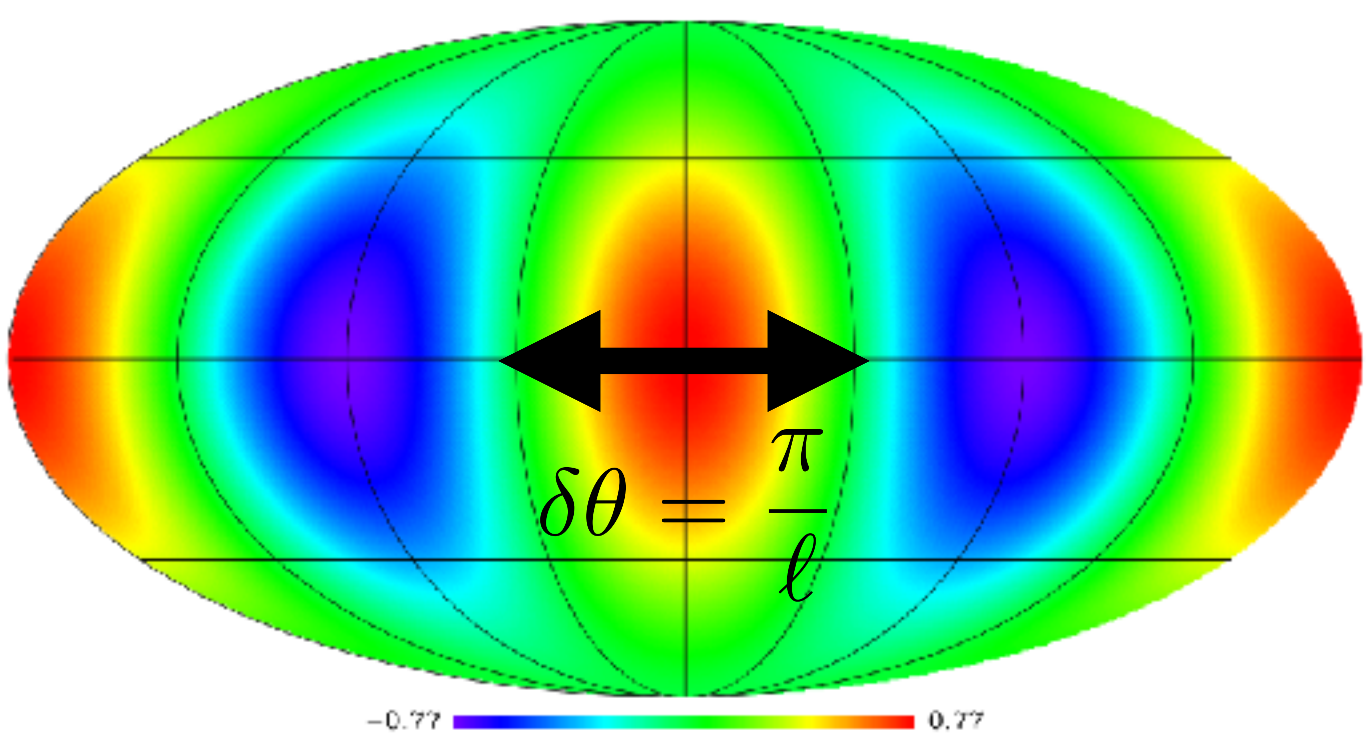
(l,m)=(2,0)



(l,m)=(2,1)

