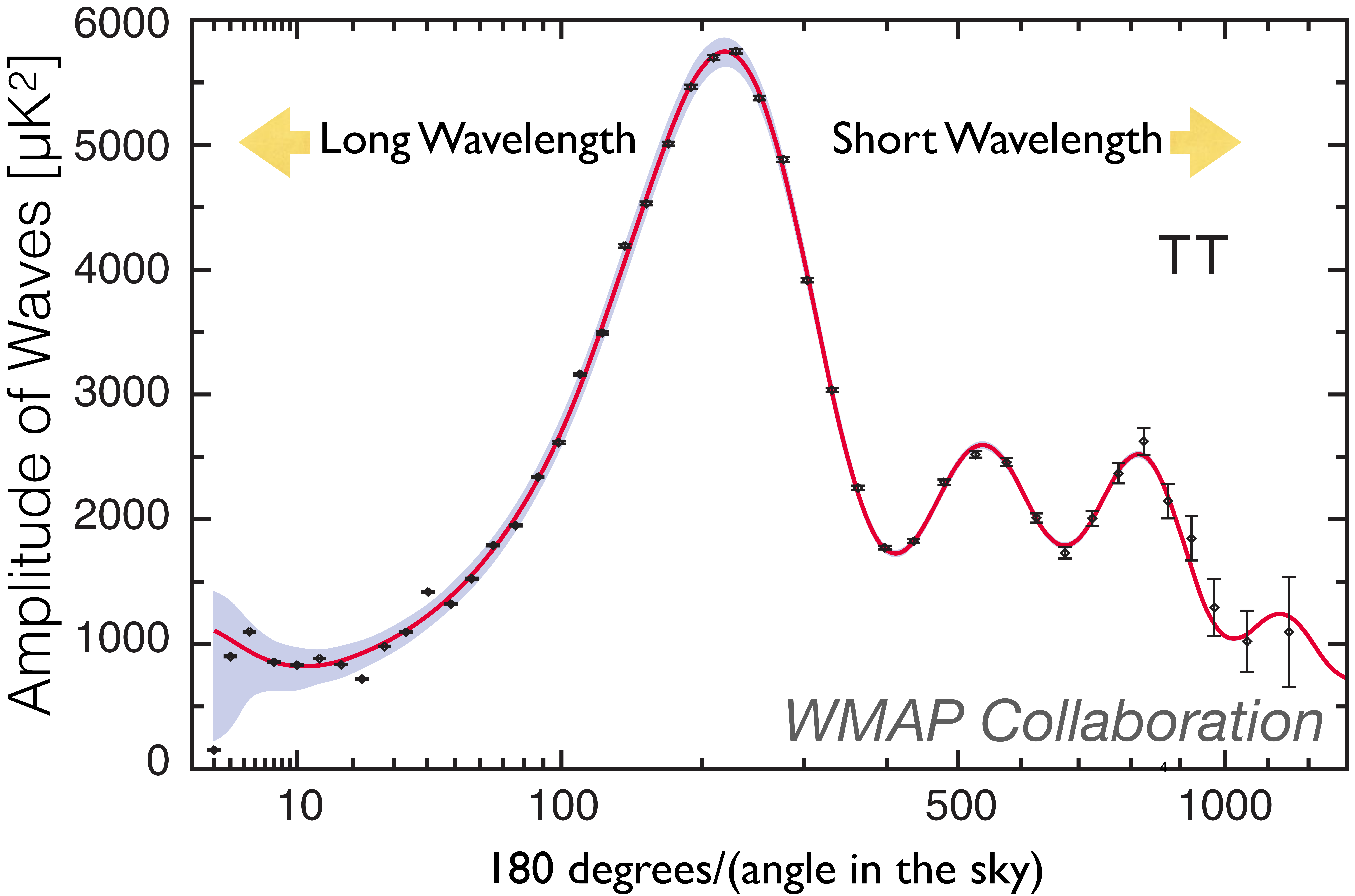
Lecture 4: Power Spectrum

How do we analyse the data like this?

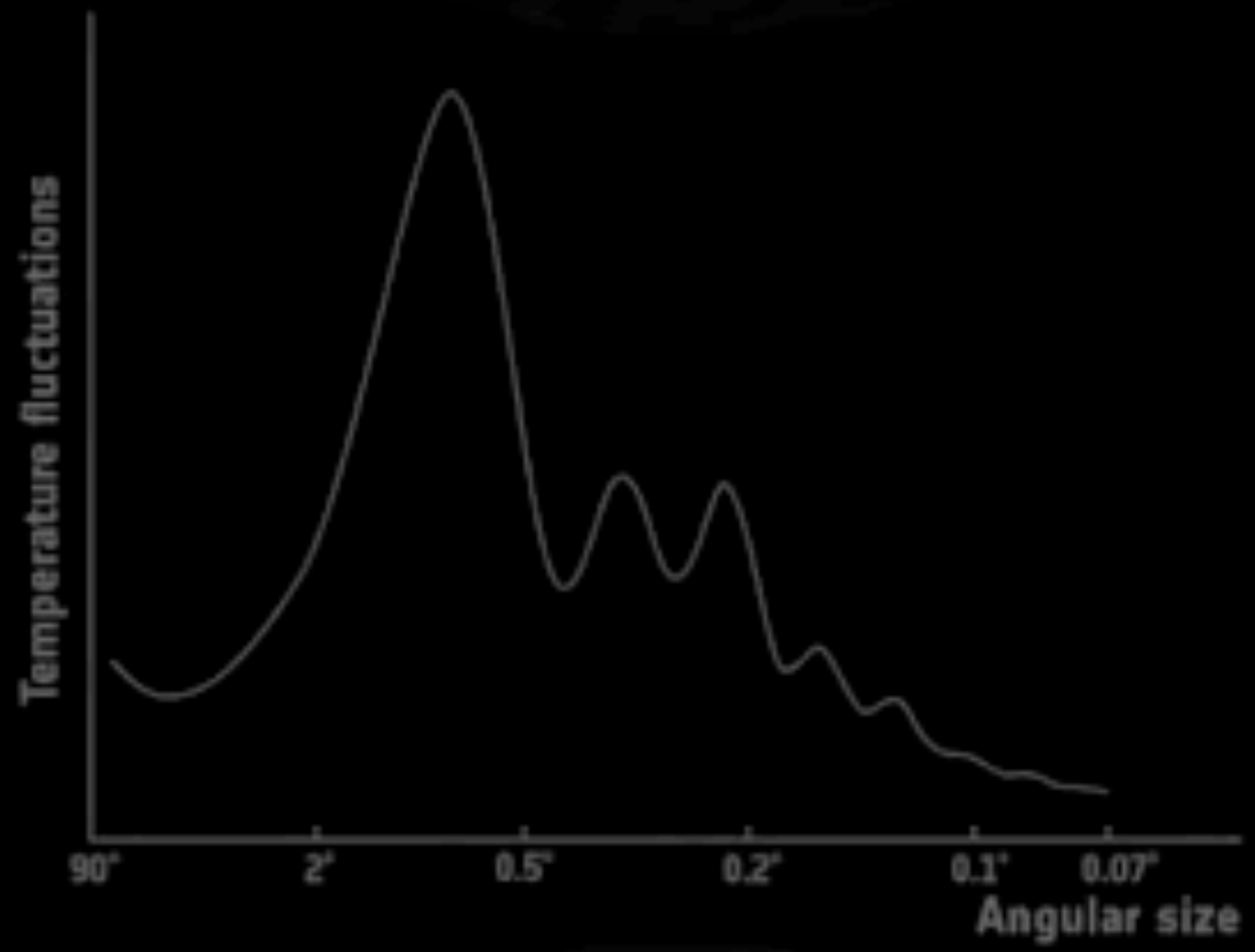
Data Analysis

- Decompose temperature fluctuations in the sky into a set of waves with various wavelengths
- Make a diagram showing the strength of each wavelength: **Power Spectrum**



Amplitude of Waves [μK²]
6000
5000
4000
3000
2000
1000
0
Long Wavelength
Short Wavelength
TT
WMAP Collaboration
10
100
500
1000
180 degrees/(angle in the sky)

Power Spectrum, Explained



Part I: Spherical Harmonics

Fourier transform?

- The simplest way to decompose fluctuations into waves is Fourier transform.
    - However, Fourier transform works only for plane waves in flat space.
- The sky is a sphere. How do we decompose fluctuations on a sphere into waves?
    - The answer: **Spherical Harmonics**.

