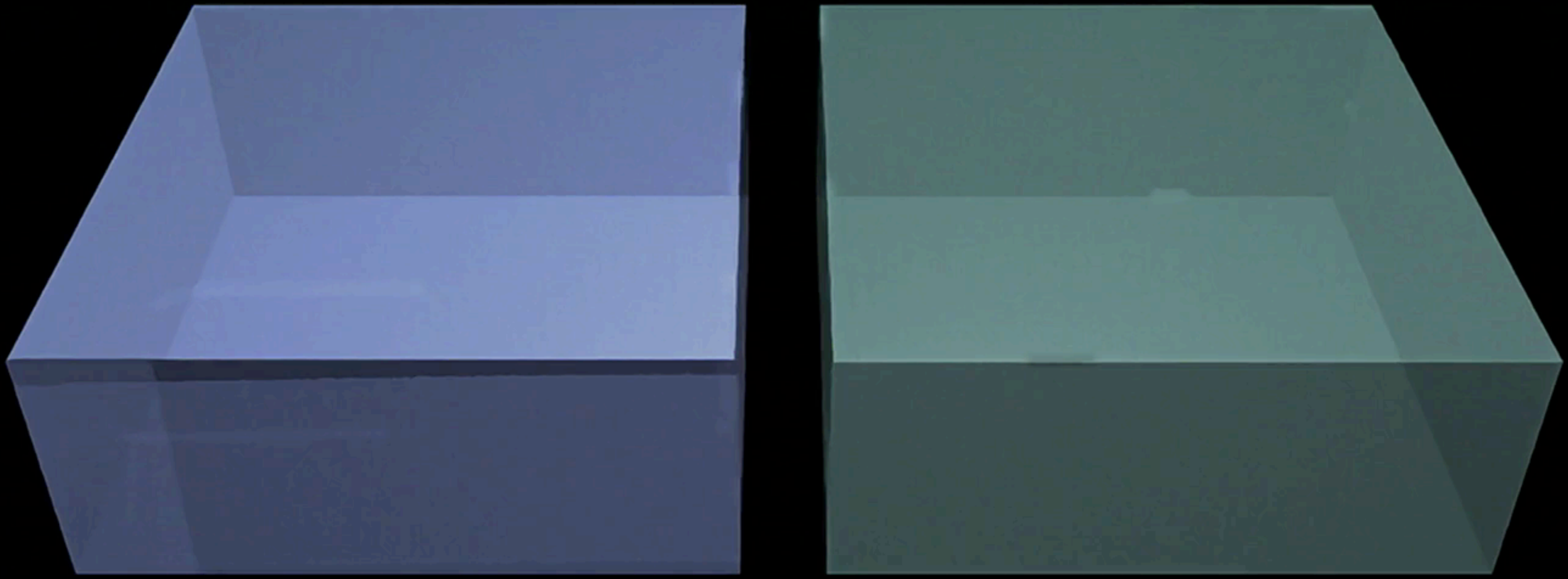
The lecture slides are available at
https://wwwmpa.mpa-garching.mpg.de/~komatsu/lectures--reviews.html

Lecture 7: Details of the Acoustic Oscillation

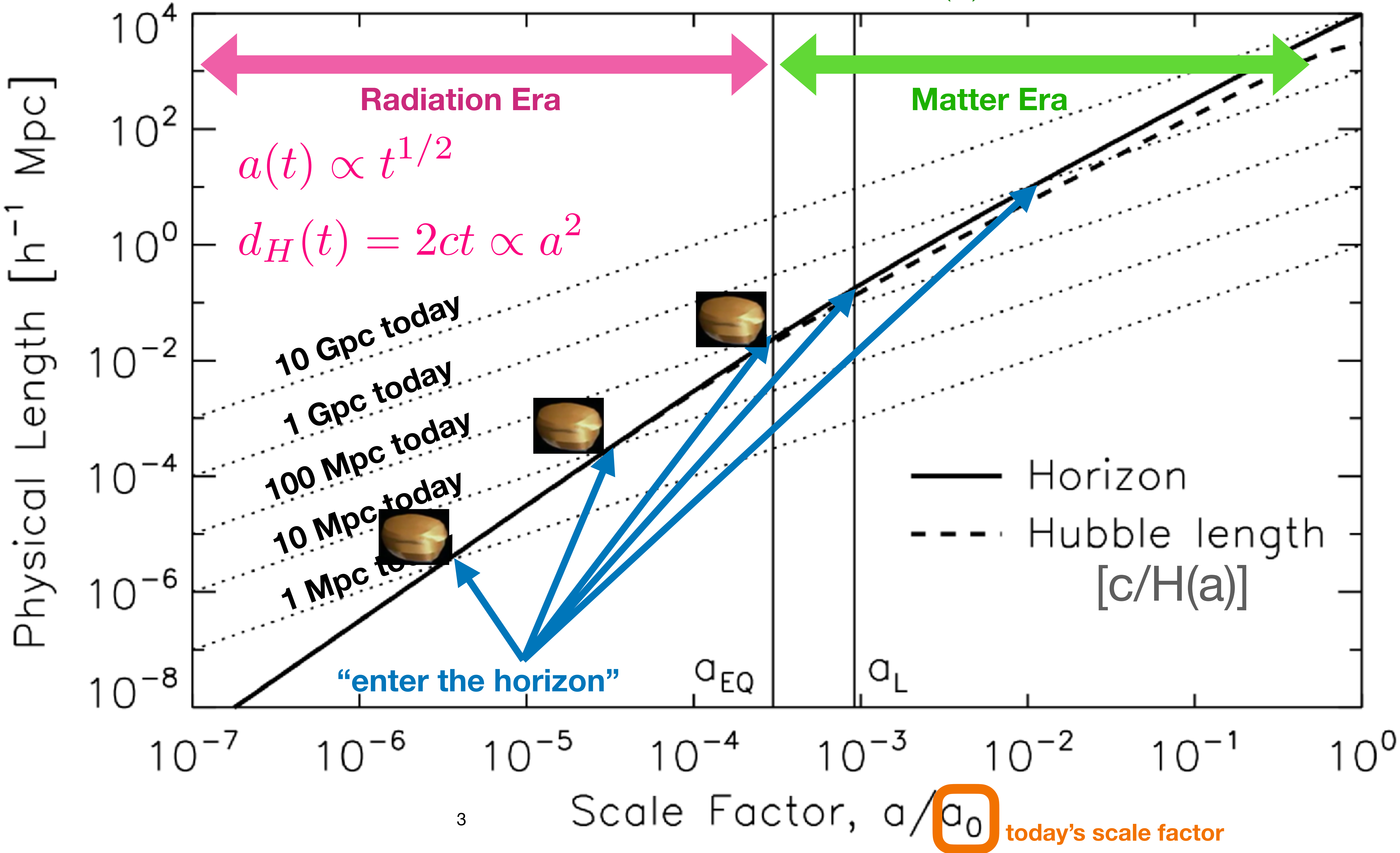


What did the stones represent?

- **A stone is dropped when a fluctuation "enters the horizon".**
  - In a decelerating Universe, we can see more of the Universe as time goes by.
  - New, longer wavelength fluctuations keep entering the horizon, perturbing the photon-baryon fluid.