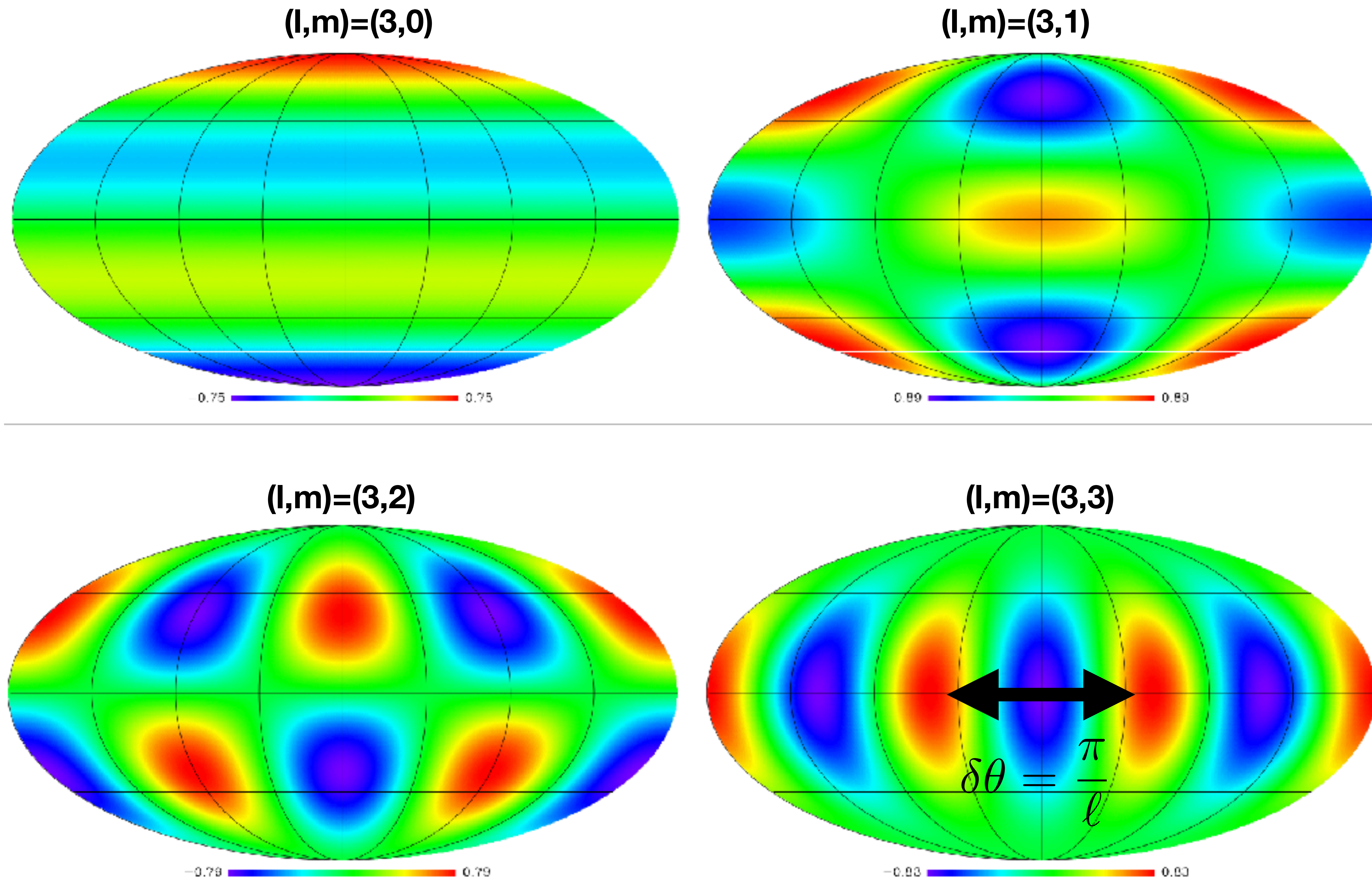


(l,m)=(2,2)



For l=m, a half-wavelength, $\lambda_\theta/2$, corresponds to π/l.

Therefore, $\lambda_\theta = 2\pi/l$



(l,m)=(3,0)
−0.75
0.75
(l,m)=(3,1)
0.89
0.89
(l,m)=(3,2)
−0.79
0.79
(l,m)=(3,3)
$\delta\theta = \frac{\pi}{\ell}$
−0.83
0.83

a_{lm} of the SW effect

- Using the inverse transform $a_{\ell m} = \int d\Omega \Delta T(\hat{n}) Y_\ell^{m*}(\hat{n})$

  on the Sachs-Wolfe (SW) formula $\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{r}_L)$

  and Fourier-transforming the potential, we obtain:

$$a_{\ell m}^{\mathrm{SW}} = \frac{T_0}{3} \int d\Omega\ Y_\ell^{m*}(\hat{n}) \int \frac{d^3q}{(2\pi)^3}\ \Phi_{\boldsymbol{q}} \exp(i\boldsymbol{q}\cdot\hat{n} r_L)$$

***q is the 3d Fourier wavenumber**

The left hand side is the coefficients of 2d spherical waves, whereas the right hand side is the coefficients of 3d plane waves. How can we make the connection?