Spherical harmonics

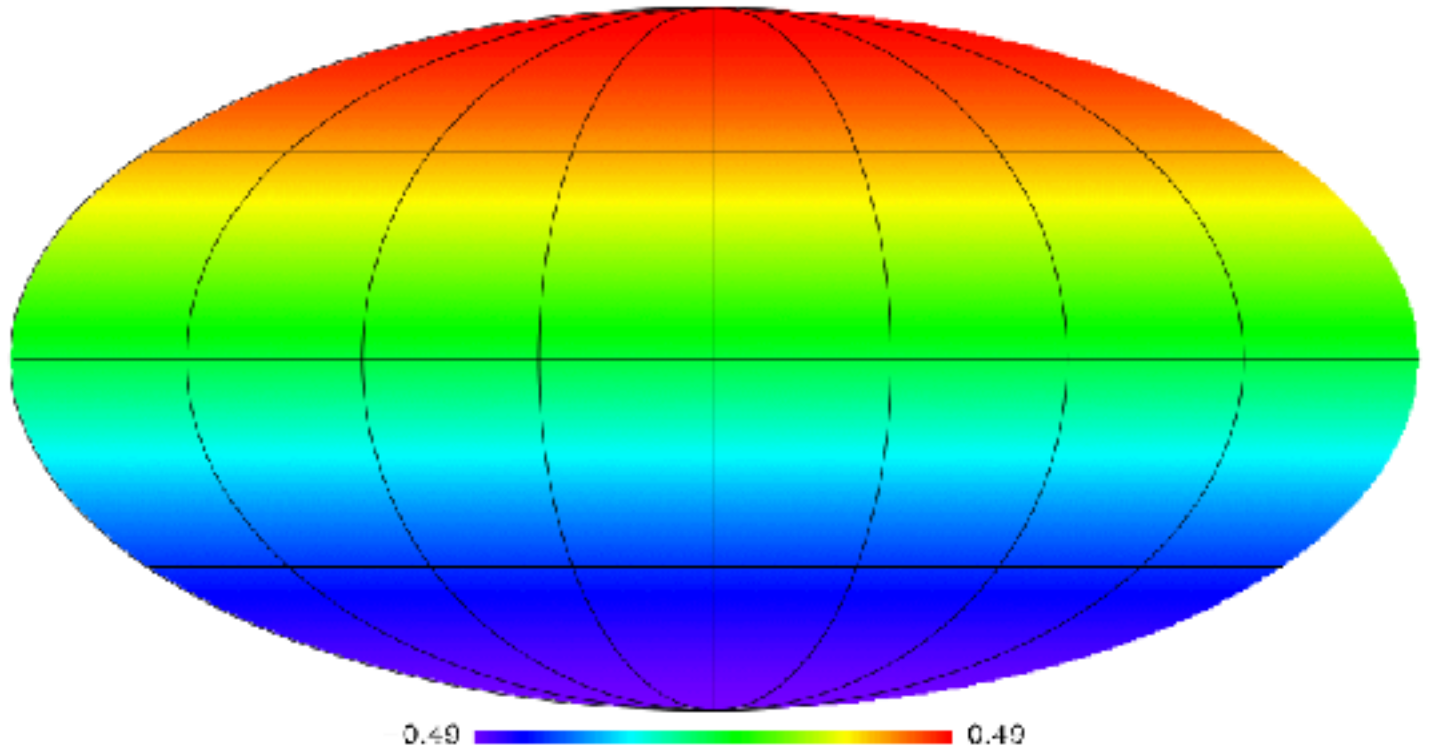
Wait, don't run! It is not as bad as you may remember from the QM class…

$$\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n})$$

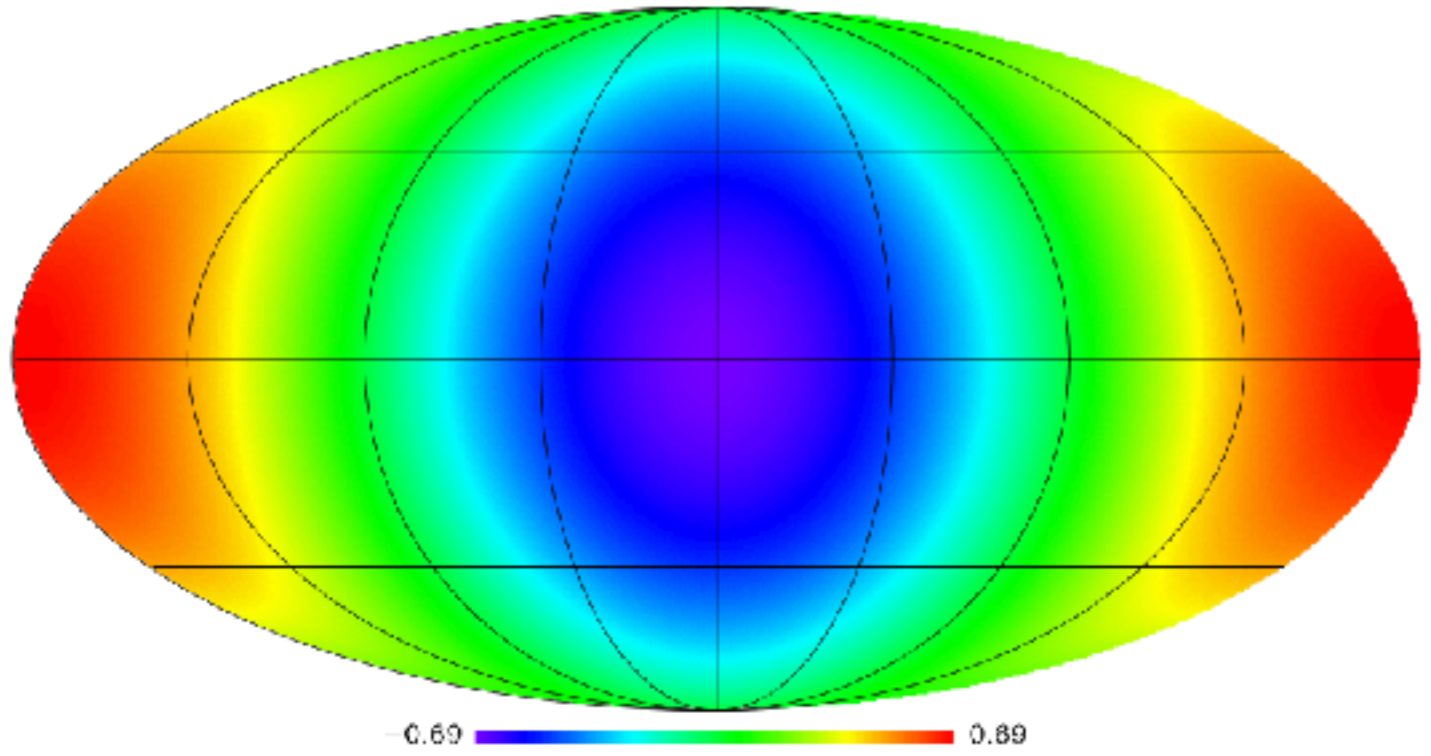
$a_{\ell -m} = (-1)^m a_{\ell m}^*$: sufficient to consider only m>=0