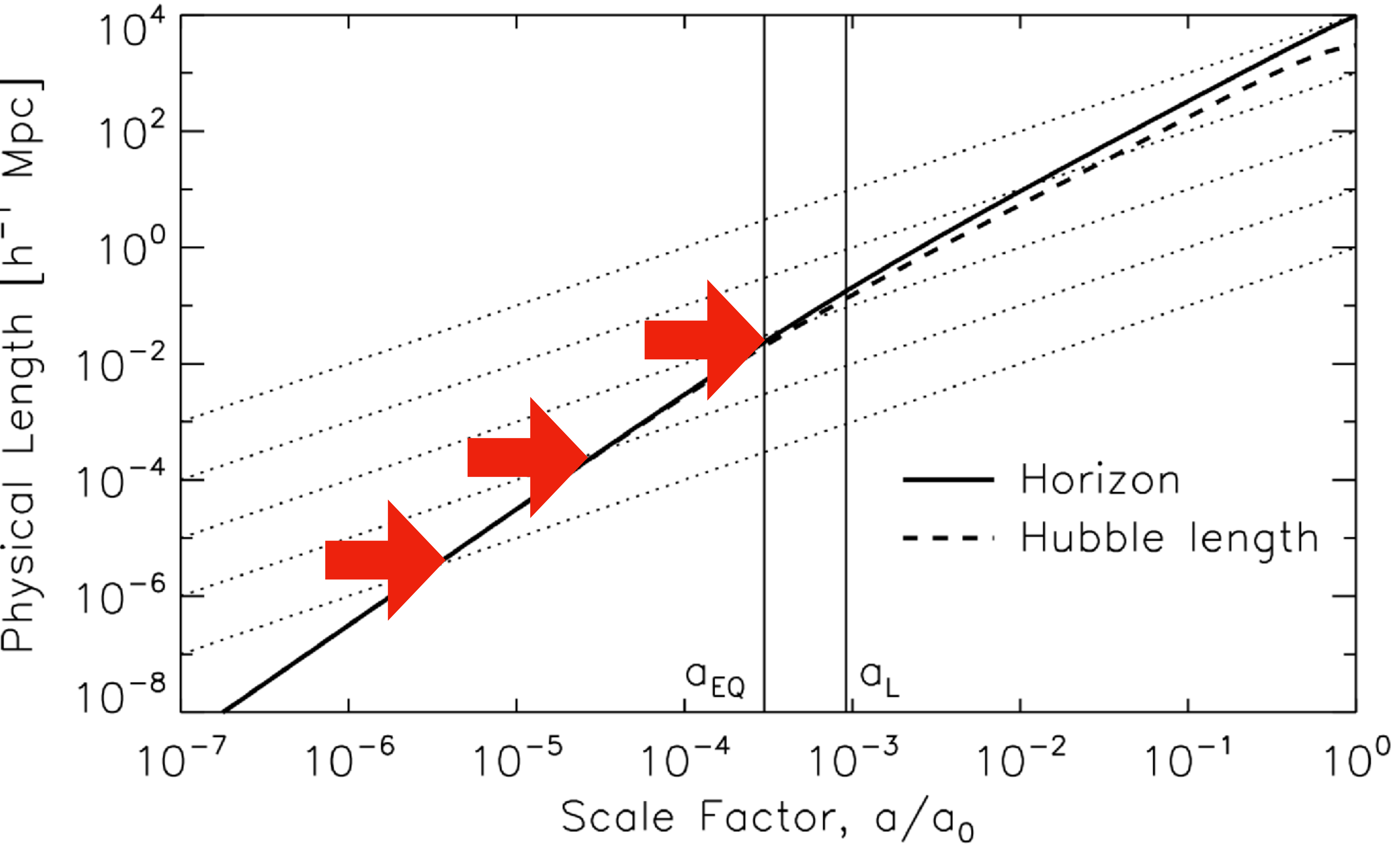
$a(t) \propto t^{2/3}$

$d_H(t) = 3ct \propto a^{3/2}$



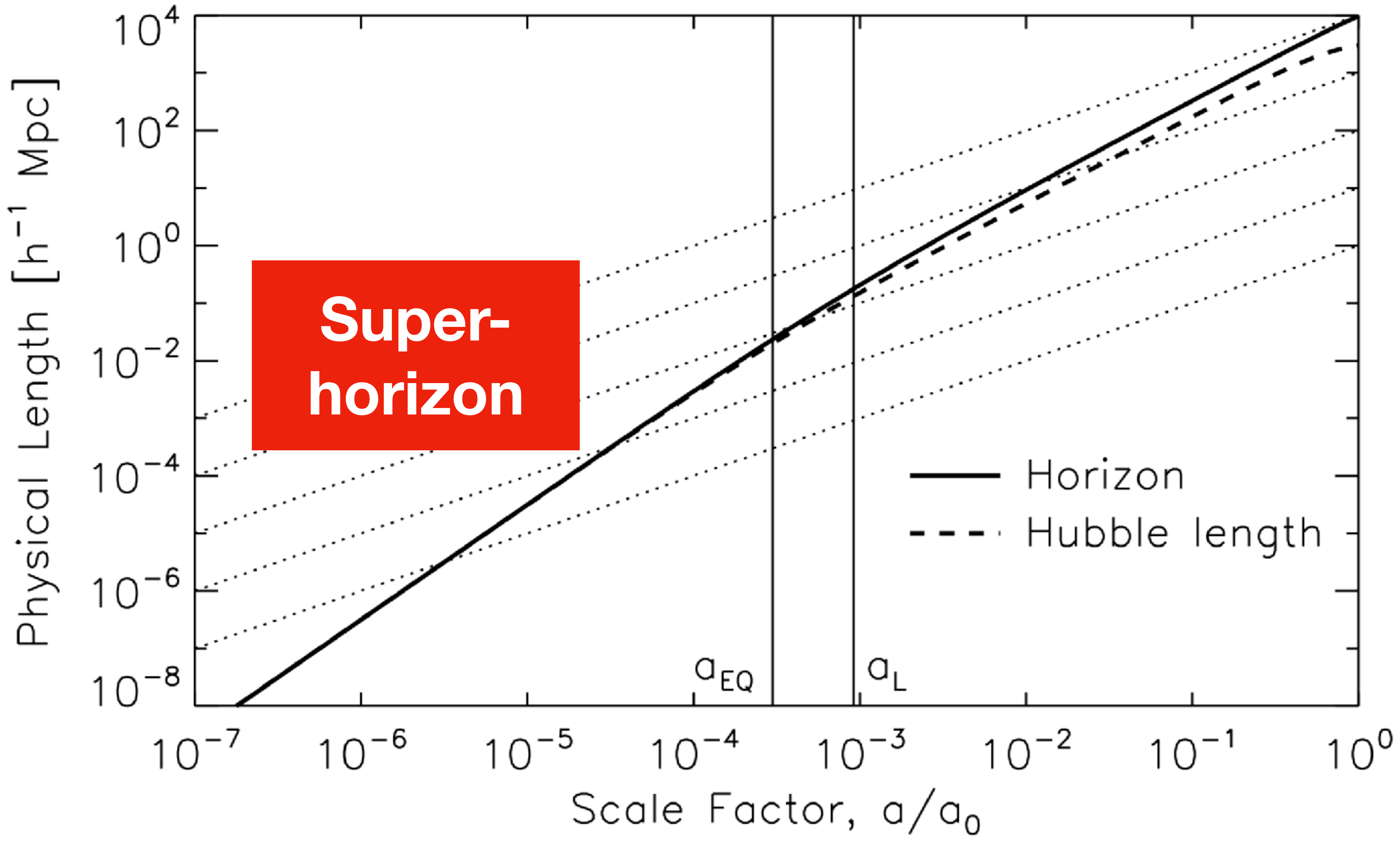
Fluctuations Entering the Horizon

- The initial impact **for a given wavelength.**

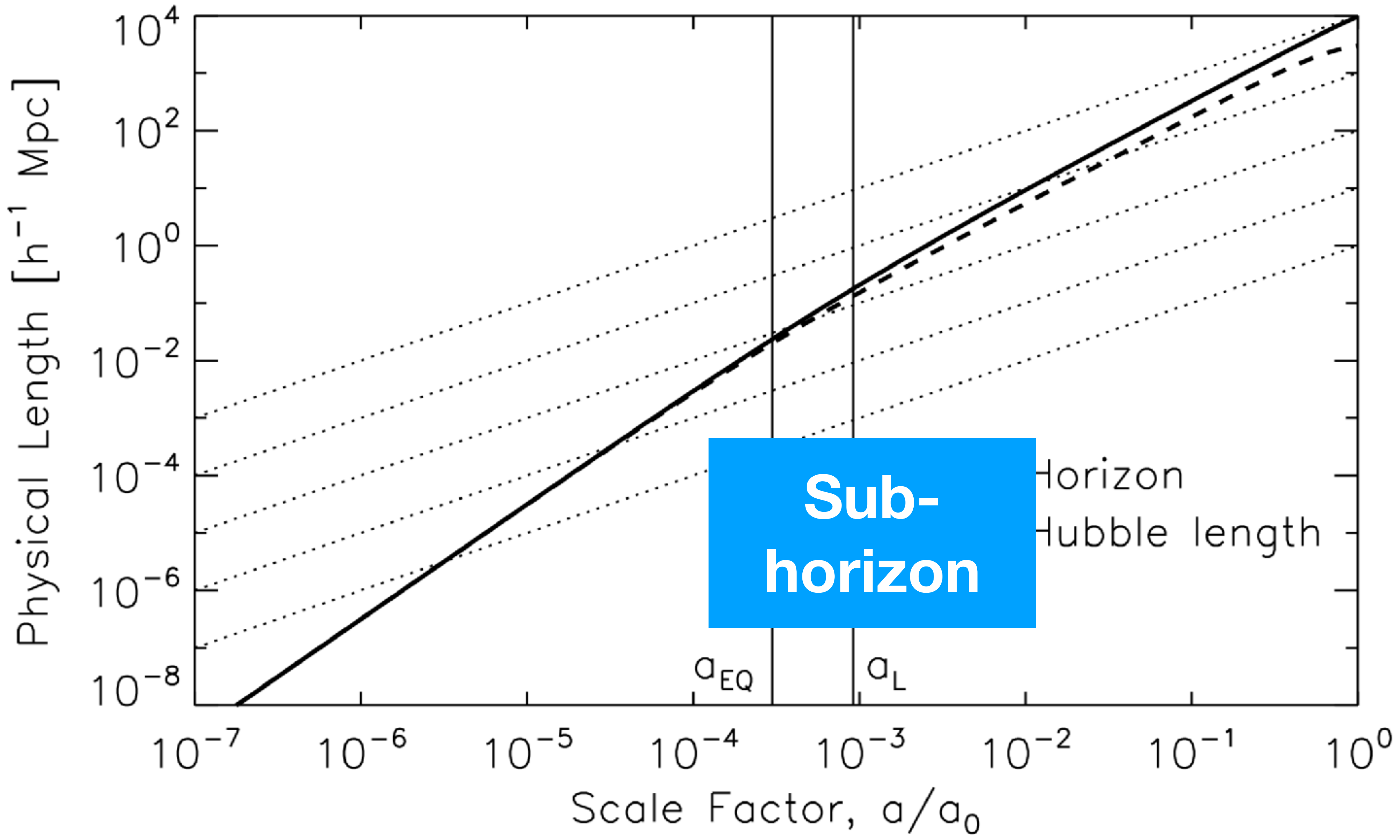


Three Regimes

- **Super-horizon scales [q < aH]**
    - Only gravity is important
    - Evolution differs from Newtonian: <u>We need GR</u>



- **Sub-horizon but super-sound-horizon [aH < q < aH/c_s]**
    - Only gravity is important
    - Evolution similar to Newtonian



- **Sub-sound-horizon scales [q > aH/c_s]**
    - Hydrodynamics important -> Sound waves

Part I: Super-horizon Scale: Conserved Curvature Perturbation

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

The Stone, "ζ"

<u>Conserved</u> quantity on the super-horizon scale, q << aH

- For the adiabatic initial condition, there exists a useful quantity, **ζ**, which **remains constant on large scales** (*super-horizon scales, q << aH*) regardless of the contents of the Universe.
    - ζ is conserved regardless of whether the Universe is radiation-dominated, matter-dominated, or whatever.
- <u>*Derivation*</u>: Energy conservation for **q << aH**

$$\delta\dot{\rho}_\alpha + \frac{\dot{a}}{a}(3\delta\rho_\alpha + 3\delta P_\alpha + \cancel{\nabla^2\pi_\alpha}) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} + \frac{1}{a^2}(\bar{\rho}_\alpha + \bar{P}_\alpha)\cancel{\nabla^2\delta u_\alpha} = 0\,,$$

$$\delta\dot{\rho}_\alpha + \frac{3\dot{a}}{a}(\delta\rho_\alpha + \delta P_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} = 0$$

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

The "ζ"

<u>Conserved</u> quantity on the super-horizon scale, q << aH

- If pressure is a function of the energy density only, i.e., $P_\alpha = P_\alpha(\rho_\alpha)$

$$\frac{1}{3}\frac{\delta\rho_\alpha(t,\boldsymbol{x})}{\bar{\rho}_\alpha(t)+\bar{P}_\alpha(t)} - \Psi(t,\boldsymbol{x}) = \underbrace{\zeta_\alpha(\boldsymbol{x})}_{\text{integration constant}}$$

Integrate

$$\delta\dot{\rho}_\alpha + \frac{3\dot{a}}{a}(\delta\rho_\alpha + \delta P_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} = 0$$

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

The "ζ"

<u>Conserved</u> quantity on the super-horizon scale, q << aH

- If pressure is a function of the energy density only, i.e., $P_\alpha = P_\alpha(\rho_\alpha)$

$$\frac{1}{3}\frac{\delta\rho_\alpha(t,\boldsymbol{x})}{\bar{\rho}_\alpha(t)+\bar{P}_\alpha(t)} - \Psi(t,\boldsymbol{x}) = \underbrace{\zeta_\alpha(\boldsymbol{x})}_{\text{integration constant}}$$

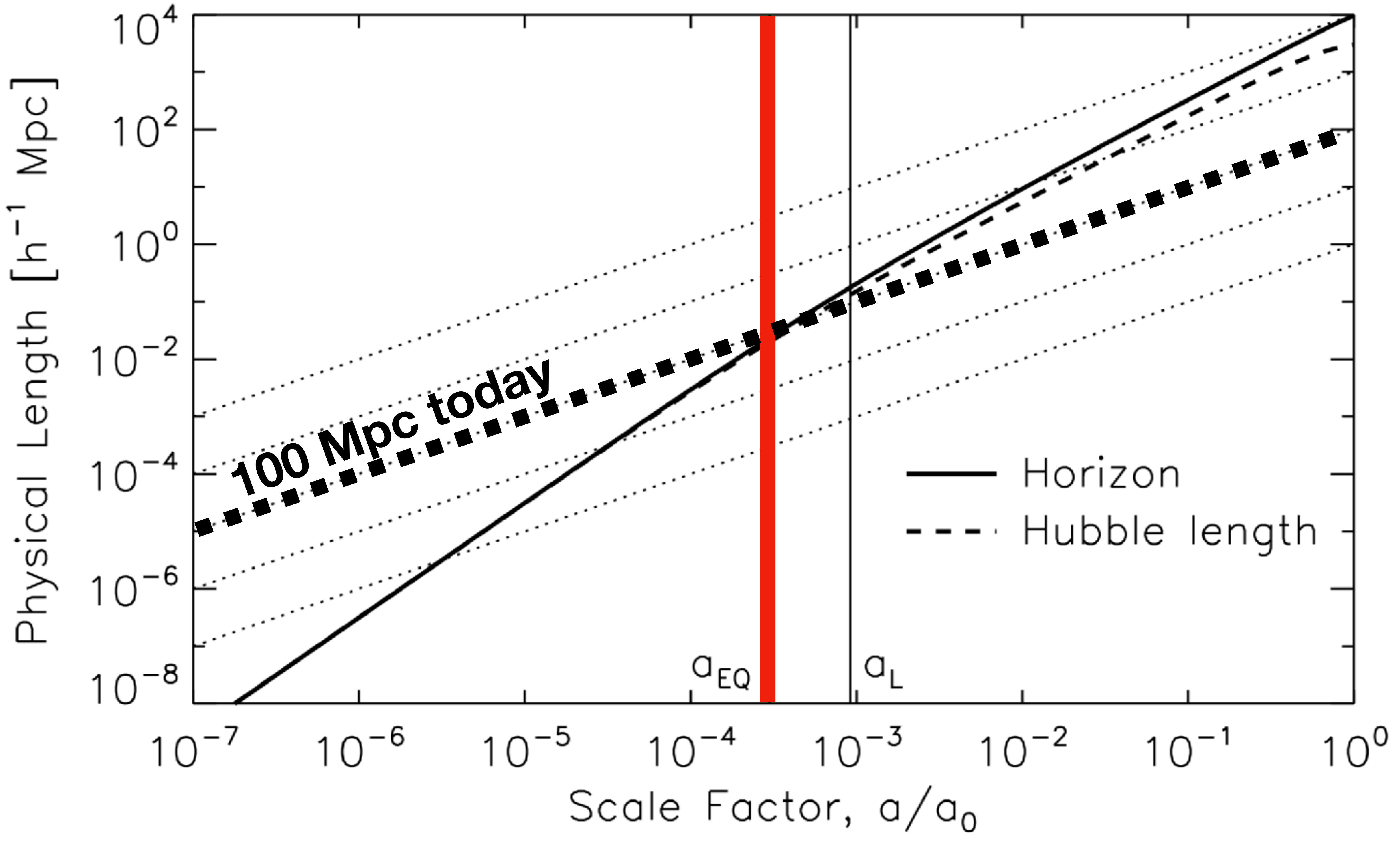
For the adiabatic initial condition, all species share the same value of ζ_α, i.e., $\zeta_\alpha = \zeta$

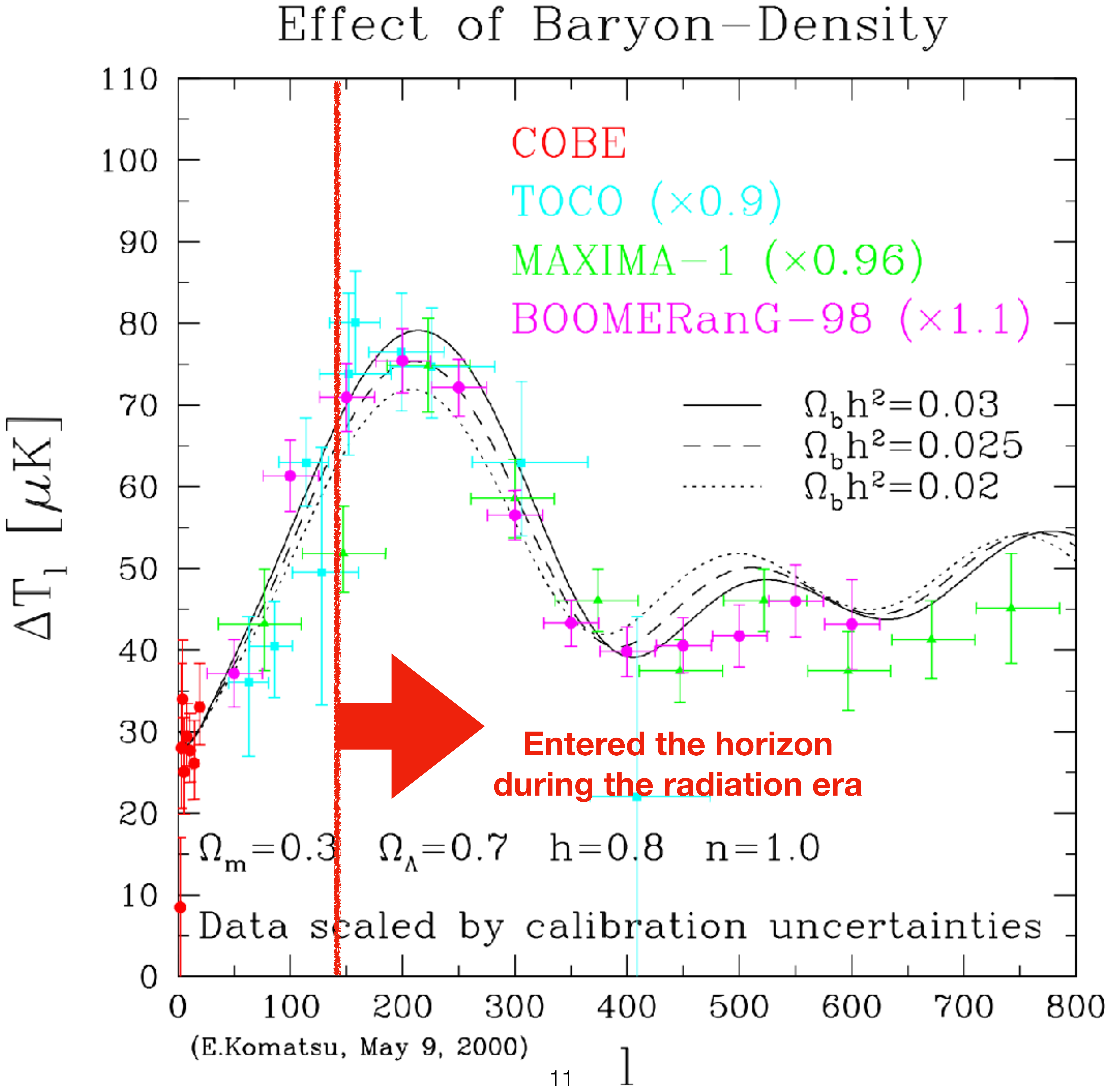
q_{EQ}

The wavenumber of the fluctuation that entered the horizon during the equality time