Spherical wave decomposition of a plane wave

$$\exp(i\boldsymbol{q}\cdot\hat{n}r_L) = 4\pi\sum_{\ell=0}^{\infty} i^\ell j_\ell(qr_L)\sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{n})Y_\ell^{m*}(\hat{q})$$

- This "partial-wave decomposition formula" (or Rayleigh's formula) then gives

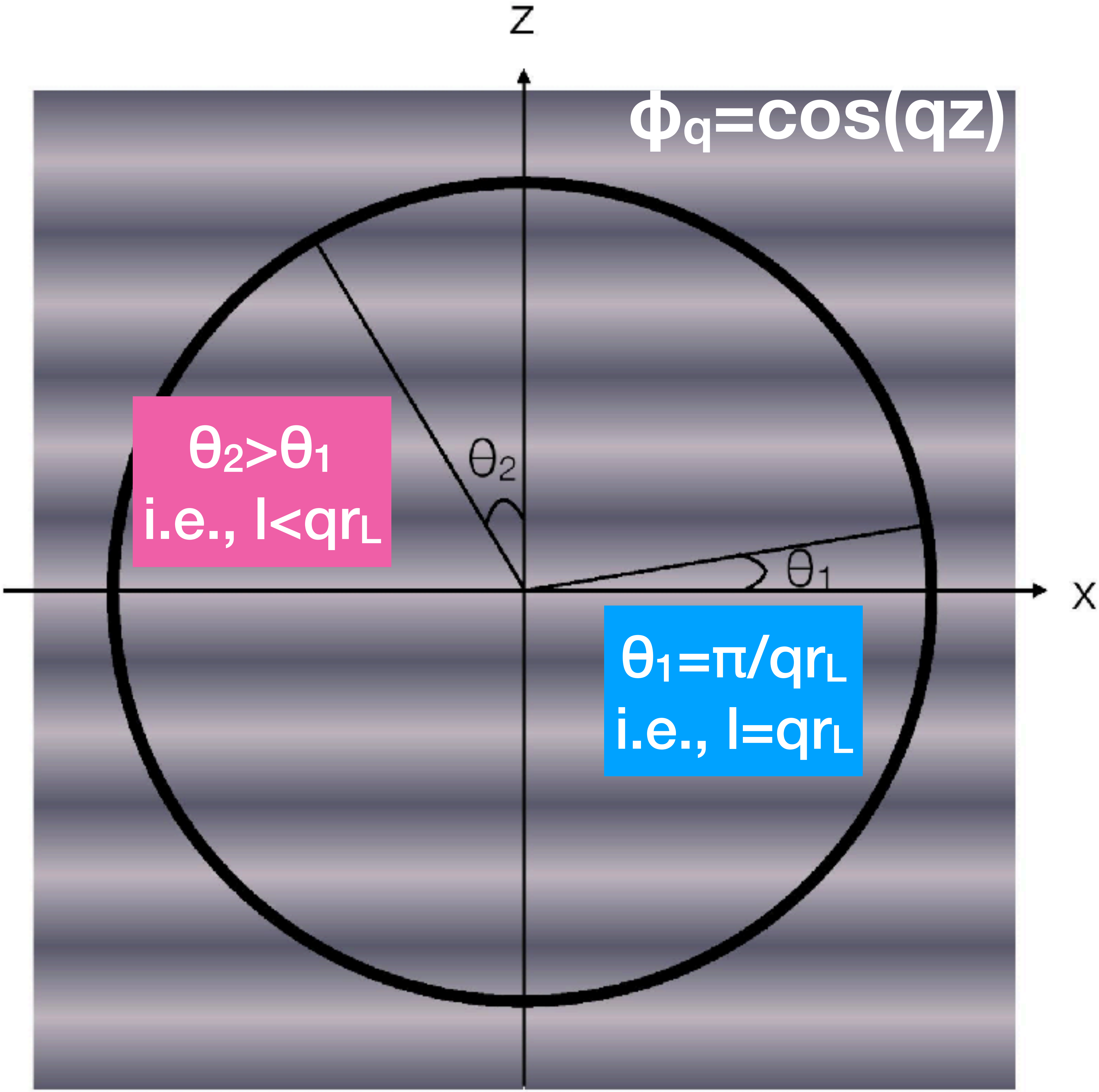
$$a_{\ell m}^{\rm SW} = \frac{4\pi T_0 i^\ell}{3}\int\frac{d^3q}{(2\pi)^3}\,\Phi_{\boldsymbol{q}} j_\ell(qr_L)Y_\ell^{m*}(\hat{q})$$