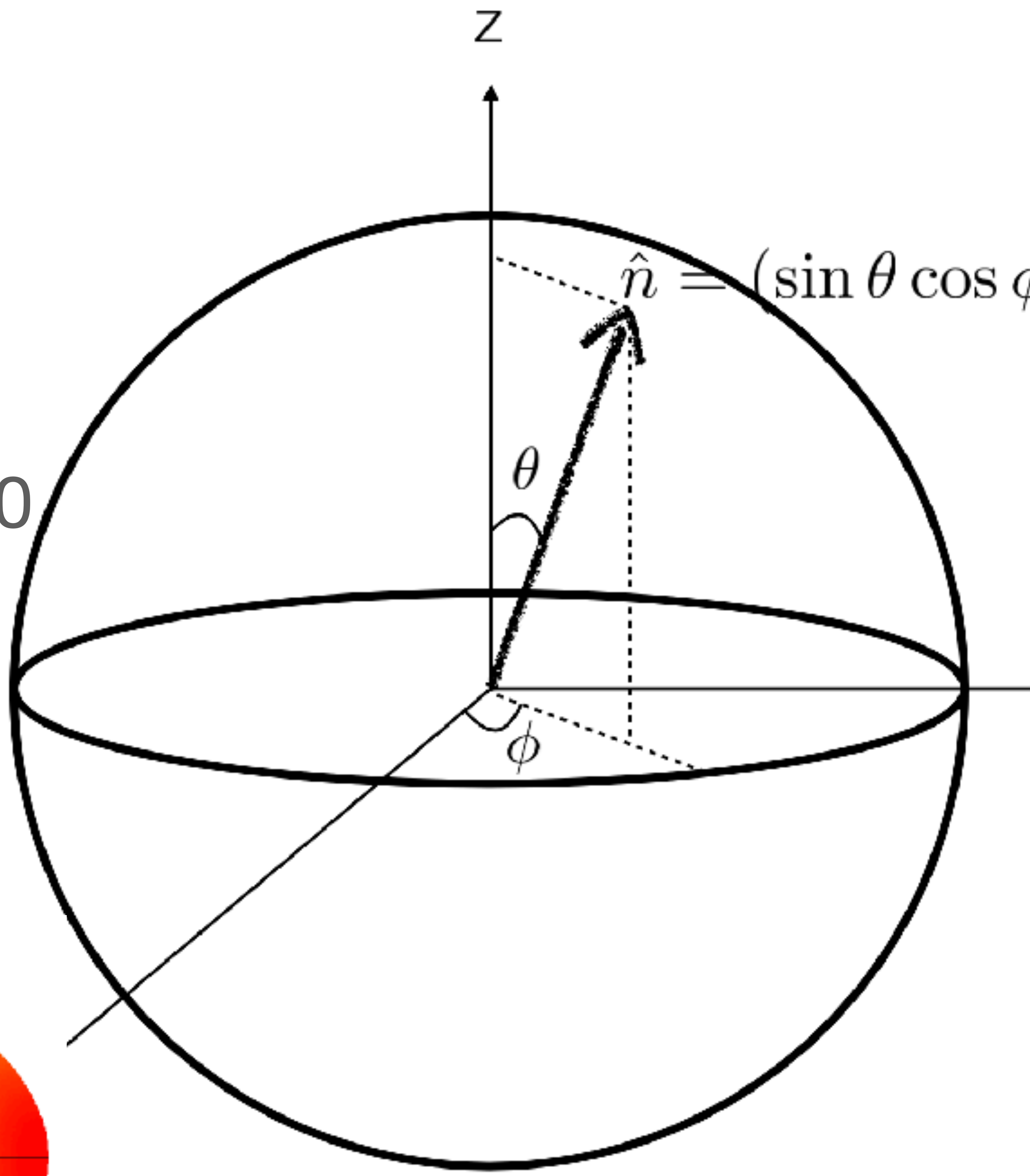
- Dipole patterns (l=1)

(l,m)=(1,0)



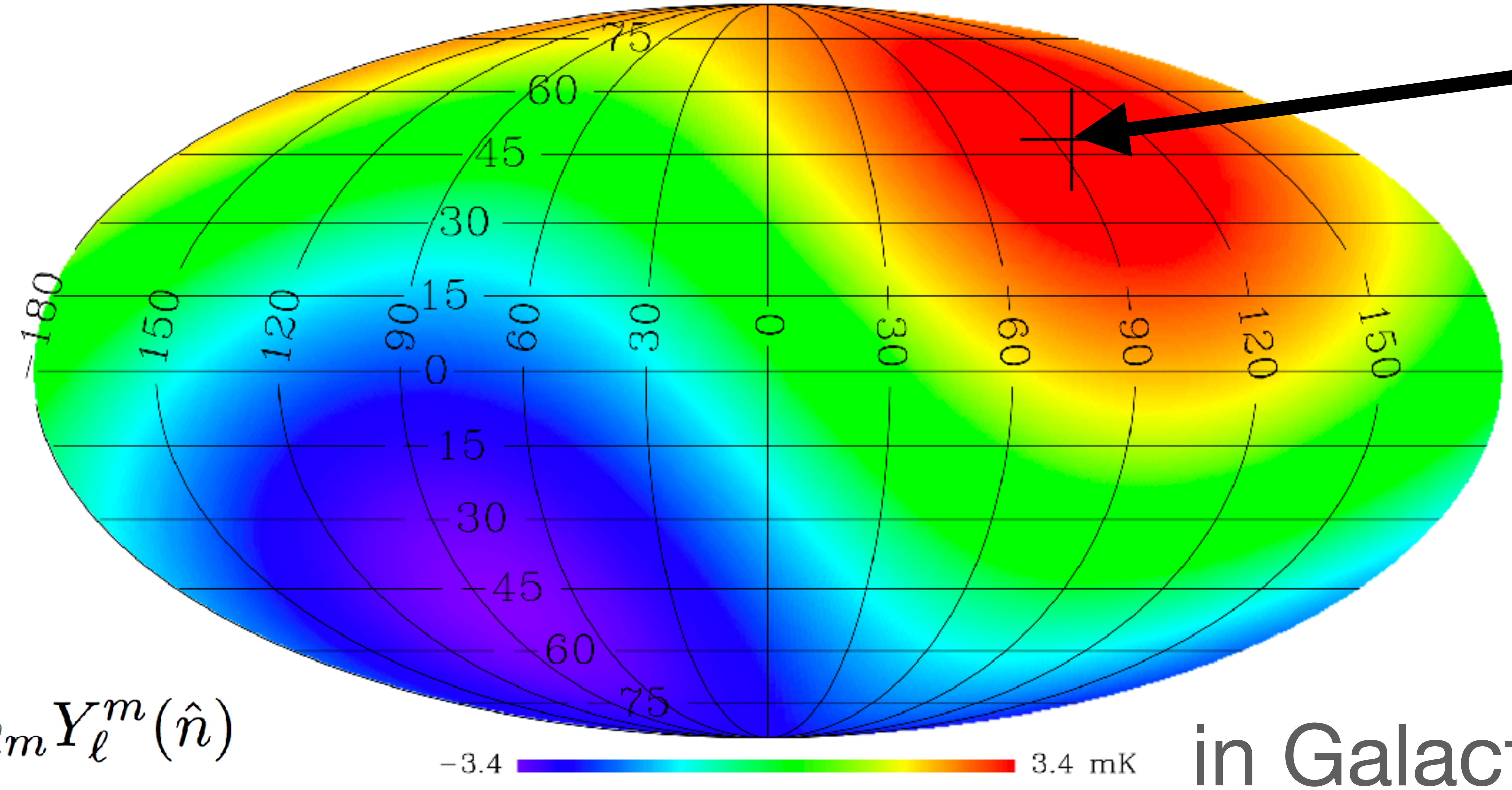
(l,m)=(1,1)





Dipole Temperature Anisotropy of the CMB

Due to the motion of Solar System with respect to the CMB rest frame



The Solar System is moving towards this direction at 369 km/s.

in Galactic coordinates

$$\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n})$$

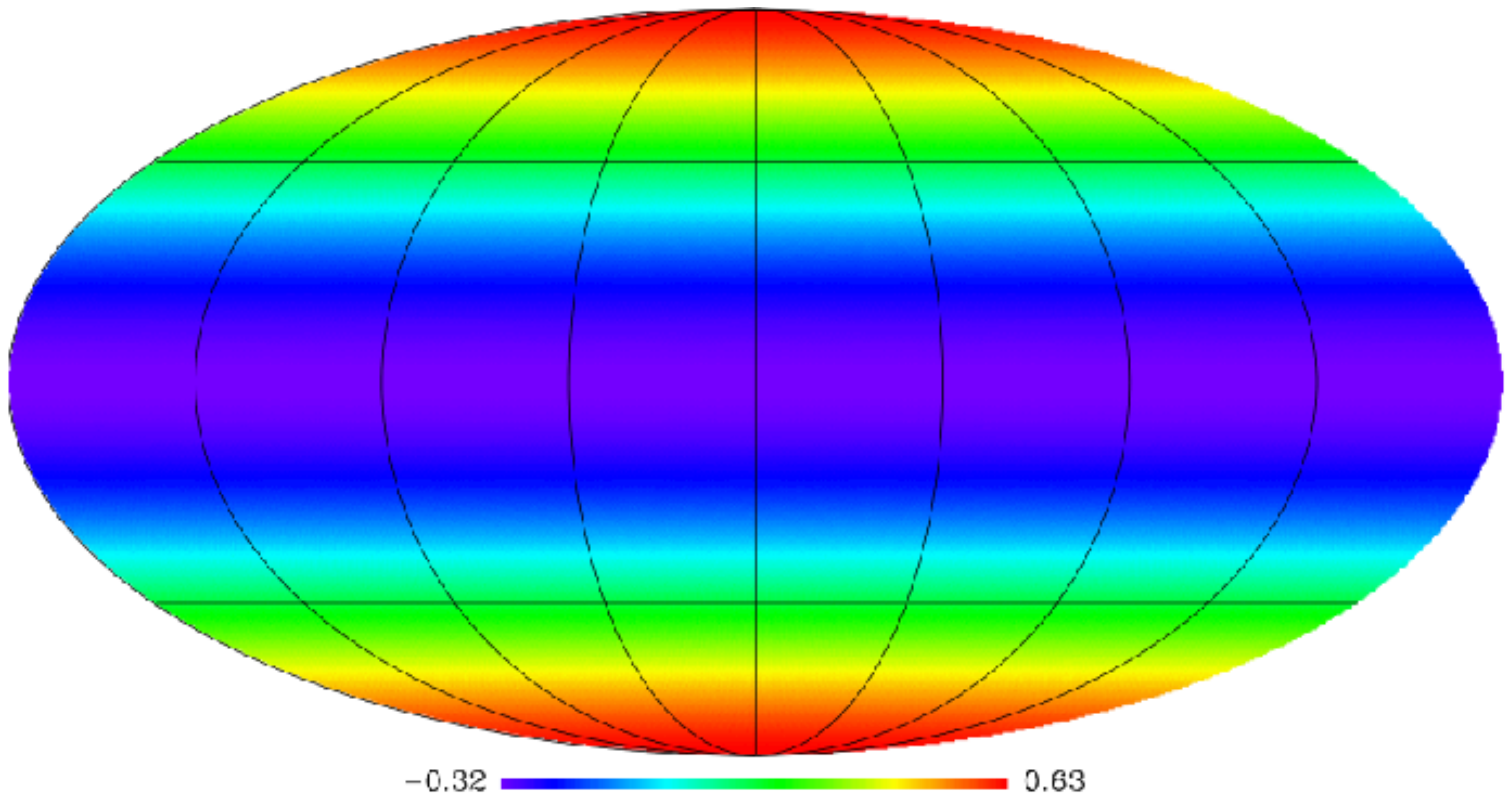
- Temperature anisotropy towards "+" is $\Delta T/T = v/c = 1.23 \times 10^{-3}$
  - Thus, **ΔT = 3.355 mK**