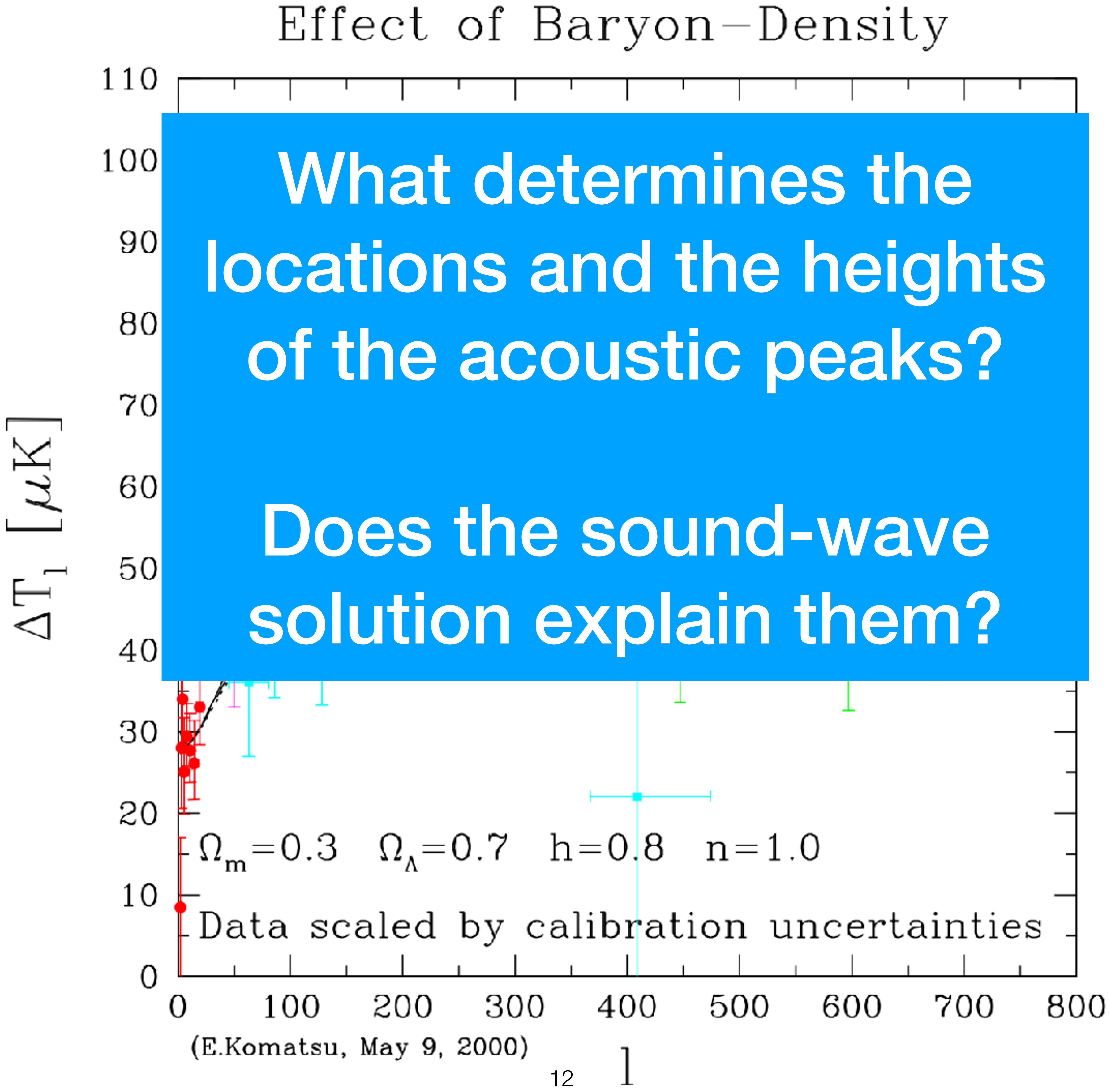
- Which fluctuation entered the horizon before the matter-radiation equality?
- $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01\ (\Omega_M h^2/0.14)\ \mathrm{Mpc}^{-1}$
- At the last scattering surface, this subtends the multipole of

$l_{EQ} = q_{EQ}r_L \sim$ **140**





Effect of Baryon-Density
COBE
TOCO (×0.9)
MAXIMA-1 (×0.96)
BOOMERanG-98 (×1.1)
$\Omega_b h^2=0.03$
$\Omega_b h^2=0.025$
$\Omega_b h^2=0.02$
Entered the horizon
during the radiation era
$\Omega_m=0.3$ $\Omega_\Lambda=0.7$ $h=0.8$ $n=1.0$
Data scaled by calibration uncertainties
ΔT_l [μK]
110
100
90
80
70
60
50
40
30
20
10
0
0
100
200
300
400
500
600
700
800
l
(E.Komatsu, May 9, 2000)



What determines the locations and the heights of the acoustic peaks?

Does the sound-wave solution explain them?

Part II: Locations of the Acoustic Peaks

Peak Locations?

<u>High-frequency solution, for q >> aH</u>

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$

- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi\right]^2$ with $q \to l/r_L$.

- <u>Question</u>: What determines the integration constants, **A** and **B**?

- <u>Answer</u>: They are determined by the initial conditions; namely, adiabatic or not.

- For the adiabatic initial condition, ***A >> B when q is large.***

[We will show this later.]

Peak Locations?

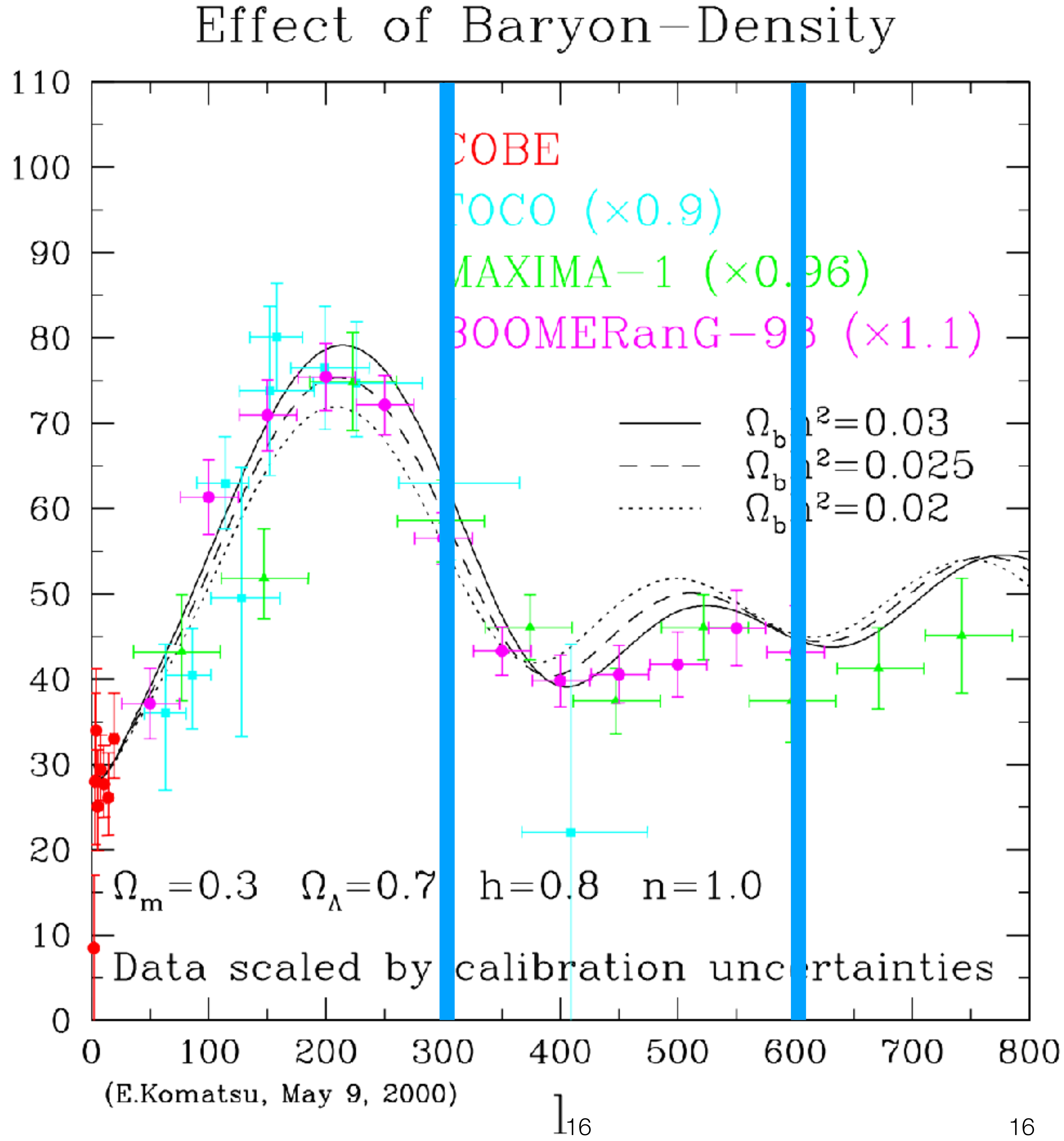
High-frequency solution, for q >> aH

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$

- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi\right]^2$ with q -> l/r_L.

- If A>>B, the locations of peaks are determined by $\mathbf{qr_s(t_L) = n\pi}$ (n=1,2,…):

$$\ell = (1,\ 2,\ \cdots)\pi r_L/r_s(t_L) = (1,\ 2,\ \cdots) \times 302$$



The simple estimates do not match!

This is because these angular scales do not satisfy $q \gg aH$, i.e, the oscillations are not pure cosine even for the adiabatic initial condition.

We need a better solution.

A Better Solution in the Radiation-dominated Era

Going back to the original tight-coupling equation:

$$\frac{1}{a(1+\cancel{R})}\frac{\partial}{\partial t}\left[a(1+\cancel{R})\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma-4\Psi)\right]+\frac{4q^2}{3a^2}\Phi+\frac{q^2}{a^2}\frac{\delta\rho_\gamma/\bar{\rho}_\gamma}{3(1+\cancel{R})}=0$$

- In the radiation-dominated era, **R << 1** as $R=\frac{3\Omega_B}{4\Omega_\gamma}\frac{a}{a_0}=0.6120\left(\frac{\Omega_B h^2}{0.022}\right)\frac{1091}{1+z}$
- Convenient to change the independent variable from the time (t) to

$$\varphi\equiv qr_s=2qt/\sqrt{3a}$$

A Better Solution in the Radiation-dominated Era

Then the equation simplifies to

$$\boxed{\partial^2 X/\partial\varphi^2 + X + \Phi + \Psi = 0}$$

where $X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$

- In the radiation-dominated era, R << 1.

- Convenient to change the independent variable from the time (t) to

$$\varphi \equiv qr_s = 2qt/\sqrt{3}a$$

A Better Solution in the Radiation-dominated Era

$$\boxed{\partial^2 X/\partial\varphi^2 + X + \Phi + \Psi = 0}$$

where $X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$

The solution is

$$X = \tilde{A}\cos\varphi + \tilde{B}\sin\varphi - \int_0^\varphi d\varphi' \, \underline{\sin(\varphi - \varphi')}(\Phi + \Psi)(\varphi')$$

We rewrite this using the formula for trigonometry:

$$\sin(\varphi - \varphi') = \sin(\varphi)\cos(\varphi') - \cos(\varphi)\sin(\varphi')$$

A Better Solution in the Radiation-dominated Era

$$\boxed{\partial^2 X/\partial\varphi^2 + X + \Phi + \Psi = 0}$$

where $X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where $\begin{cases} \Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi')\,, \\ \Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi') \end{cases}$

Einstein’s Equations

- Now we need to know Newton’s gravitational potential, ϕ, and the scalar curvature perturbation, ψ.

- Einstein’s equations - let’s look up any text books:

$$\nabla^2 \Psi = 4\pi G a^2 \sum_\alpha \left[\delta\rho_\alpha - \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha \right]$$

$$\dot{\Psi} + \frac{\dot{a}}{a}\Phi = -4\pi G \sum_\alpha (\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha$$

$$\partial_i \partial_j (\Phi - \Psi) = -8\pi G a^2 \partial_i \partial_j \sum_\alpha \pi_\alpha$$

(C.f., Newtonian (Poisson equation) $\nabla^2 \Phi = 4\pi G \delta\rho$)

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$$\partial_i\partial_j(\Phi - \Psi) = -8\pi G a^2 \partial_i\partial_j \boxed{\sum_\alpha \pi_\alpha} \, ?$$

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$$\dot{\Psi} + \frac{\dot{a}}{a}\Phi = -4\pi G \sum_\alpha (\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha$$

$$\partial_i\partial_j(\Phi - \Psi) = -8\pi G a^2 \partial_i\partial_j \boxed{\pi_\nu}$$

Will come back to this later. For now, let's ignore any viscosity.

Einstein's Equations

- Now we need to know Newton's gravitational potential, ϕ, and the scalar curvature perturbation, ψ.

- Einstein's equations - let's look up any text books:

$$\nabla^2\Psi = 4\pi G a^2 \sum_\alpha \left[\delta\rho_\alpha - \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha\right]$$

$$\dot{\Psi} + \frac{\dot{a}}{a}\Phi = -4\pi G \sum_\alpha (\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha$$

$$\Phi = \Psi$$

Will come back to this later.
For now, let's ignore any viscosity.

Einstein's Equations in the Radiation-dominated Era

- Now we need to know Newton's gravitational potential, ϕ, and the scalar curvature perturbation, ψ.

- Combine Einstein's equations:

$$\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{4}{\varphi}\frac{\partial\Phi}{\partial\varphi} + \Phi = \frac{3}{2\varphi^2}\frac{\boxed{\delta\mathcal{P}}}{\bar{\rho}_R}$$

Decompose the total pressure perturbation into the total energy density perturbation and the rest.

$$\sum_\alpha \delta P_\alpha(t,\boldsymbol{x}) = \frac{\sum_\alpha \dot{\bar{P}}_\alpha(t)}{\sum_\alpha \dot{\bar{\rho}}_\alpha(t)} \sum_\alpha \delta\rho_\alpha(t,\boldsymbol{x}) + \boxed{\delta\mathcal{P}(t,\boldsymbol{x})}$$

"non-adiabatic" pressure

Einstein's Equations in the Radiation-dominated Era

- Now we need to know Newton's gravitational potential, ϕ, and the scalar curvature perturbation, ψ.
- **Choose the adiabatic solution!**

$$\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{4}{\varphi}\frac{\partial \Phi}{\partial \varphi} + \Phi = \cancel{\frac{3}{2\varphi^2}\frac{\boxed{\delta \mathcal{P}}}{\bar{\rho}_R}}$$

We shall ignore this

Decompose the total pressure perturbation into the total energy density perturbation and the rest.

$$\sum_\alpha \delta P_\alpha(t, \boldsymbol{x}) = \frac{\sum_\alpha \dot{\bar{P}}_\alpha(t)}{\sum_\alpha \dot{\bar{\rho}}_\alpha(t)} \sum_\alpha \delta\rho_\alpha(t, \boldsymbol{x}) + \cancel{\boxed{\delta \mathcal{P}(t, \boldsymbol{x})}}$$

"non-adiabatic" pressure

Kodama & Sasaki (1986, 1987)

Adiabatic Solution in the Radiation-dominated Era

$$\Phi_{\mathbf{ADI}} = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

where

$$\varphi \equiv q r_s = 2qt/\sqrt{3}a$$

- Low-frequency limit (*super-sound-horizon scales, $qr_s << 1$*)
  - Φ_{ADI} -> $-2\zeta/3$ = **constant**
- High-frequency limit (*sub-sound-horizon scales, $qr_s >> 1$*)
  - Φ_{ADI} -> $2\zeta \cos\varphi/\varphi^2 \propto a^{-2}$

The potential decays -> The integrated Sachs-Wolfe Effect

Adiabatic Solution in the Radiation-dominated Era

$$\Phi_{\mathrm{ADI}} = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

wh

Poisson Equation

$$-q^2\Phi = 4\pi G a^2 \delta\rho$$

& oscillation solution for radiation

$$\delta\rho_R/\bar{\rho}_R \propto \cos\varphi$$

- Low-frequency l … *ales, $qr_s << 1$*)
  - Φ_{ADI} -> $-2\zeta/3$ =
- High-frequency li … *les, $qr_s >> 1$*)
  - Φ_{ADI} -> $2\zeta \boxed{\cos\varphi/\varphi^2} \propto a^{-2}$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi'),$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