- This is the exact formula relating 3d potential at the last scattering surface onto a_{lm}. **How do we understand this?**

q -> l projection

$$a_{\ell m}^{\mathrm{SW}} = \frac{4\pi T_0 i^\ell}{3} \int \frac{d^3q}{(2\pi)^3}\, \Phi_q\, j_\ell(qr_L)\, Y_\ell^{m*}(\hat{q})$$

- A half wavelength, $\lambda/2$, at the last scattering surface subtends an angle of $\lambda/2r_L$. Since $q=2\pi/\lambda$, the angle is given by $\delta\theta=\pi/qr_L$. Comparing this with the relation $\delta\theta=\pi/l$ (for l=m), we obtain **$l=qr_L$**. How can we see this?
- For l>>1, the spherical Bessel function, **$j_l(qr_L)$, peaks at $l=qr_L$** and falls gradually toward $qr_L>l$. Thus, a given q mode contributes to large angular scales too.



z
$\phi_q = \cos(qz)$
θ_2
$\theta_2 > \theta_1$
i.e., $l < qr_L$
θ_1
x
$\theta_1 = \pi / qr_L$
i.e., $l = qr_L$

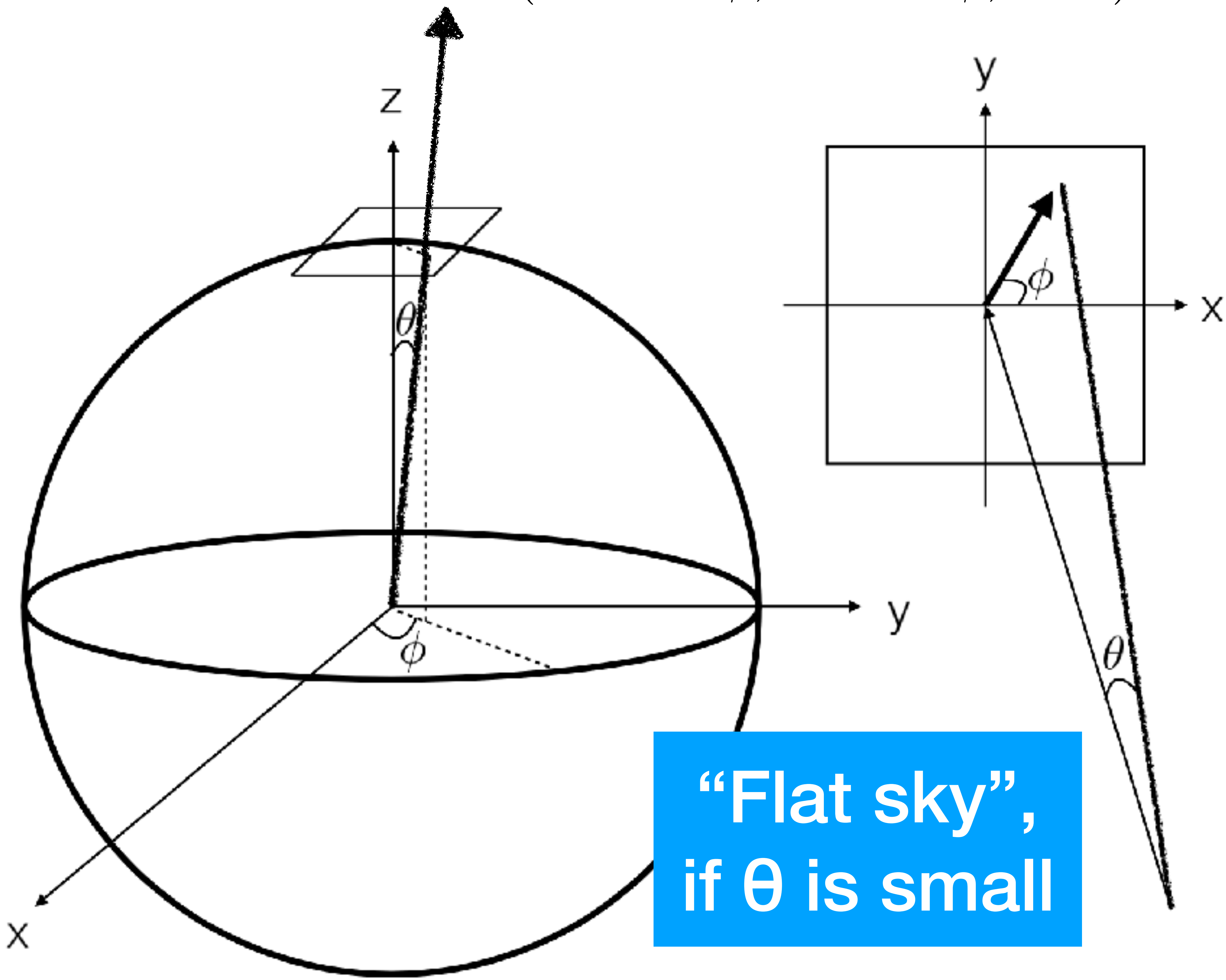
More intuitive approach: Flay-sky Approximation