$$a_{10} = 5.124 \text{ mK str}^{1/2},$$

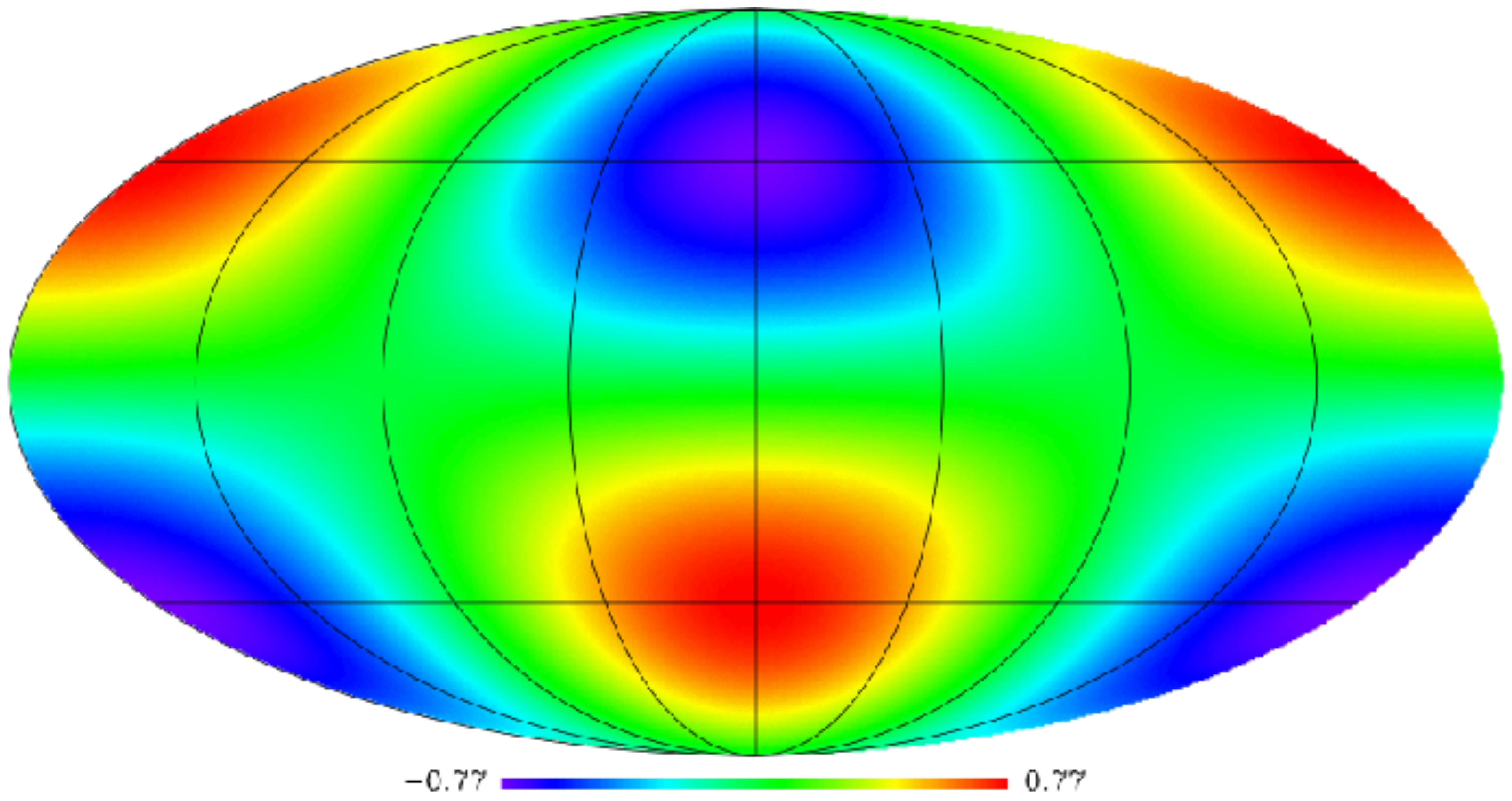
$$a_{11} = 0.3384 - 3.215i \text{ mK str}^{1/2},$$

$$a_{1-1} = -a_{11}^*$$

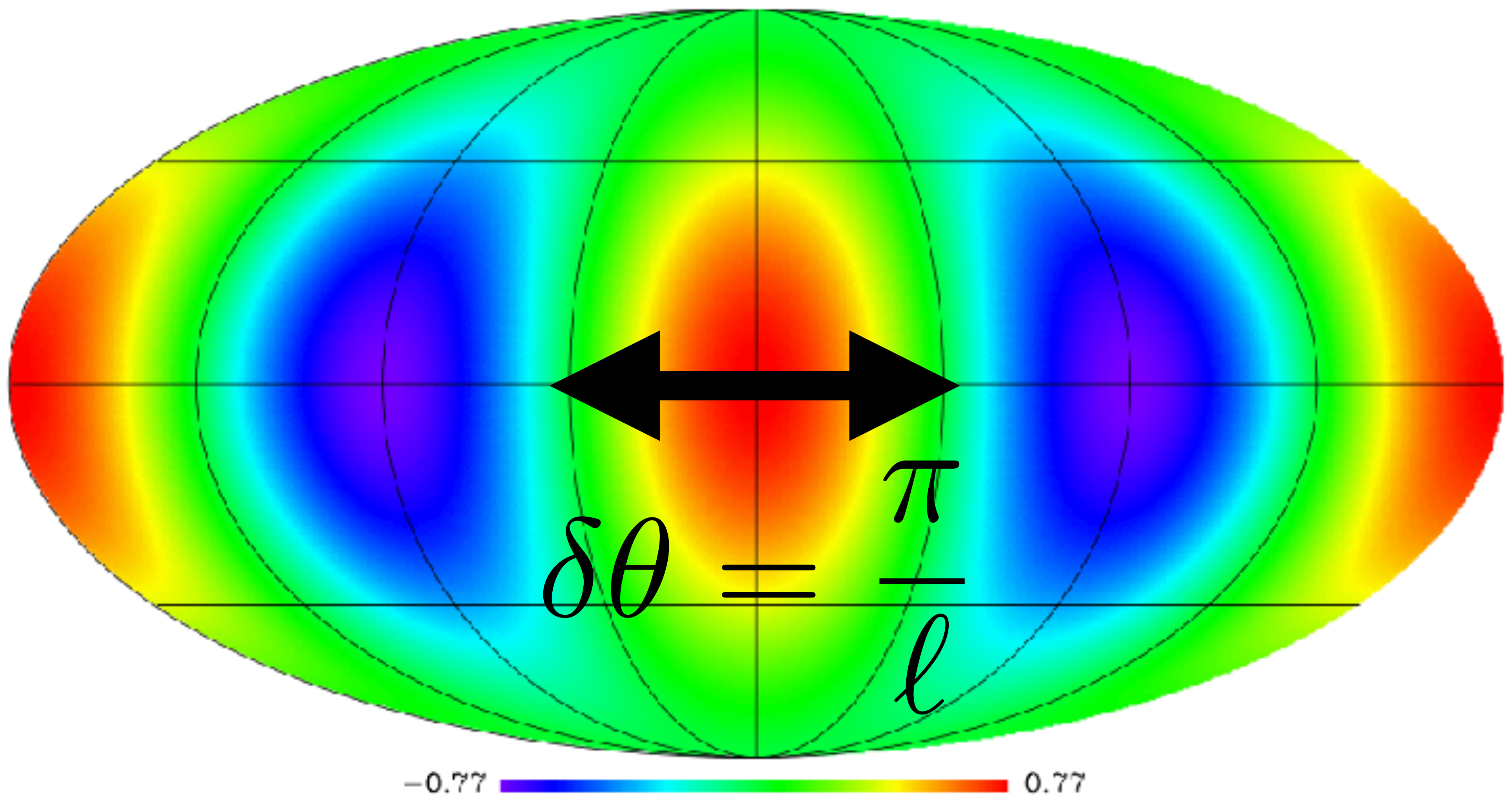
(l,m)=(2,0)