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Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2)$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2$$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi \xrightarrow{\varphi \ll 1} \zeta$$

i.e., $\tilde{A}_{\text{ADI}} = \zeta, \ \tilde{B}_{\text{ADI}} = 0$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2)$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2$$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

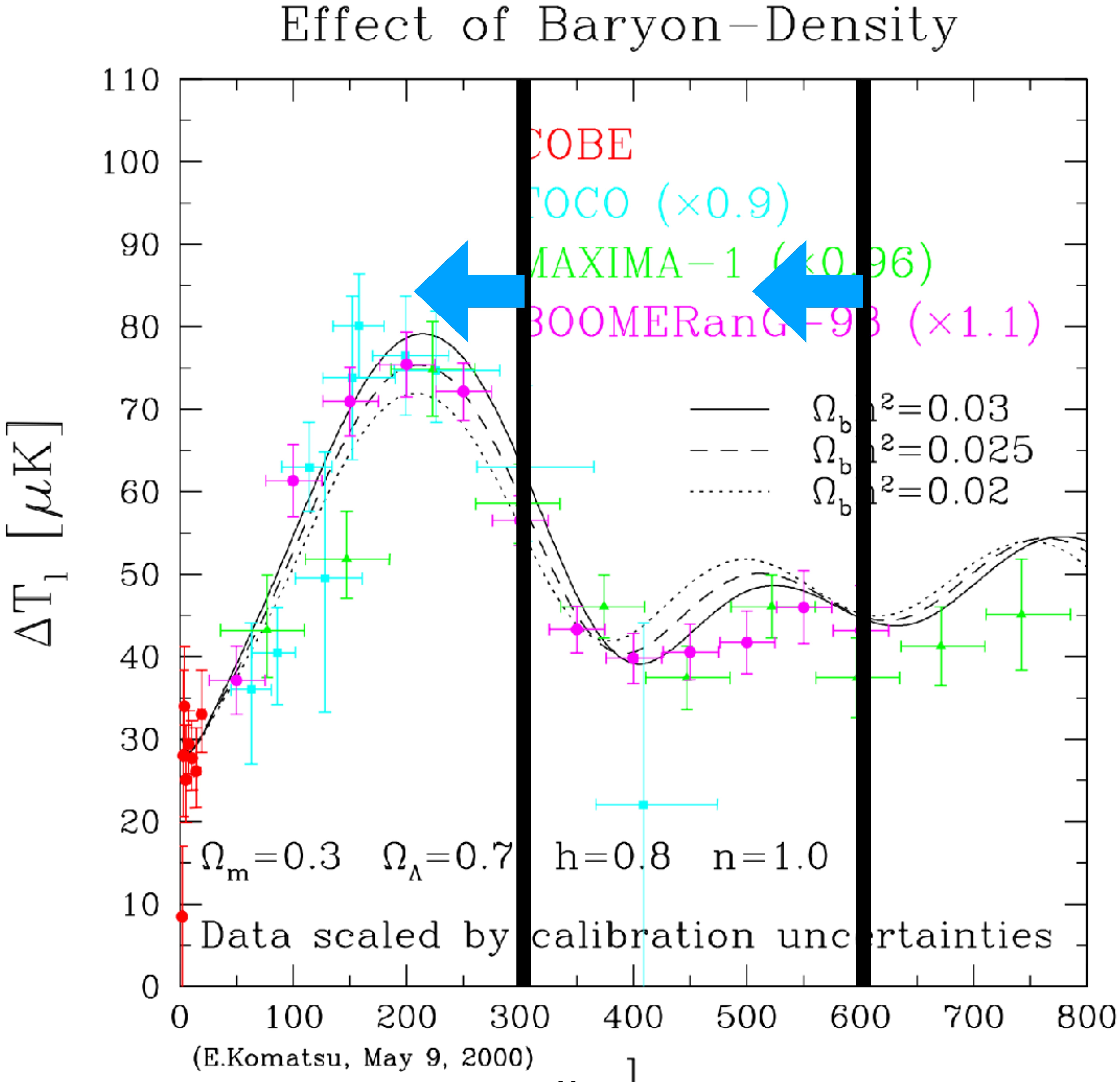
The complete adiabatic solution is

$$X = \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta\left(-\cos\varphi + \frac{2}{\varphi}\sin\varphi\right)$$

with

$$\Phi = \Psi = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

Therefore, the solution is a **pure cosine** only in the **high-frequency** limit!



Effect of Baryon−Density
COBE
TOCO (×0.9)
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$\Omega_b h^2=0.02$
$\Omega_m=0.3$ $\Omega_\Lambda=0.7$ $h=0.8$ $n=1.0$
Data scaled by calibration uncertainties
ΔT_l [μK]
l
(E.Komatsu, May 9, 2000)

Roles of viscosity

- **Neutrino viscosity: Gravitational Impact**
  - Modify potentials:

$$\partial_i \partial_j (\Phi - \Psi) = -8\pi G a^2 \partial_i \partial_j \boxed{\pi_\nu}$$

- **Photon viscosity: Hydrodynamical Impact**
  - Viscous photon-baryon fluid: **damping of sound waves**

Silk (1968) "Silk damping"

$$a\frac{\partial}{\partial t}(\delta u_\gamma / a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2 \boxed{\pi_\gamma}}{2\bar{\rho}_\gamma} = \sigma_T \bar{n}_e (\delta u_B - \delta u_\gamma)$$

Part III: Damping of the Sound Waves

Photon Viscosity

Origin of the Silk damping

- In the tight-coupling approximation, the photon viscosity damps exponentially.

- To take into account a non-zero photon viscosity, we need go **higher order** in the tight-coupling approximation.

Peebles & Yu (1970)

The previous lecture: The 1st-order Tight-coupling Approximation

- When the Thomson scattering is efficient, photons and baryons "move together"; thus, their relative velocity is small. We write

$$\delta u_B - \delta u_\gamma = d/\sigma_T \bar{n}_e$$

[d is an arbitrary dimensionless variable]

- And take $\sigma_T \bar{n}_e \to \infty$ (*). We obtain

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} = d\,, \qquad \delta\dot{u}_\gamma + \Phi = -\frac{d}{R}$$

(*) In this limit, viscosity π_γ is exponentially suppressed. This result comes from the Boltzmann equation but we do not derive it here. It makes sense physically.

Today: The 2nd-order Tight-coupling Approximation

- When the Thomson scattering is efficient, photons and baryons "move together"; thus, their relative velocity is small. We write

$$\delta u_B - \delta u_\gamma = \underline{d_1}/\sigma_T\bar{n}_e + \underline{qd_2/(\sigma_T\bar{n}_e)^2}$$

$$d_1 = -R(\delta\dot{u}_\gamma + \Phi)$$

[the 1st-order solution]

[d_2 is an arbitrary dimensionless variable]

- And take $\sigma_T\bar{n}_e \to \infty$. We obtain

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = -R(\delta\dot{u}_\gamma + \Phi) + \frac{q}{\sigma_T\bar{n}_e}d_2$$

$$\frac{\partial}{\partial t}\left[\frac{R(\delta\dot{u}_\gamma + \Phi)}{\sigma_T\bar{n}_e}\right] = \frac{q}{R\sigma_T\bar{n}_e}d_2$$

The 2nd-order Tight-coupling Approximation

- Eliminating d_2 and using the fact that R is proportional to the scale factor, we obtain

$$a\frac{\partial}{\partial t}\left[(1+R)\delta u_\gamma/a\right]+(1+R)\Phi+\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}-\frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma}+R\frac{\partial}{\partial t}\left[\frac{R(\delta\dot{u}_\gamma+\Phi)}{\sigma_{\mathcal{T}}\bar{n}_e}\right]=0$$

- Getting π_γ requires an approximate solution of the Boltzmann equation in the 2nd-order tight coupling. We do not derive it here. The answer is

$$\Delta T_{ij}=a^2\partial_i\partial_j\pi_\gamma=-\frac{32}{45}\frac{\bar{\rho}_\gamma}{\sigma_{\mathcal{T}}\bar{n}_e}\partial_i\partial_j\delta u_\gamma$$

Kaiser (1983)

The 2nd-order Tight-coupling Approximation

- Eliminating d_2 and using the fact that R is proportional to the scale factor, we obtain

$$a\frac{\partial}{\partial t}\left[(1+R)\delta u_\gamma/a\right]+(1+R)\Phi+\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}-\frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma}+R\frac{\partial}{\partial t}\left[\frac{R(\delta\dot{u}_\gamma+\Phi)}{\sigma_T\bar{n}_e}\right]=0$$

- Getting π_γ requires an approxim equation in the 2nd-order tight here. The answer is

Given by the **spatial gradient of the velocity field**

- a well-known result in fluid dynamics

$$\Delta T_{ij}=a^2\partial_i\partial_j\pi_\gamma=-\frac{32}{45}\frac{\bar{\rho}_\gamma}{\sigma_T\bar{n}_e}\partial_i\partial_j\delta u_\gamma$$

Kaiser (1983)

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for q >> aH,

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

where

$$\Gamma(q,t)\equiv\frac{q^2}{6a^2\sigma_T\bar{n}_e}\left[\frac{16}{15(1+R)}+\frac{R^2}{(1+R)^2}\right]$$

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for $q \gg aH$,

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

Important for high frequencies (large multipoles)

where

$$\Gamma(q,t)\equiv\frac{q^2}{6a^2\sigma_T\bar{n}_e}\left[\frac{16}{15(1+R)}+\frac{R^2}{(1+R)^2}\right]$$

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for q >> aH,

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+\underline{2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)}+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

The new solution is

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=\left[A\cos(q\tilde{r}_s)+B\sin(q\tilde{r}_s)\right]\underline{\exp\left[-\int_0^t dt'\ \Gamma(q,t')\right]}-R\Phi$$

Exponential dampling!

$$\approx\exp\left(-q^2/\sigma_T\bar{n}_eH\right)$$

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for q >> aH,

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+\underline{2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)}+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

<u>The new solution is</u>

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=\left[A\cos(q\tilde{r}_s)+B\sin(q\tilde{r}_s)\right]\underline{\exp(-q^2/q^2_{\rm Silk})}-R\Phi$$

Exponential Silk dampling!

$$a/q_{\rm Silk}\approx(\sigma_T\bar{n}_eH)^{-1/2}$$

"diffusion length"
= length traveled by photon's random walks

The Diffusion Length

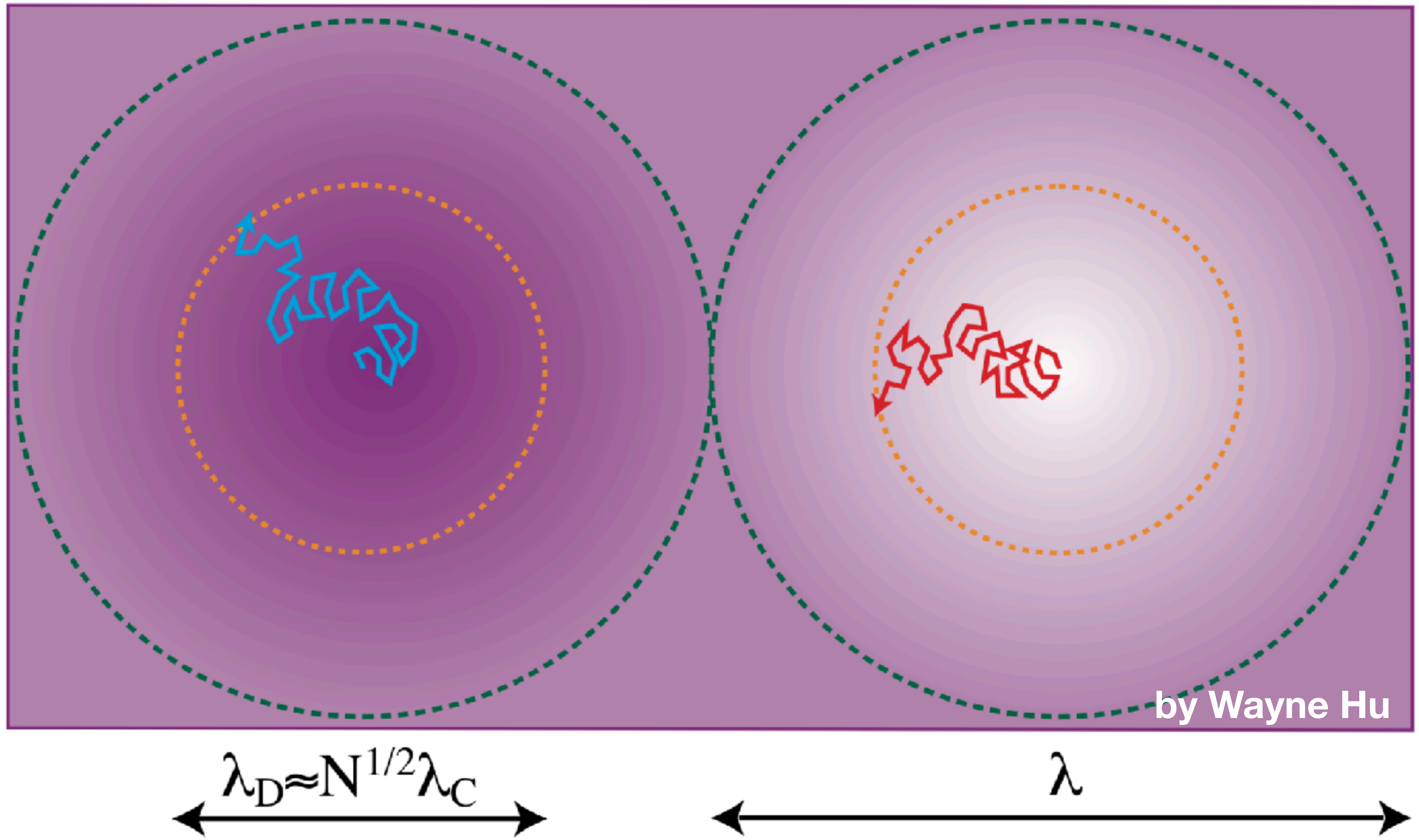
$$\boxed{a/q_{\text{Silk}} \approx (\sigma_T \bar{n}_e H)^{-1/2}}$$