- Not all of us are familiar with spherical bessel functions…
  - The fundamental complication here is that we are trying to relate a 3d plane wave with a spherical wave.
  - More intuitive approach would be to relate a 3d plane wave with **a 2d plane wave**

Decomposition

- **Full sky**
  - Decompose temperature fluctuations using **spherical harmonics**
- **Flat sky**
  - Decompose temperature fluctuations using **Fourier transform**
- The former approaches the latter in the small-angle limit

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



"Flat sky", if θ is small

2d Fourier Transform

$$\Delta T(\hat{n}) = \int \frac{d^2\ell}{(2\pi)^2}\, a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

$$= \int_0^\infty \frac{\ell d\ell}{2\pi} \int_0^{2\pi} \frac{d\phi_\ell}{2\pi}\, a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

C.f.,

$$\left(\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n}) \right)$$

a(l) of the SW effect

- Using the inverse 2d Fourier transform on the Sachs-Wolfe (SW) formula

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{r}_L)$$

and Fourier-transforming the potential, we obtain:

$$a_{\boldsymbol{\ell}}^{\mathrm{SW}} = \frac{T_0}{3}\int d^2\theta \ \exp(-i\boldsymbol{\ell}\cdot\boldsymbol{\theta}) \times \int \frac{d^3q}{(2\pi)^3}\, \varPhi_{\boldsymbol{q}} \ \exp(i\boldsymbol{q}_\perp r_L \cdot \boldsymbol{\theta} + i q_{\|} r_L \boxed{\cos\theta})$$

$\cos\theta \rightarrow 1$

flat-sky approx.