(l,m)=(2,1)



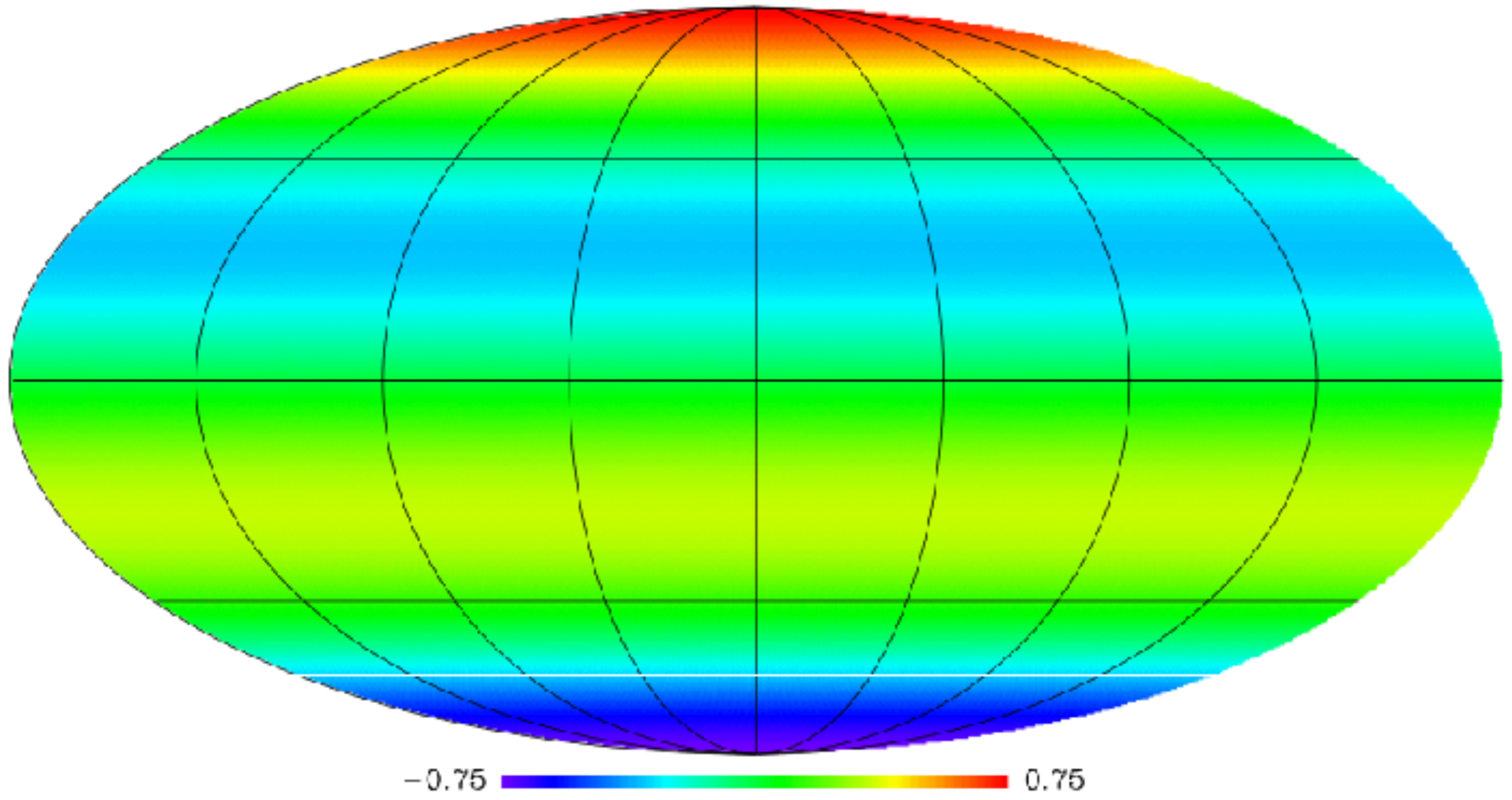
(l,m)=(2,2)



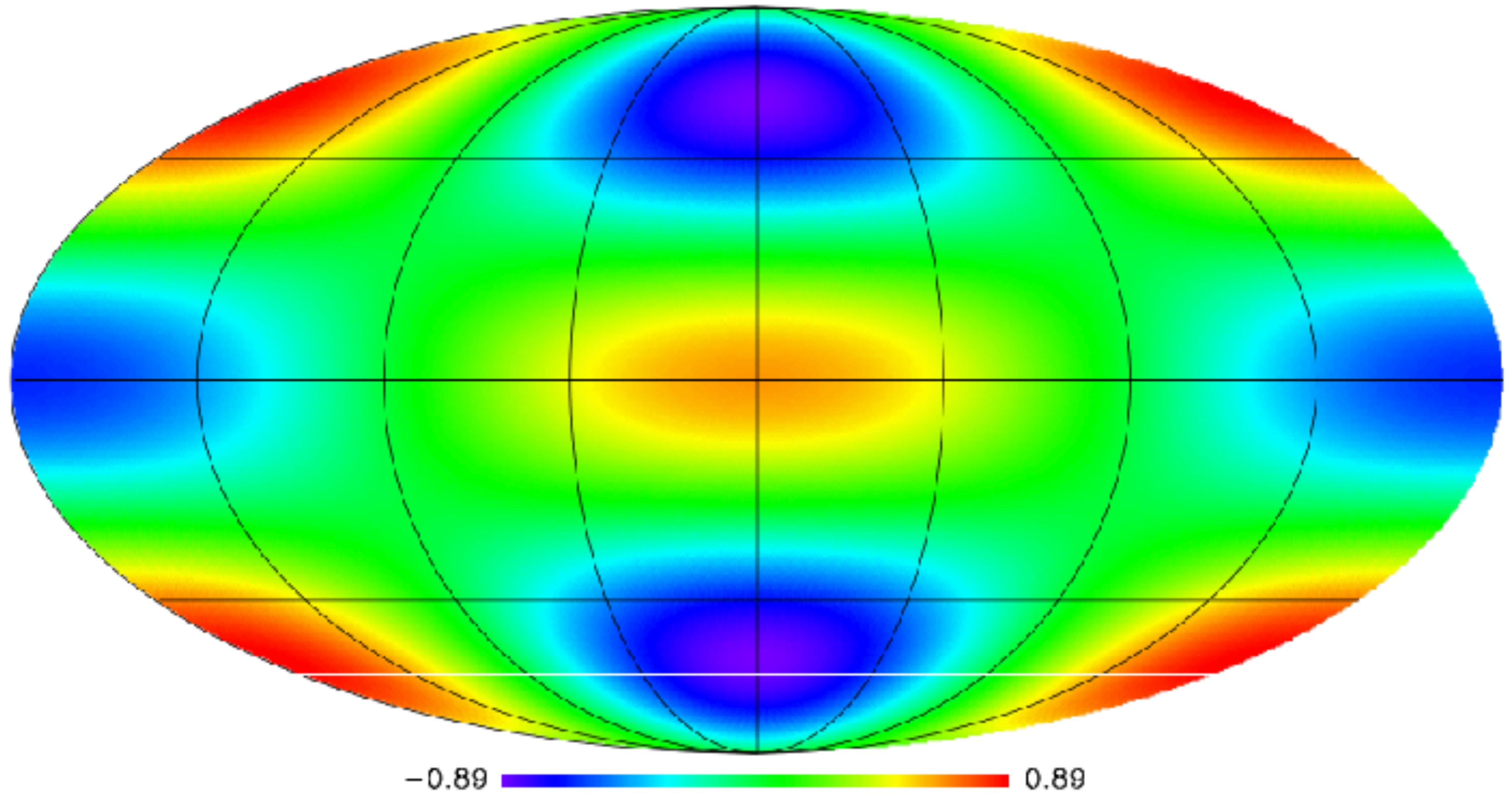
For l=m, a half-wavelength, $\lambda_\theta/2$, corresponds to π/l.