"diffusion length"
= length traveled by photon's random walks

Random walk

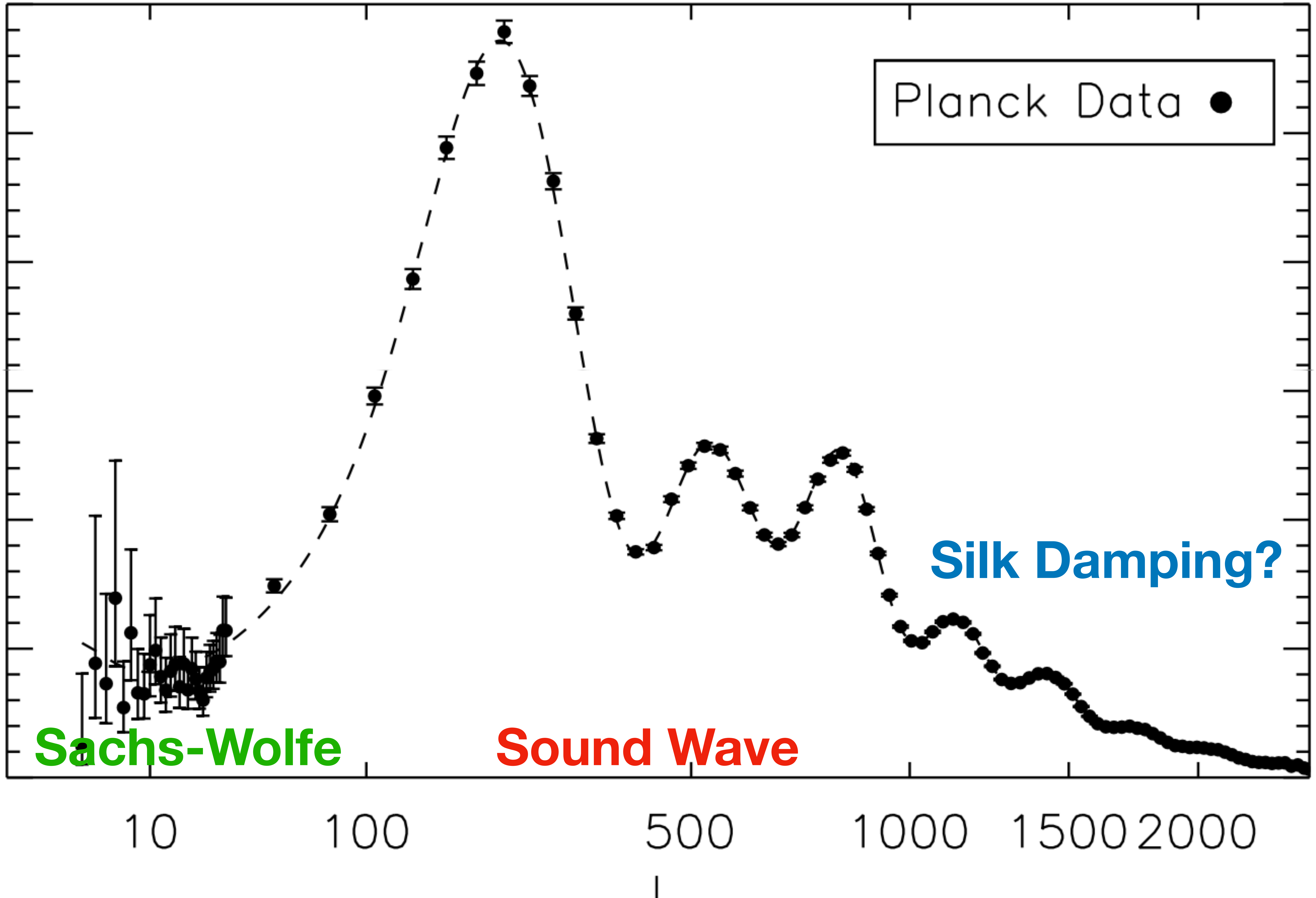
- The mean free path of the photon between scatterings is $(\sigma_T n_e)^{-1}$.
  - Below this scale, you do not have a photon-baryon fluid: they are individual particles.
- The number of scatterings per Hubble time is $N_{\text{scattering}}=\sigma_T n_e/H$.
- Then, the length traveled by photons by random walks within the Hubble time is $(\sigma_T n_e)^{-1}$ times $\sqrt{N_{\text{scatterings}}}$
  - The diffusion length is thus $(\sigma_T n_e)^{-1}$ times $\sqrt{N_{\text{scatterings}}} =$ **$(\sigma_T n_e H)^{-1/2}$**.

The Diffusion Damping



- Diffusion mixes hot and cold photons -> Damping of anisotropies

Planck Collaboration (2016)



Additional Damping

- The power spectrum is $\left|\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi\right|^2$ with q -> l/r_L. The damping factor is thus $\exp(-\mathbf{2}q^2/q_{silk}^2)$.

- $q_{silk}(t_L)$ = 0.139 Mpc^{-1}. This corresponds to a multipole of **$l_{silk} \sim q_{silk}\, r_L/\sqrt{2} = 1370$.** Seems too large, compared to the exact calculation.

- There is an additional damping due to a finite width of the last scattering surface, **σ~250 K.**
  - "Fuzziness damping" – Bond (1996); "Landau damping" - Weinberg (2001)

$$q_{\text{fuzziness}}^{-2} = \frac{3\sigma^2 t_L^2}{8a_0^2T_0^2(1+R_L)} \approx \left(0.20\ \mathrm{Mpc}^{-1}\right)^{-2}$$

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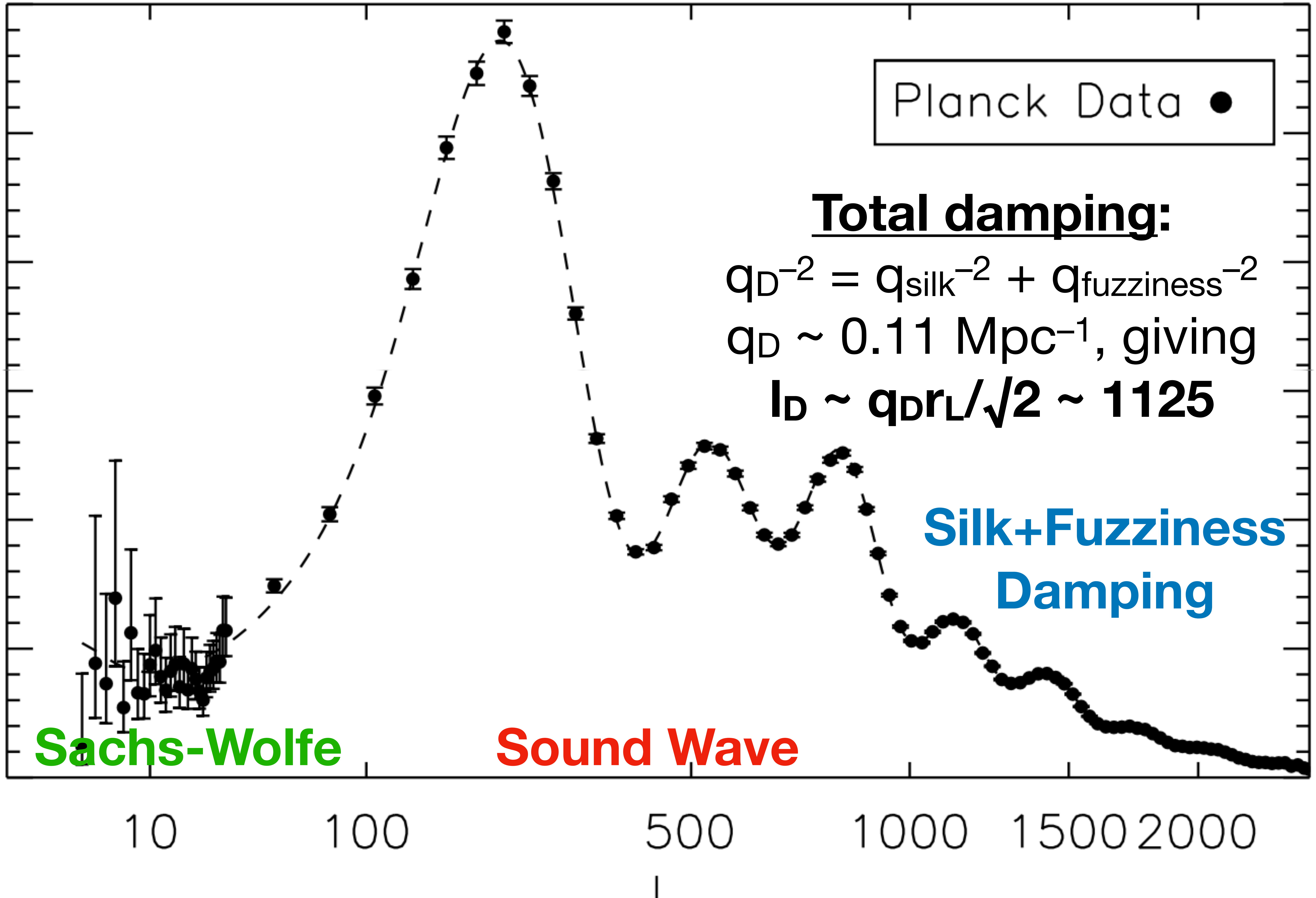
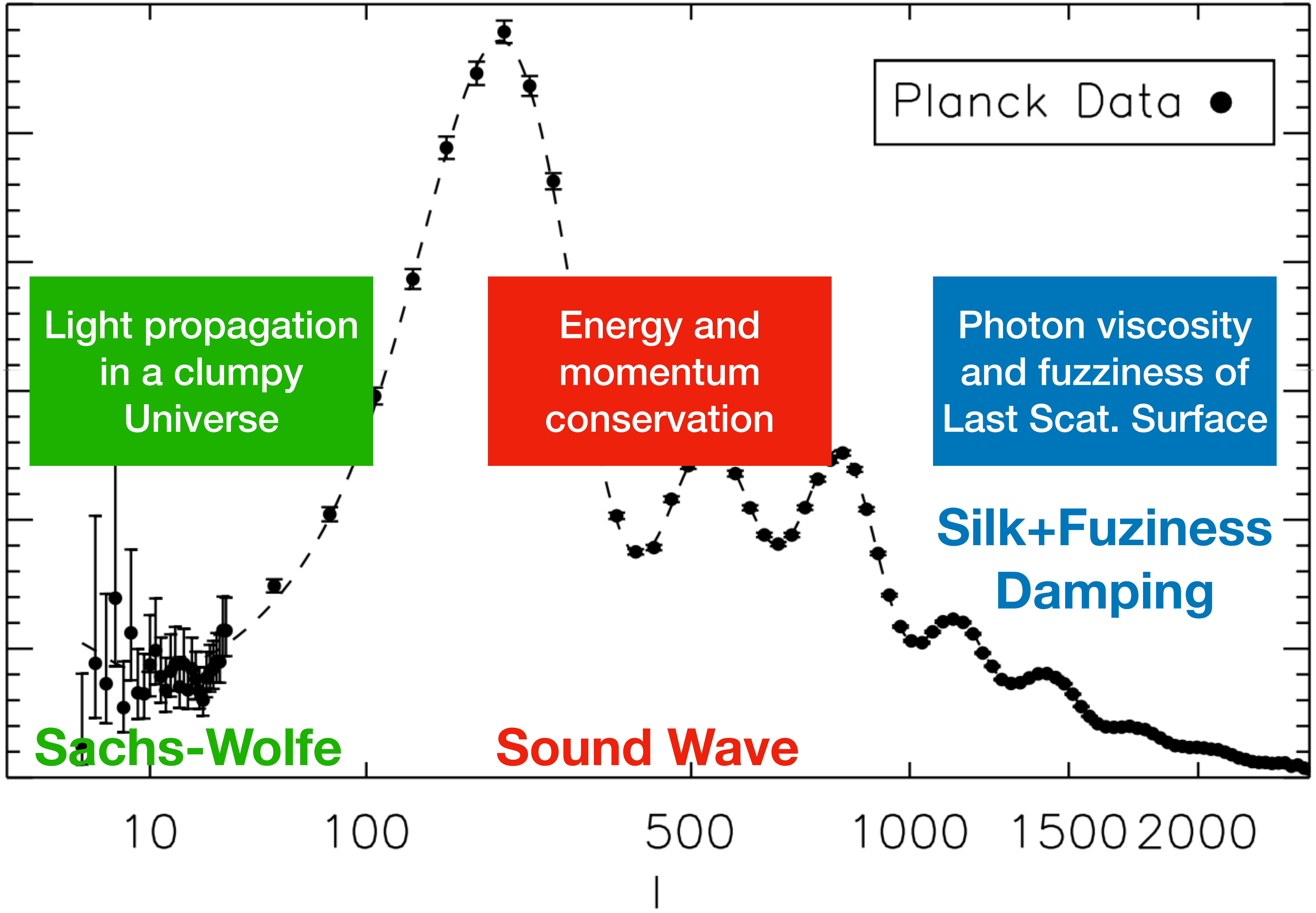


Recap

- The basic structure of the temperature power spectrum is
  - **The Sachs-Wolfe "plateau"** at low multipoles, $l(l+1)C_l \sim l^{n-1}$
  - **Sound waves** at intermediate multipoles
    - The 1st-order tight-coupling approximation
  - **Silk damping and Fuzziness damping** at high multipoles
    - The 2nd-order tight-coupling approximation

Part IV: The Acoustic Oscillation at the Last-scattering Surface

Planck Collaboration (2016)



Matching Solutions: Radiation- and Matter-dominated Eras

- We have a very good analytical solution valid at low and high frequencies **during the radiation era**: $\varphi \equiv q r_s$

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta\left(-\cos\varphi + \frac{2}{\varphi}\sin\varphi\right)$$

- Now, match this to a high-frequency solution valid **at the last-scattering surface** (when R is no longer small)

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(q r_s) + B\sin(q r_s) - R\Phi$$

$$R \equiv 3\bar{\rho}_B / 4\bar{\rho}_\gamma$$

Matching Solutions: Radiation- and Matter-dominated Eras

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- Now, match this to a high-frequency solution valid **at the last-scattering surface** (when R is no longer small)

Slightly improved solution, with a weak time dependence of R using the WKB method [*Peebles & Yu (1970)*]

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = (1+R)^{-1/4}[A\cos(qr_s) + B\sin(qr_s)] - R\Phi$$

Solution at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

where *T*(q), *S*(q), *θ*(q) are "transfer functions" that smoothly interpolate two limits as

$q \ll q_{EQ}$: $\mathcal{S} \to 1,\ \mathcal{T} \to 1,\ \theta \to 0$

$q \gg q_{EQ}$: $\mathcal{S} \to 5,\ \mathcal{T} \propto \ln q/q^2,\ \theta \to 0.062\pi$

"EQ" for "matter-radiation Equality epoch"

<u>with $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01\ \mathrm{Mpc}^{-1}$, giving $l_{EQ}=q_{EQ}r_L \sim 140$</u>

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales

Solution at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

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q << q$_{EQ}$: $\mathcal{S} \to 1,\ \mathcal{T} \to 1,\ \theta \to 0$

q >> q$_{EQ}$: $\boxed{\mathcal{S} \to 5,\ \mathcal{T} \propto \ln q/q^2}, \ \theta \to 0.062\pi$

"EQ" for "matter-radiation Equality epoch"

with q$_E$ … r_L ~ 140

Due to the decay of gravitational potential during the radiation dominated era

- (*) To a … solution is given by … not importa… at large scales

Solution at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

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- (*) To a good approximation, the low-f given by setting R=0 because sound v important at large scales

Due to the neutrino anisotropic stress (see "Appendix" of the last lecture slides)

Low-frequency Limit

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

$$\xrightarrow{\textbf{q -> 0(*)}} \quad -\frac{\zeta}{5}$$

This should agree with the Sachs-Wolfe result: Φ/3; thus,

$$\Phi = -3\zeta/5$$ **in the matter-dominated era**

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales

High-frequency Limit

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

$$\xrightarrow{q/q_{EQ} \gg 1} -(1+R)^{-1/4}\zeta\cos[qr_s + \theta(q)]$$

- The amplitude of the oscillation on small scales is a factor of $5(1+R)^{-1/4}$ times the Sachs-Wolfe plateau!