Flat-sky Result

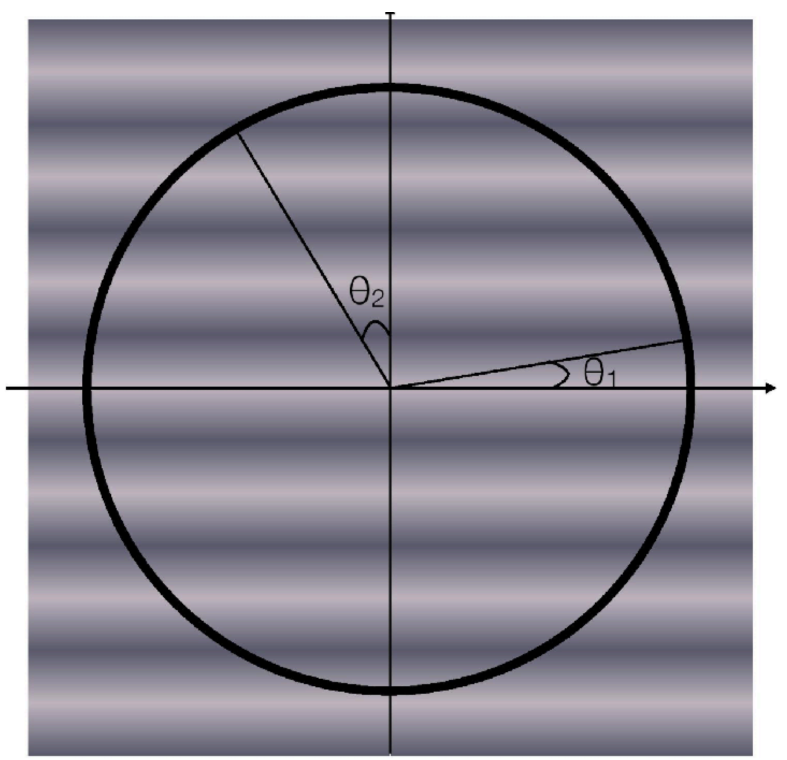
$$a_{\boldsymbol{\ell}}^{\rm SW} = \frac{T_0}{3r_L^2}\int_{-\infty}^{\infty}\frac{dq_\parallel}{2\pi}\,\Phi_{\boldsymbol{q}}\left(\boldsymbol{q}_\perp = \frac{\boldsymbol{\ell}}{r_L}, q_\parallel\right)\exp(iq_\parallel r_L)$$

$q = \sqrt{\ell^2/r_L^2 + q_\parallel^2}$ **i.e.,** $q \geq \ell/r_L$

C.f.,

$$\left(a_{\ell m}^{\rm SW} = \frac{4\pi T_0 i^\ell}{3}\int\frac{d^3q}{(2\pi)^3}\,\Phi_{\boldsymbol{q}} j_\ell(qr_L) Y_\ell^{m*}(\hat{q}) \right)$$

- It is **now manifest** that only the perpendicular wavenumber contributes to l, i.e., **$l = q_{perp} r_L$**, giving $l < qr_L$



Angular Power Spectrum

- The angular power spectrum, C_l, quantifies how much correlation power we have at a given angular separation.

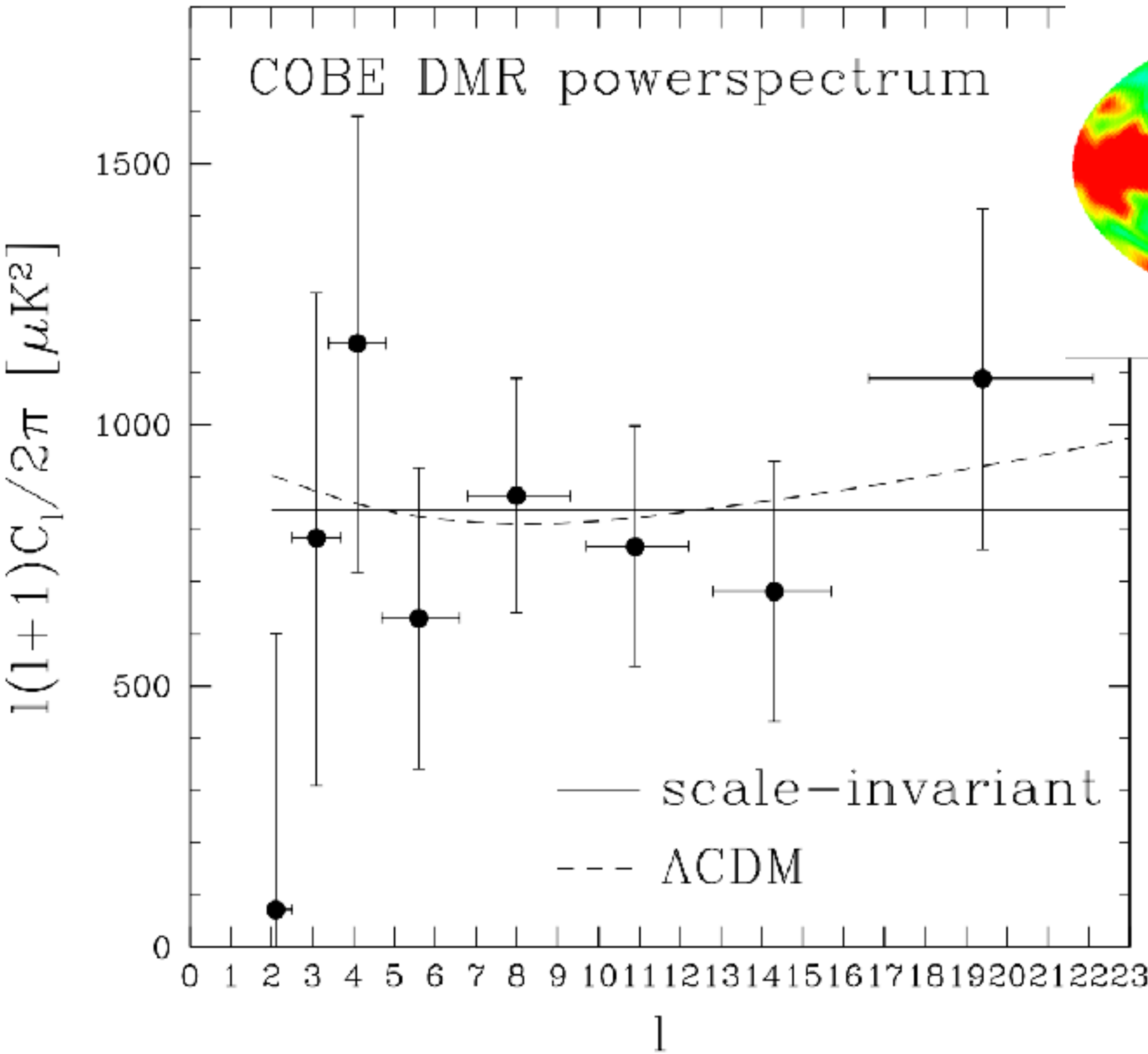
$$C_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^*$$