Therefore, $\lambda_\theta=2\pi/l$

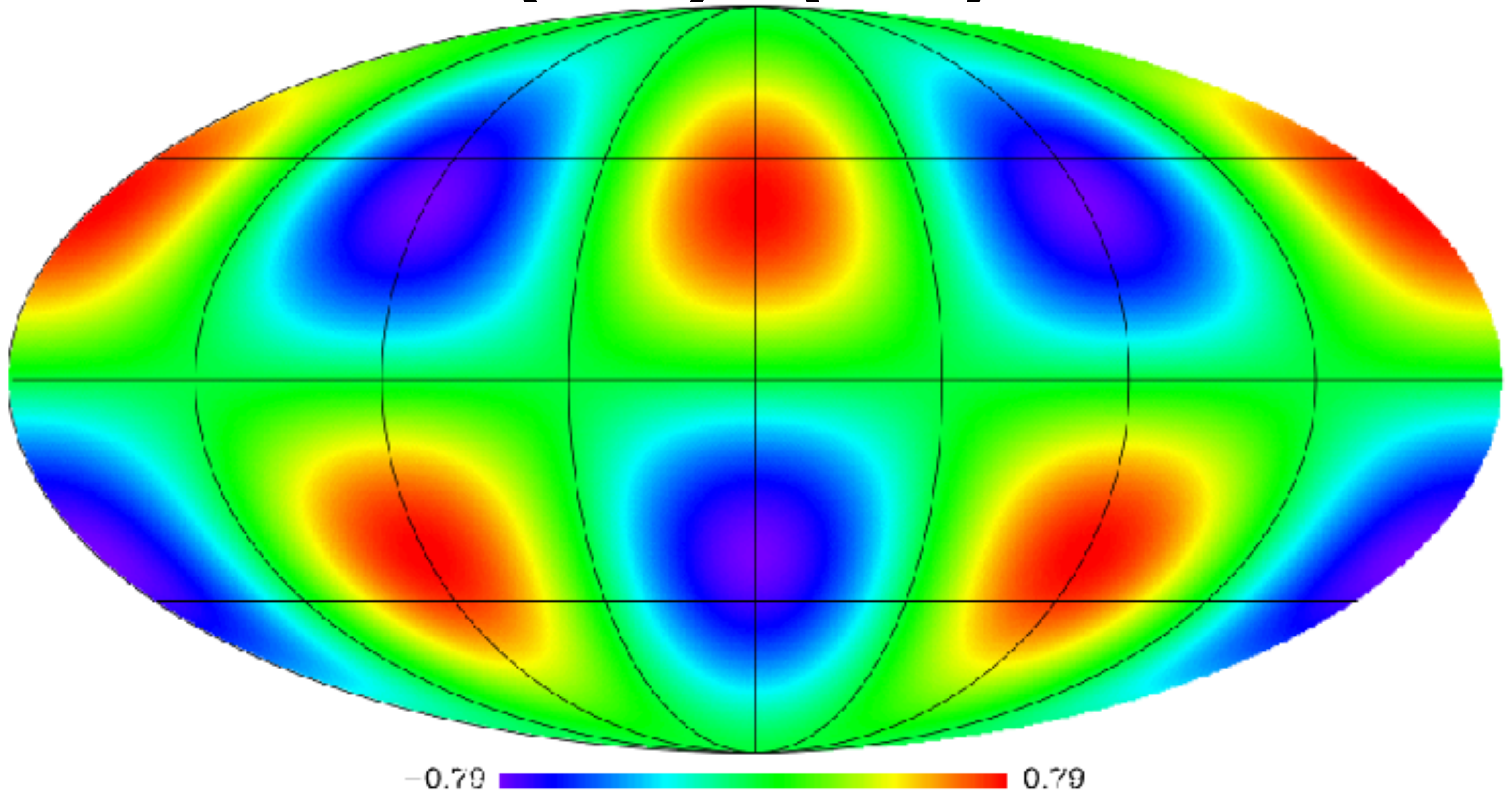
(l,m)=(3,0)



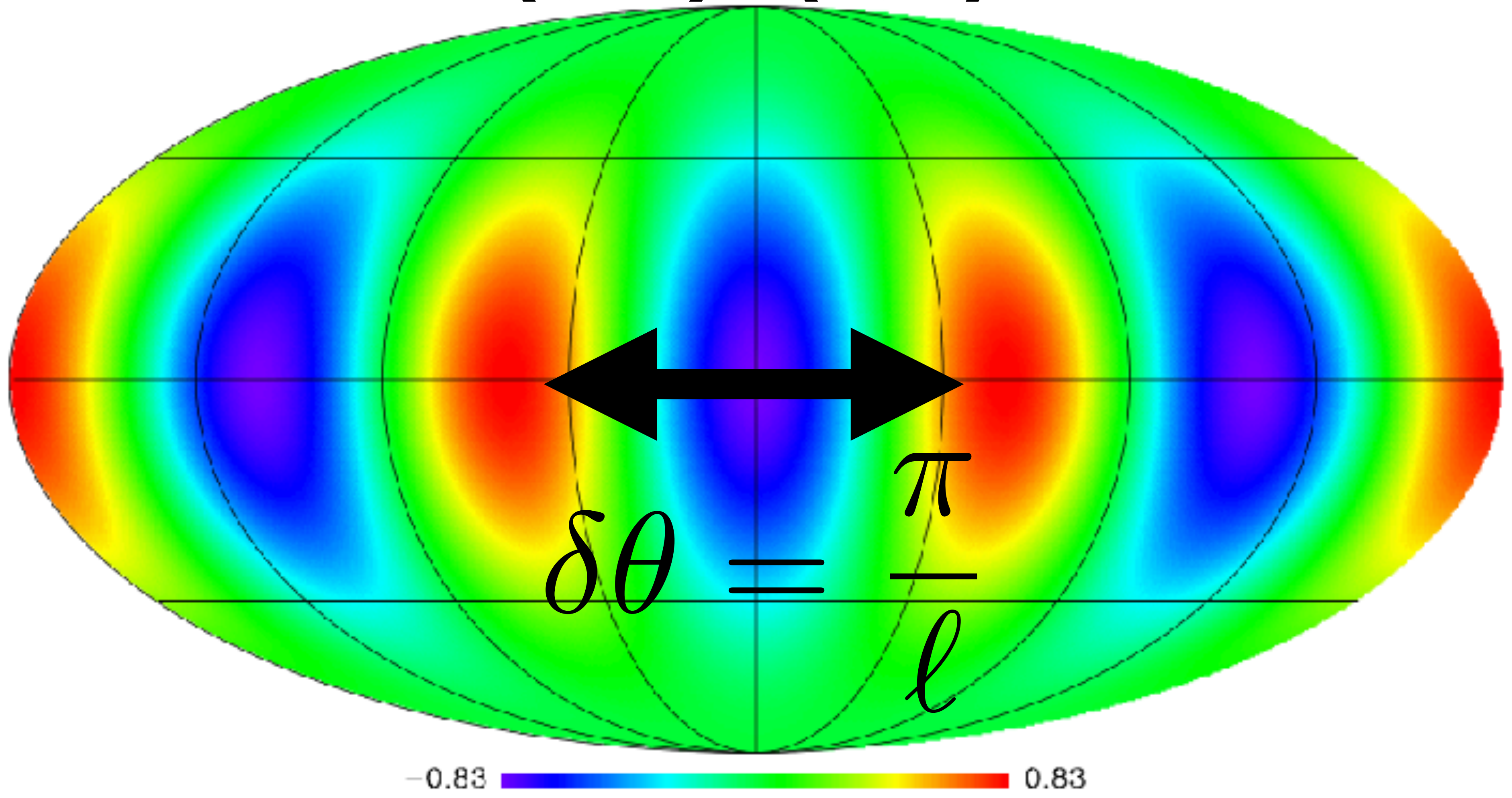
(l,m)=(3,1)