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Effect of Baryons

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{\boxed{3R\mathcal{T}(q)} - \boxed{(1+R)^{-1/4}}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

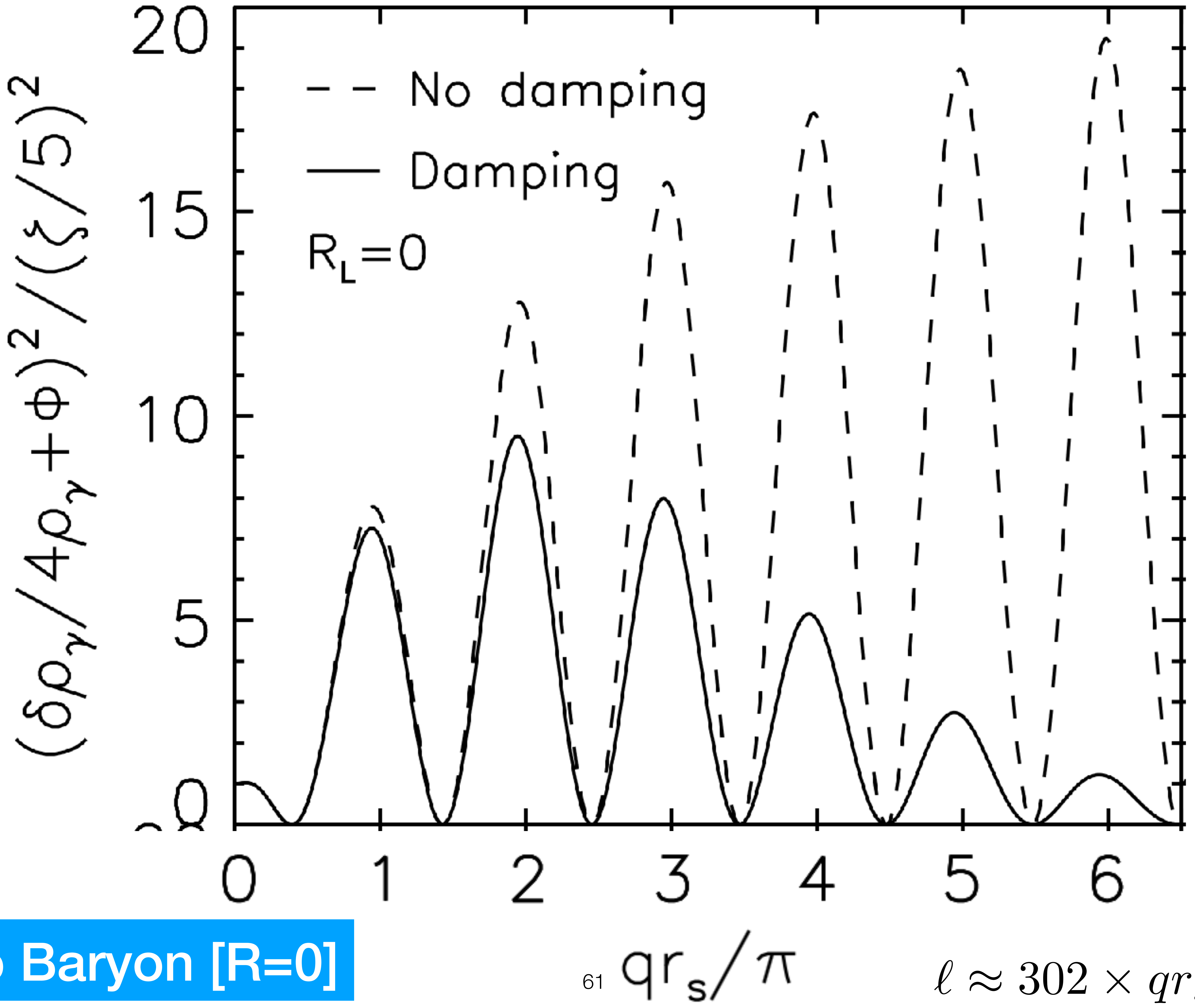
Shift the zero-point of oscillations

Reduce the amplitude of oscillations

The CMB power spectrum is sensitive to this combination of the parameters.

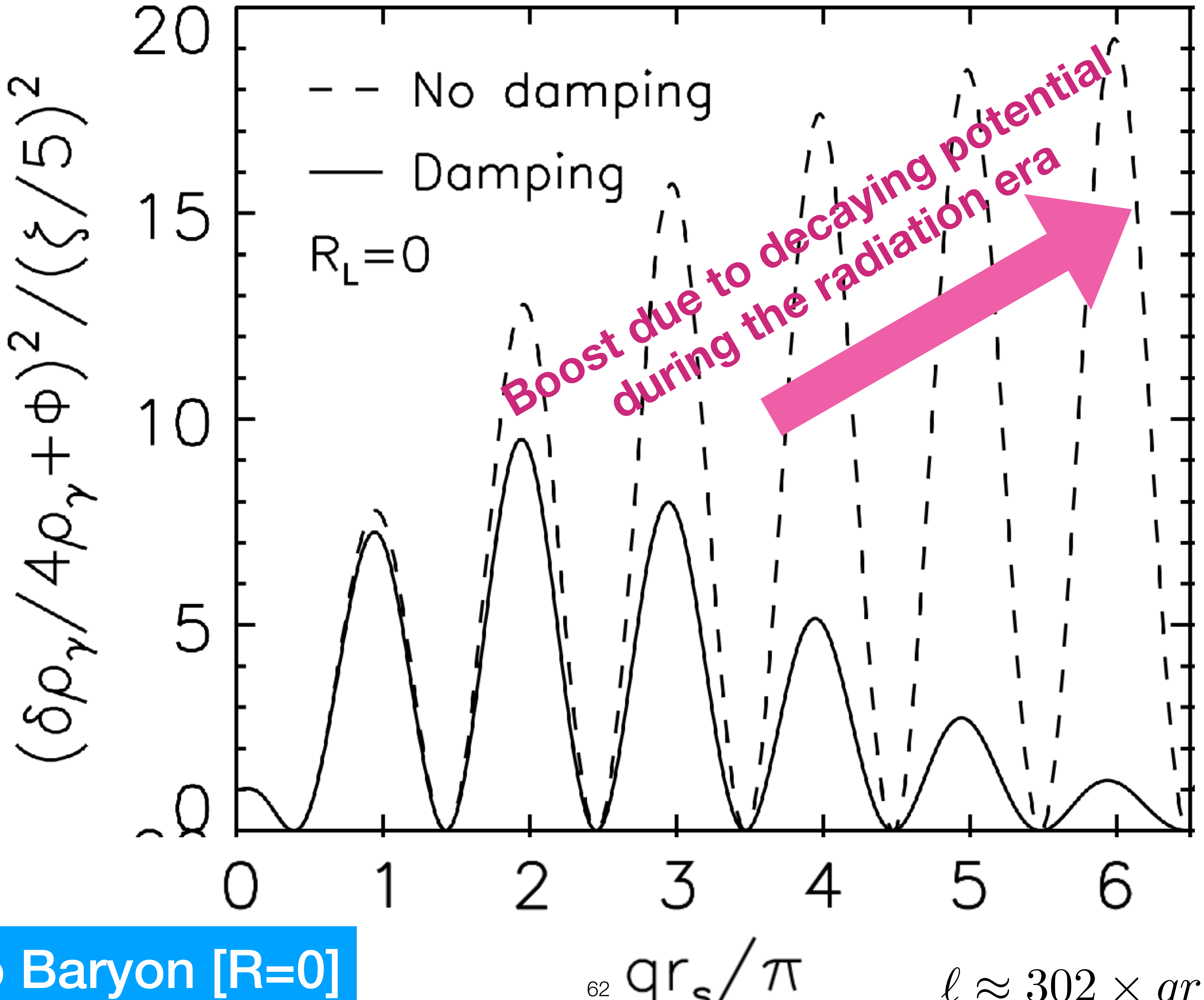
$$R = \frac{3\Omega_B}{4\Omega_\gamma}\frac{a}{a_0} = 0.6120\left(\frac{\boxed{\Omega_B h^2}}{0.022}\right)\frac{1091}{1+z}$$

$$\Omega_\gamma \equiv \frac{8\pi G\rho_{\gamma 0}}{3H_0^2} = 2.471\times 10^{-5}\ h^{-2}$$ for $T_0 = 2.725\ \mathrm{K}$



No Baryon [R=0]

$\ell \approx 302 \times qr_s/\pi$



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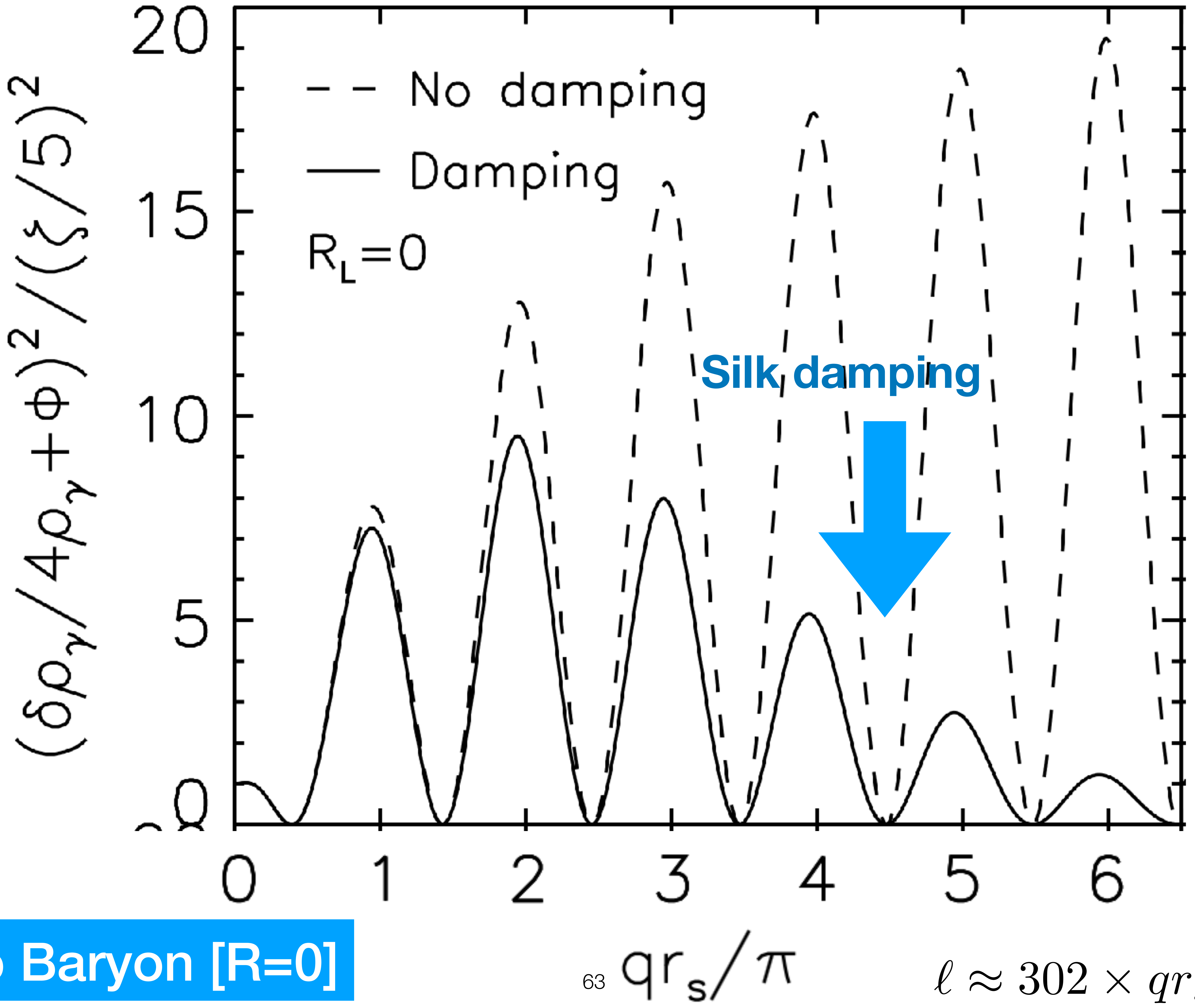

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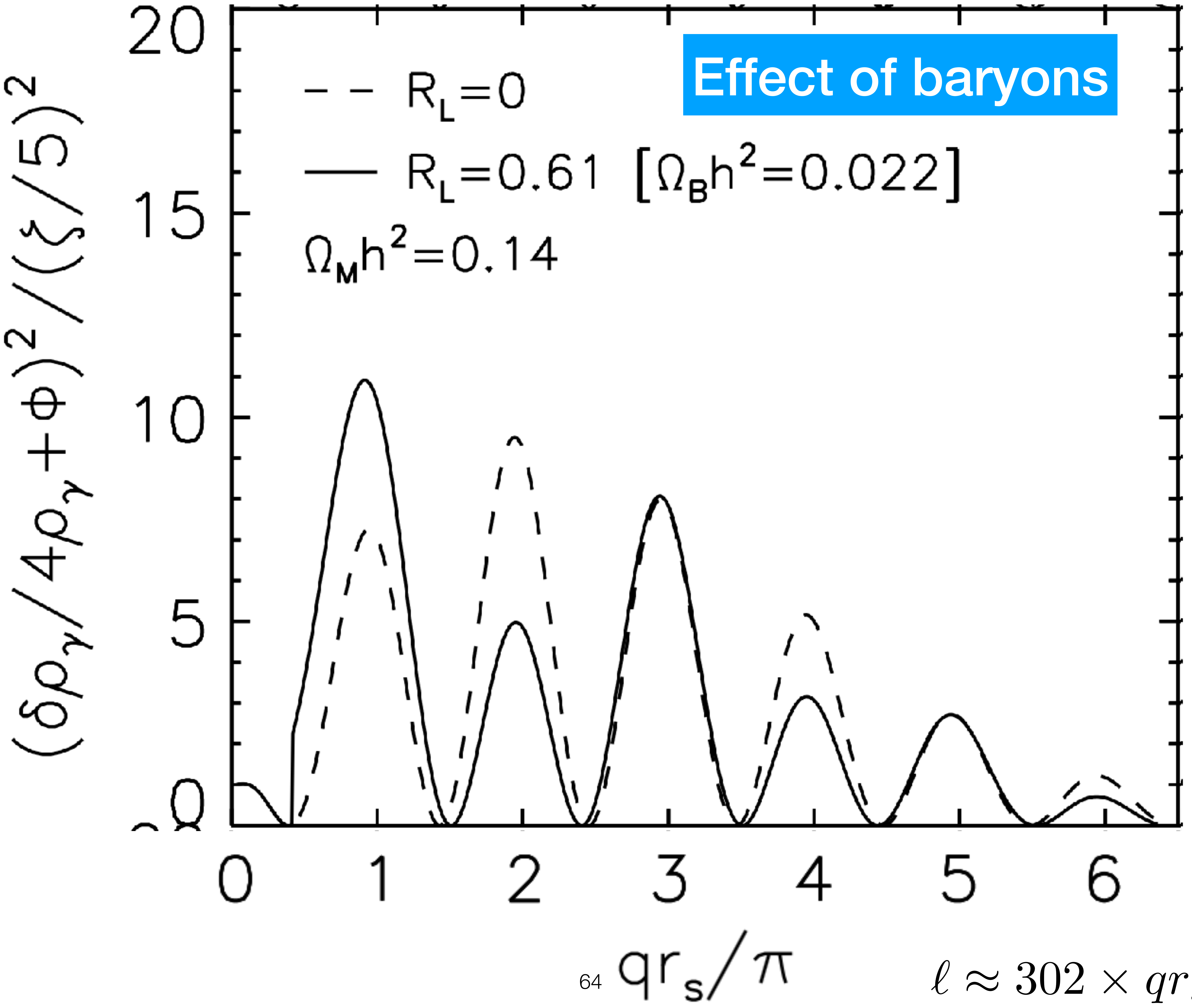