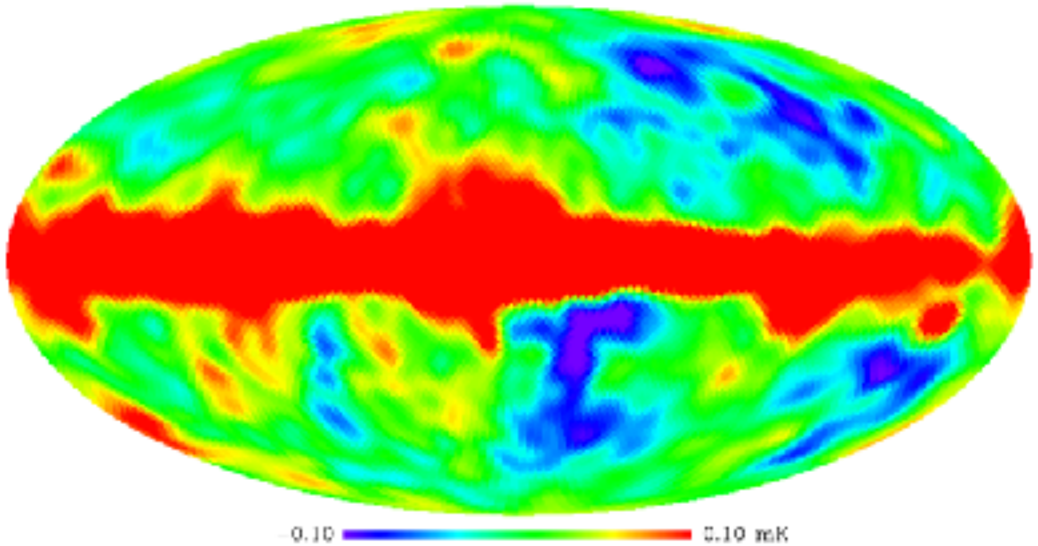
- More precisely: it is **$l(2l+1)C_l/4\pi$** that gives the fluctuation power at a given angular separation, ~π/l. We can see this by computing **variance**:

$$\int \frac{d\Omega}{4\pi} \Delta T^2(\hat{n}) = \frac{1}{4\pi} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^* = \sum_{\ell=2}^{\infty} \frac{2\ell+1}{4\pi} C_\ell$$

Bennett et al. (1996)

COBE 4-year Power Spectrum





The SW formula allows us to determine the 3d power spectrum of ϕ at the last scattering surface from C_l.

But how?