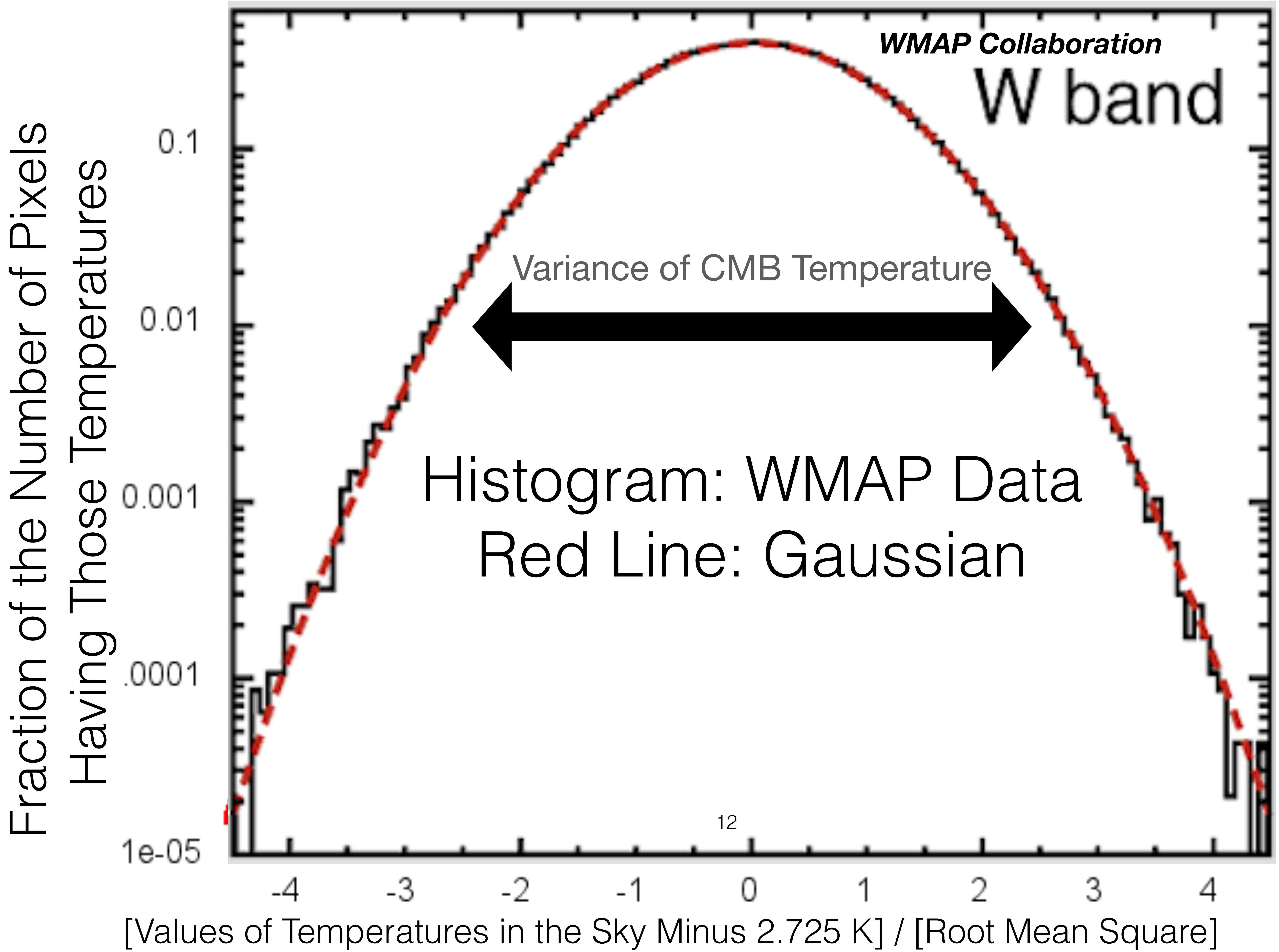


(l,m)=(3,2)



(l,m)=(3,3)




WMAP Collaboration
W band
Variance of CMB Temperature
Histogram: WMAP Data
Red Line: Gaussian
0.1
0.01
0.001
.0001
1e-05
-4
-3
-2
-1
0
1
2
3
4
Fraction of the Number of Pixels
Having Those Temperatures
[Values of Temperatures in the Sky Minus 2.725 K] / [Root Mean Square]

Angular Power Spectrum

- The angular power spectrum, C_l, quantifies how much correlation power we have at a given angular separation.

$$C_\ell \equiv \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} a_{\ell m}a^*_{\ell m}$$

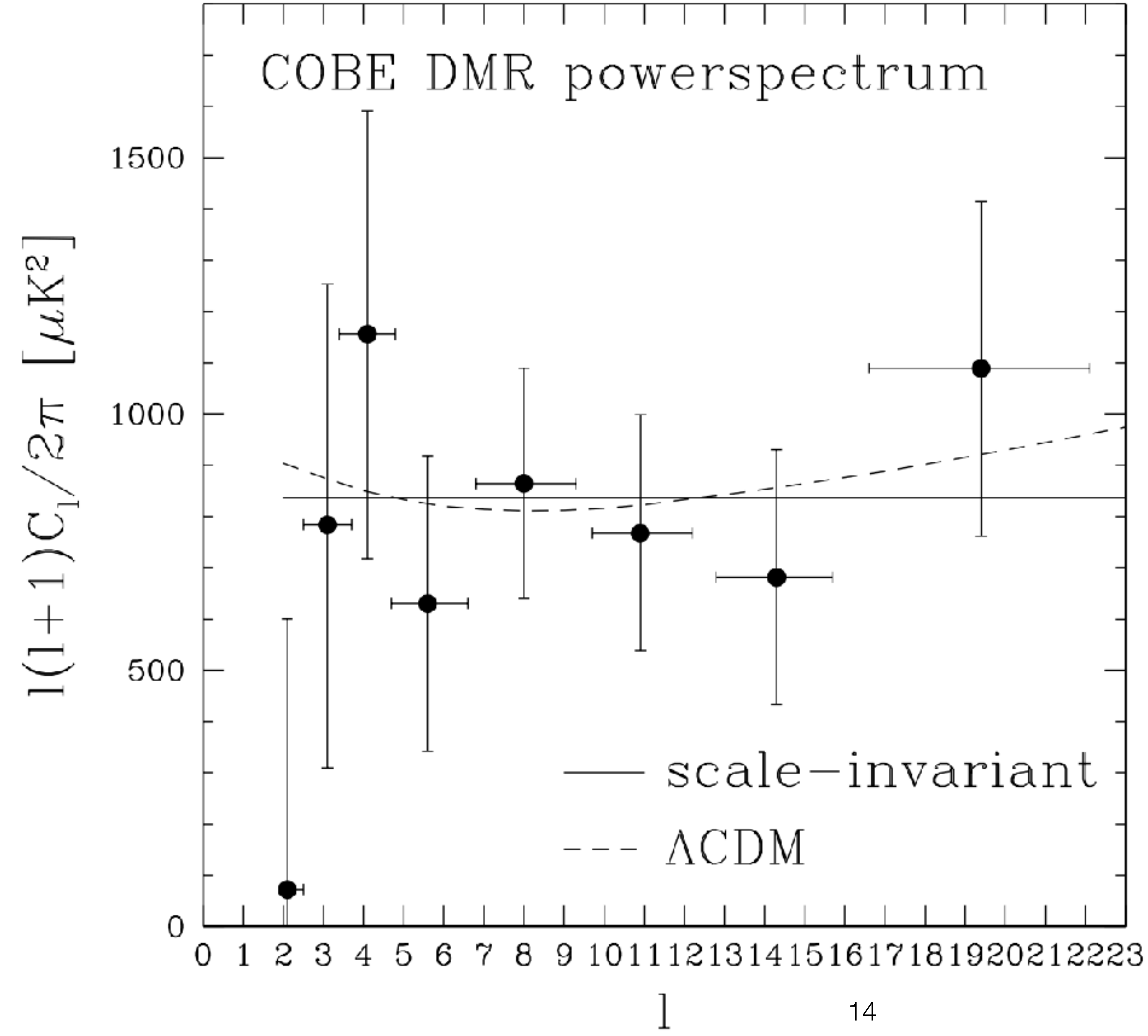
- Values of a_{lm} depend on coordinates, but the squared amplitude, $\sum_{m=-\ell}^{\ell} a_{\ell m}a^*_{\ell m}$, does not depend on coordinates