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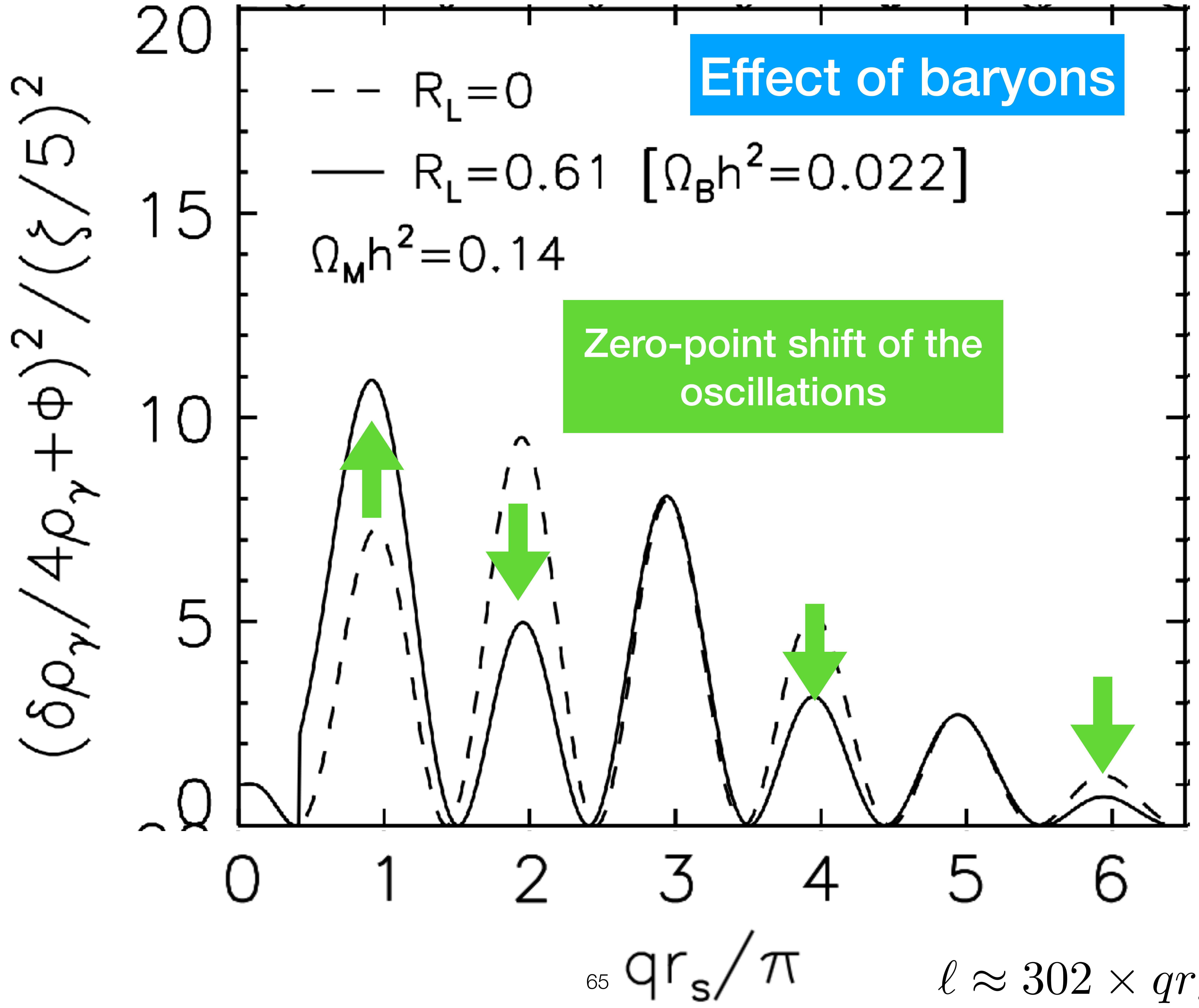

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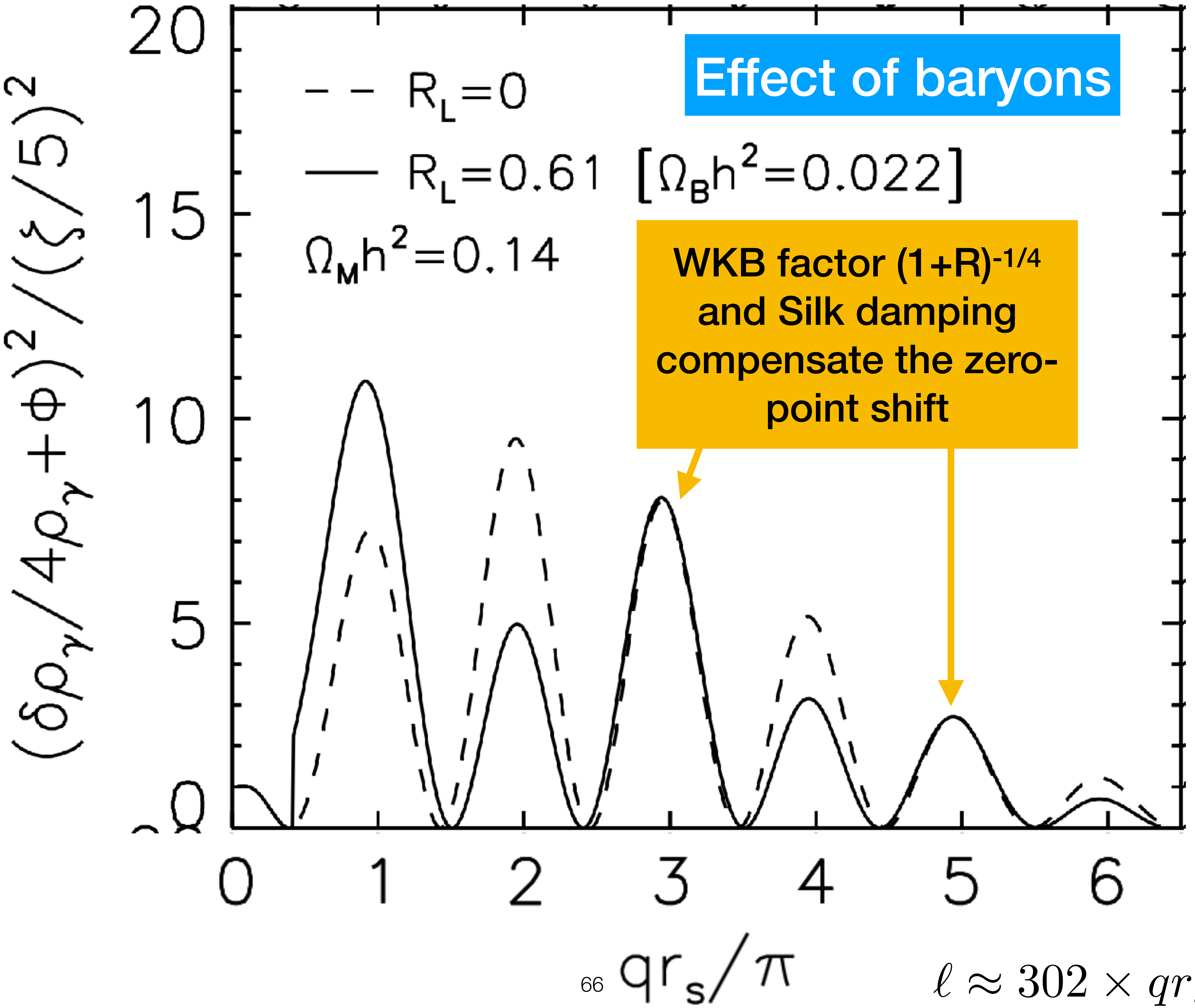

$\ell \approx 302 \times qr_s/\pi$

Effect of Total Matter

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

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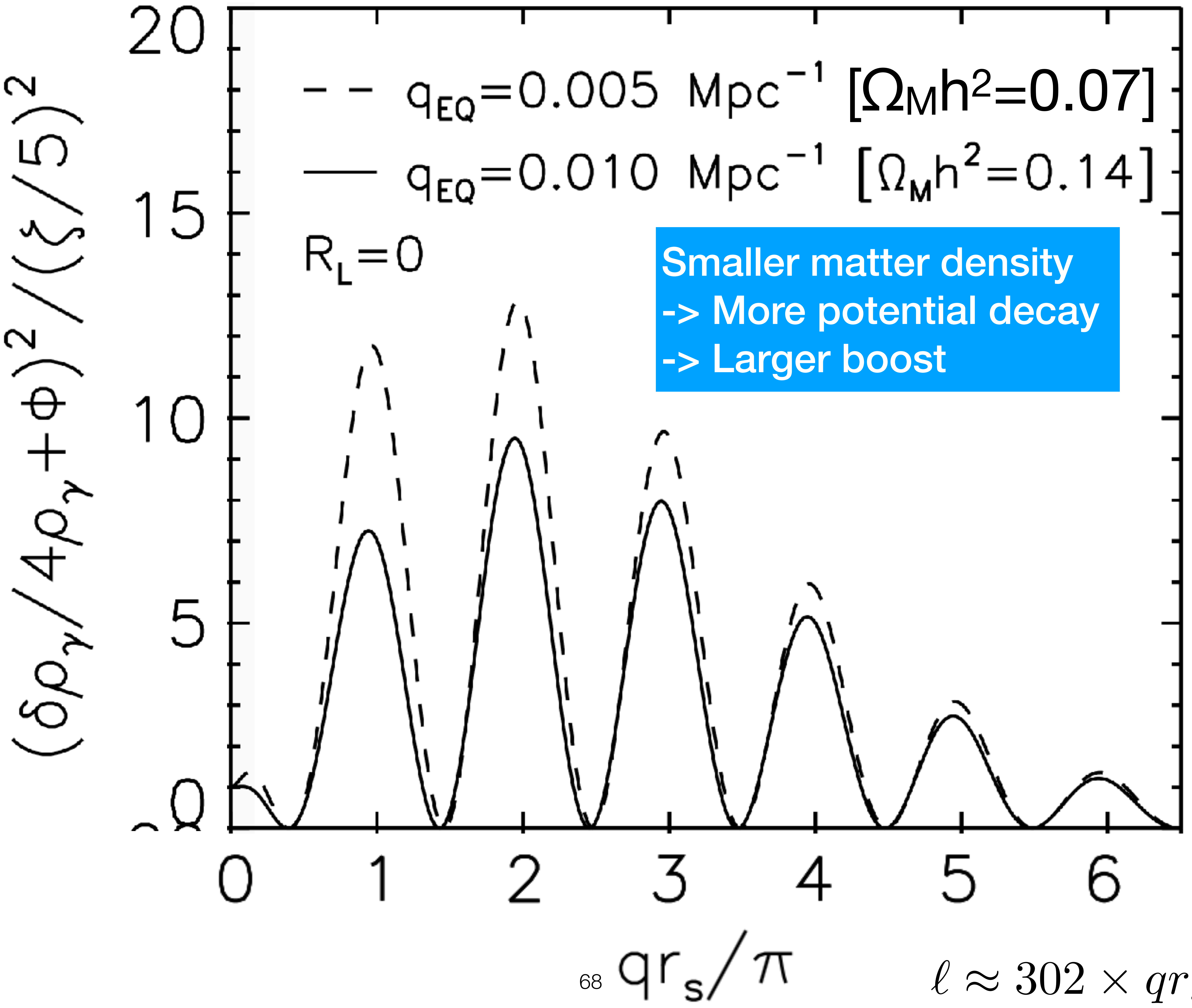
The CMB power spectrum is sensitive to this combination of the parameters.



$\ell \approx 302 \times qr_s/\pi$

Recap

- Decay of the gravitational potential boosts the temperature anisotropy dT/T at high multipoles by **$5(1+R)^{-1/4}$ compared to the Sachs-Wolfe plateau.**
    - Where this boost starts depends on the total matter density => **We can measure $\Omega_M h^2$ using this.**
- Baryon density shifts the zero-point of the oscillation, boosting the heights of the odd peaks relative to those of the even peaks => **We can measure $\Omega_B h^2$ using this.**
    - However, the WKB factor $(1+R)^{-1/4}$ and damping make the **boosting of the 3rd and 5th peaks not so obvious.**

Not quite there yet…

- **The first peak is too low**
  - We need to include the “integrated Sachs-Wolfe effect”
- **How to fill zeros between the peaks?**
  - We need to include the Doppler shift of light

The Doppler Shift of Light

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n}r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n}r_L) \boxed{- \hat{n}\cdot\boldsymbol{v}_B(t_L, \hat{n}r_L)}$$

V_B is the bulk velocity of a baryon fluid

- Using the velocity potential, we write $-\hat{n}\cdot\nabla\delta u_B/a$
- In tight coupling, $\delta u_B = \delta u_\gamma$
- Using the energy conservation,

$$\delta u_\gamma = (3a^2/q^2)\partial(\delta\rho_\gamma/4\bar{\rho}_\gamma)/\partial t$$

<u>**Line-of-sight direction**</u>

$$\hat{n}^i = -\gamma^i$$

<u>**Coming distance (r)**</u>

$$x^i = \hat{n}^i r$$

$$r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

The Doppler Shift of Light

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n}r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n}r_L) \boxed{- \hat{n}\cdot\boldsymbol{v}_B(t_L, \hat{n}r_L)}$$

V_B is the bulk velocity of a baryon fluid

- Using the velocity potential, we write $-\hat{n}\cdot\nabla\delta u_B/a$
- In tight coupling, $\delta u_B = \delta u_\gamma$
- Using the energy conservation,

$$\delta u_\gamma = (3a^2/q^2)\boxed{\partial(\delta\rho_\gamma/4\bar{\rho}_\gamma)/\partial t}$$

Velocity potential is a

time-derivative

of the energy density:

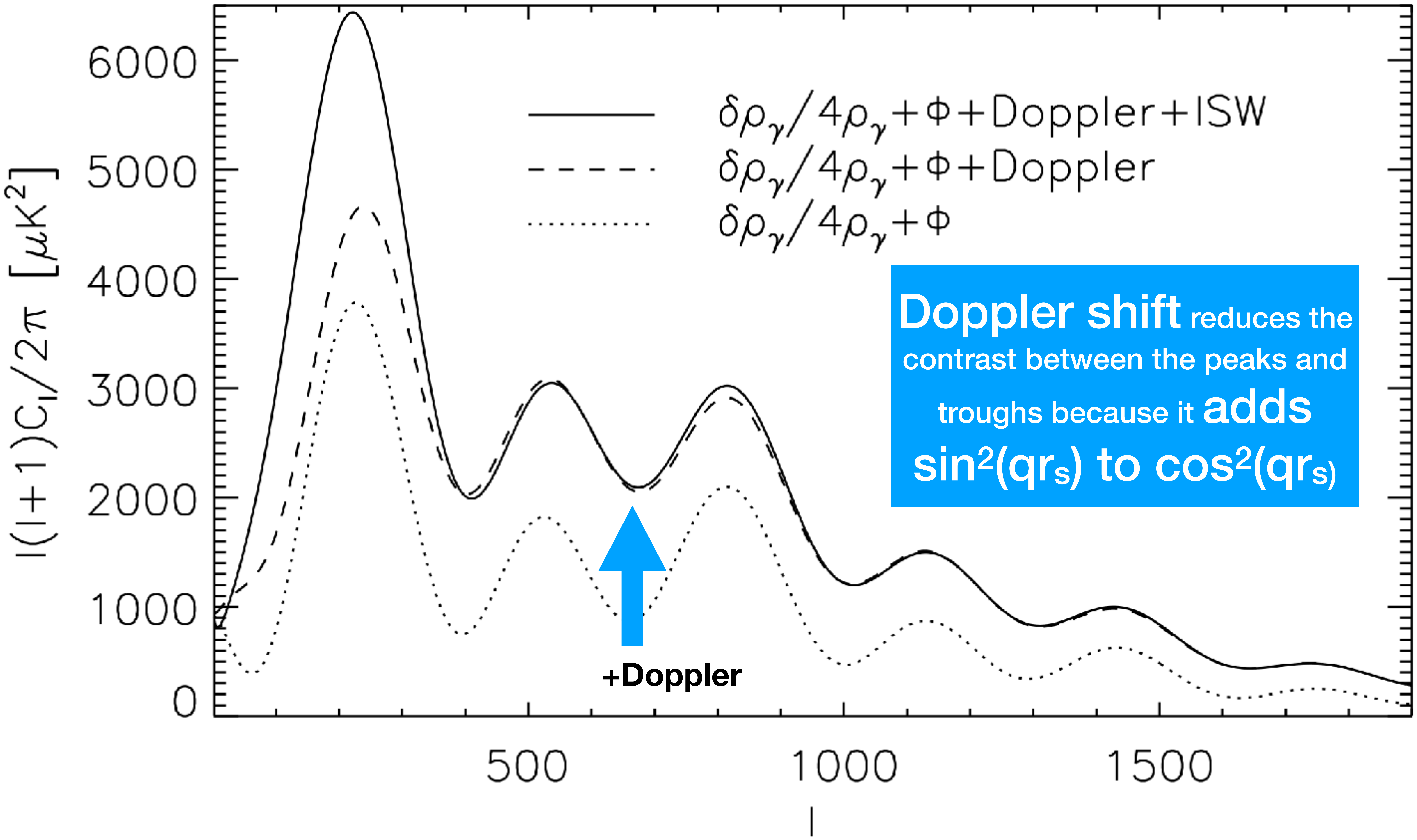
cos(qr$_s$) becomes sin(qr$_s$)!

Temperature Anisotropy from the Doppler Shift

$$\frac{q}{a}\delta u_\gamma = \frac{\sqrt{3}\zeta}{5}(1+R)^{-3/4}\mathcal{S}(\kappa)\sin[qr_s+\theta(\kappa)]$$

- To this, we should multiply the damping factor

$$\exp(-q^2/q^2_{\text{Damp}})$$

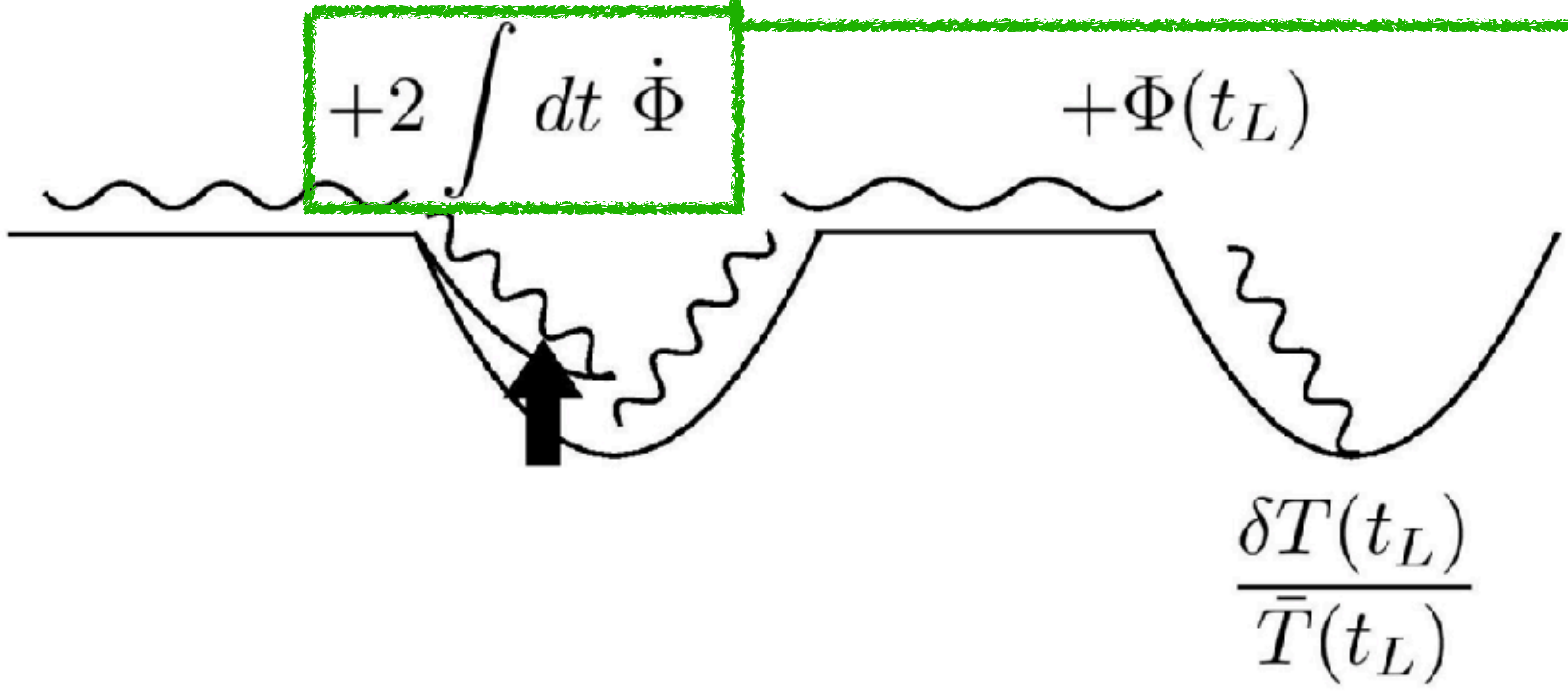

$\delta\rho_\gamma/4\rho_\gamma+\Phi+\text{Doppler}+\text{ISW}$
$\delta\rho_\gamma/4\rho_\gamma+\Phi+\text{Doppler}$
$\delta\rho_\gamma/4\rho_\gamma+\Phi$
$l(l+1)C_l/2\pi\ [\mu K^2]$
6000
5000
4000
3000
2000
1000
0
500
1000
1500
l
+Doppler
Doppler shift reduces the contrast between the peaks and troughs because it adds $\sin^2(qr_s)$ to $\cos^2(qr_s)$

(Early) ISW

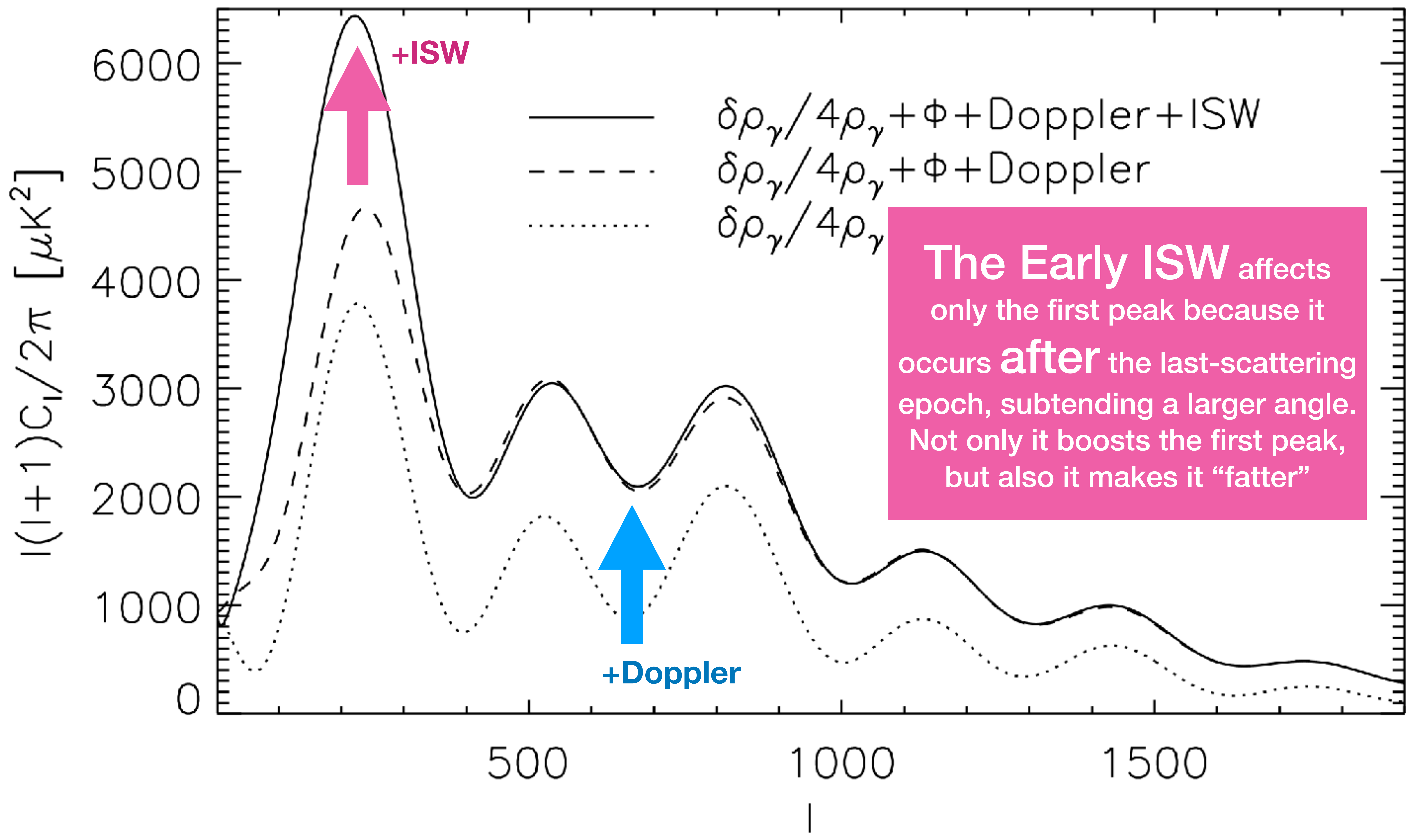
$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0)$$

“integrated Sachs-Wolfe” (ISW) effect

$$\boxed{+ \int_{t_L}^{t_0} dt \; (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)}$$



Gravitational potentials decay after the last-scattering because the Universe was not yet completely matter-dominated.


+ISW
$\delta\rho_\gamma/4\rho_\gamma+\Phi+\text{Doppler}+\text{ISW}$
$\delta\rho_\gamma/4\rho_\gamma+\Phi+\text{Doppler}$
$\delta\rho_\gamma/4\rho_\gamma$
The Early ISW affects only the first peak because it occurs after the last-scattering epoch, subtending a larger angle. Not only it boosts the first peak, but also it makes it “fatter”
+Doppler
6000
5000
4000
3000
2000
1000
0
500
1000
1500
$l(l+1)C_l/2\pi\ [\mu K^2]$
l

We are ready!

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n} r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n} r_L) - \hat{n}\cdot \boldsymbol{v}_B(t_L, \hat{n} r_L)$$

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(\kappa) - (1+R)^{-1/4}\mathcal{S}(\kappa)\cos[q r_s + \theta(\kappa)]\right\}$$

$$\frac{q}{a}\delta u_\gamma = \frac{\sqrt{3}\zeta}{5}(1+R)^{-3/4}\mathcal{S}(\kappa)\sin[q r_s + \theta(\kappa)]$$

- We are ready to understand the effects of all the cosmological parameters.

➡ Next Lecture!

Appendix: Neutrino Viscosity

High-frequency solution without neutrino viscosity

The solution is

$$X = (\zeta + \Delta A)\cos\varphi + (\qquad \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi'(\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2)$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi'(\Phi + \Psi)(\varphi') \qquad \xrightarrow[\varphi \gg 1]{} -2\zeta$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2 \xrightarrow[\varphi \gg 1]{} 0$$

Chluba & Grin (2013)

High-frequency solution with neutrino viscosity

The solution is

$$X = (-\zeta + \Delta A_\nu) \cos\varphi + \Delta B_\nu \sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A_\nu \xrightarrow[\varphi \gg 1]{} 0.338 R_\nu \zeta$$

$$\Delta B_\nu \xrightarrow[\varphi \gg 1]{} 0.418 R_\nu \zeta$$

non-zero value!

$$\boxed{\begin{aligned} R_\nu &\equiv \bar{\rho}_\nu / (\bar{\rho}_\gamma + \bar{\rho}_\nu) \\ &\approx 0.409 \end{aligned}}$$

High-frequency solution **with** neutrino viscosity

Using the formula for trigonometry, we write

$$X = -C\cos(\varphi + \theta)$$

where

$$C \equiv \sqrt{(-\zeta + \Delta A_\nu)^2 + \Delta B_\nu^2}$$

$$\approx \zeta(1 + 4R_\nu/15)^{-1}$$ (Hu & Sugiyama 1996)

$$\tan\theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi$$ **Phase shift!**

(Bashinsky & Seljak 2004)

High-frequency solution with neutrino viscosity

The s

X

where

C

Thus, the neutrino viscosity will:

(1) **Reduce the amplitude** of sound waves at large multipoles

(2) **Shift the peak positions** of the temperature power spectrum

$$\tan\theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi$$

Phase shift!

(Bashinsky & Seljak 2004)