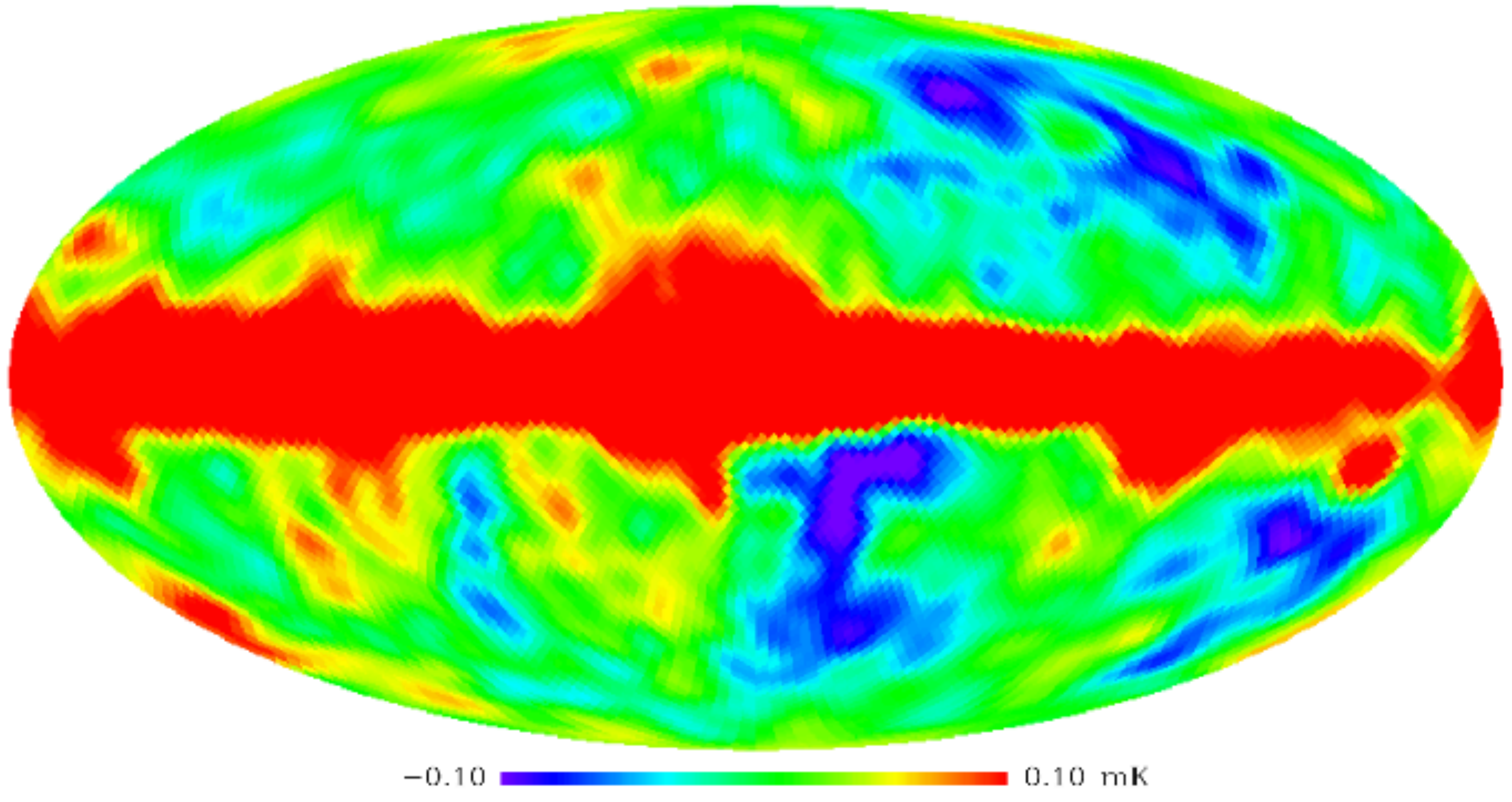
- More precisely: it is **$l(2l+1)C_l/4\pi$** that gives the fluctuation power at a given angular separation, ~π/l. We can see this by computing **variance**:

$$\int \frac{d\Omega}{4\pi}\Delta T^2(\hat{n}) = \frac{1}{4\pi}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}a^*_{\ell m} = \sum_{\ell=2}^{\infty}\frac{2\ell+1}{4\pi}C_\ell$$

Bennett et al. (1996)

COBE 4-year Power Spectrum





What physics can we learn from this measurement?

Φ!!

Gravitational Potential in 3D to Temperature in 2D

More generally: *How is a plane wave in 3D projected on the sky?*

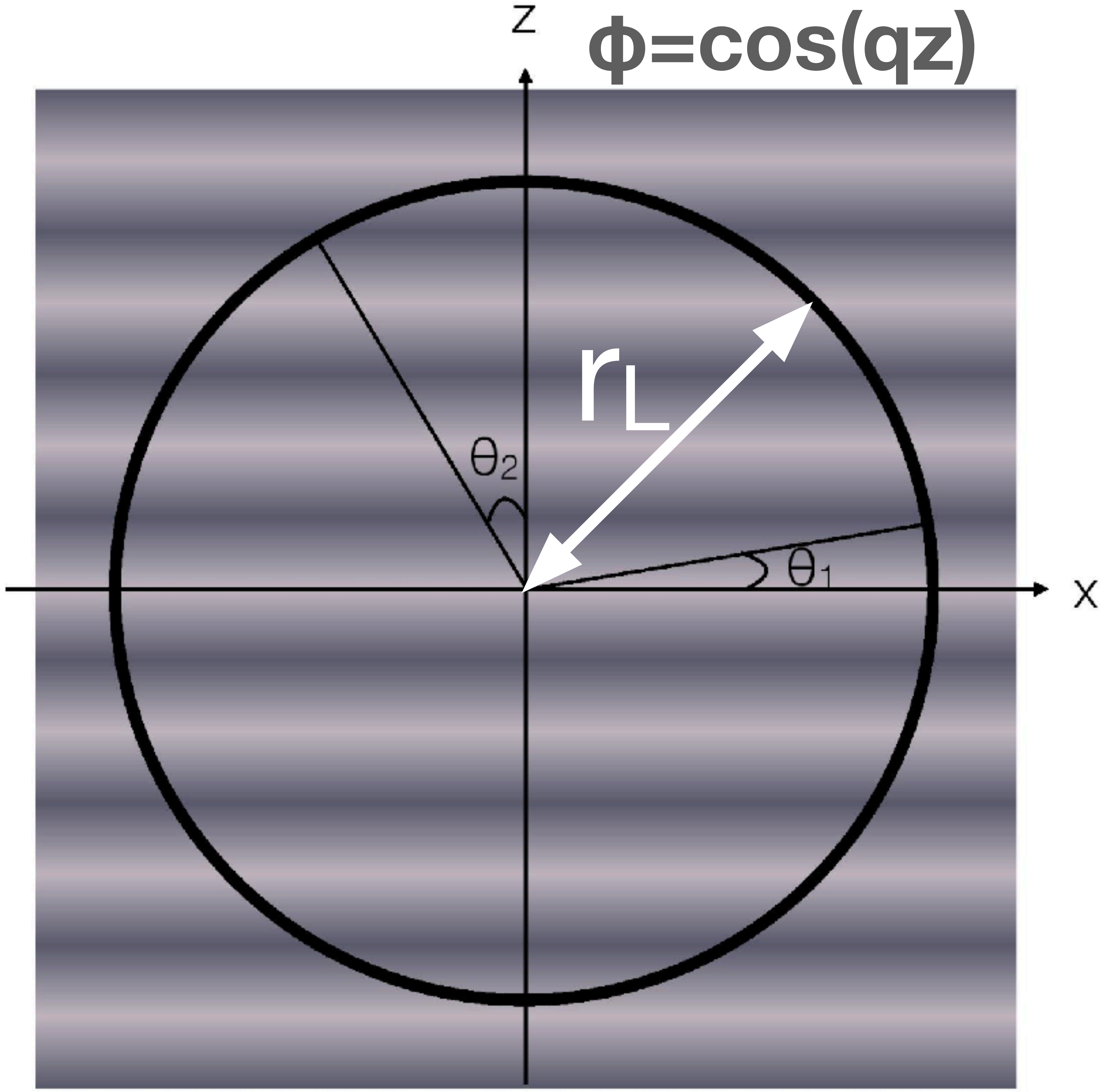
Let's use the Sachs-Wolfe formula for the adiabatic initial condition:

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{n} r_L)$$

- Take a single plane wave for the potential, going in the z direction:

$$\Phi(t_L, \mathbf{x}) \propto A(t_L)\cos(qz)$$

- $A(t_L)$: Amplitude
- q: Wavenumber in 3D



Gravitational Potential in 3D to Temperature in 2D

More generally: *How is a plane wave in 3D projected on the sky?*

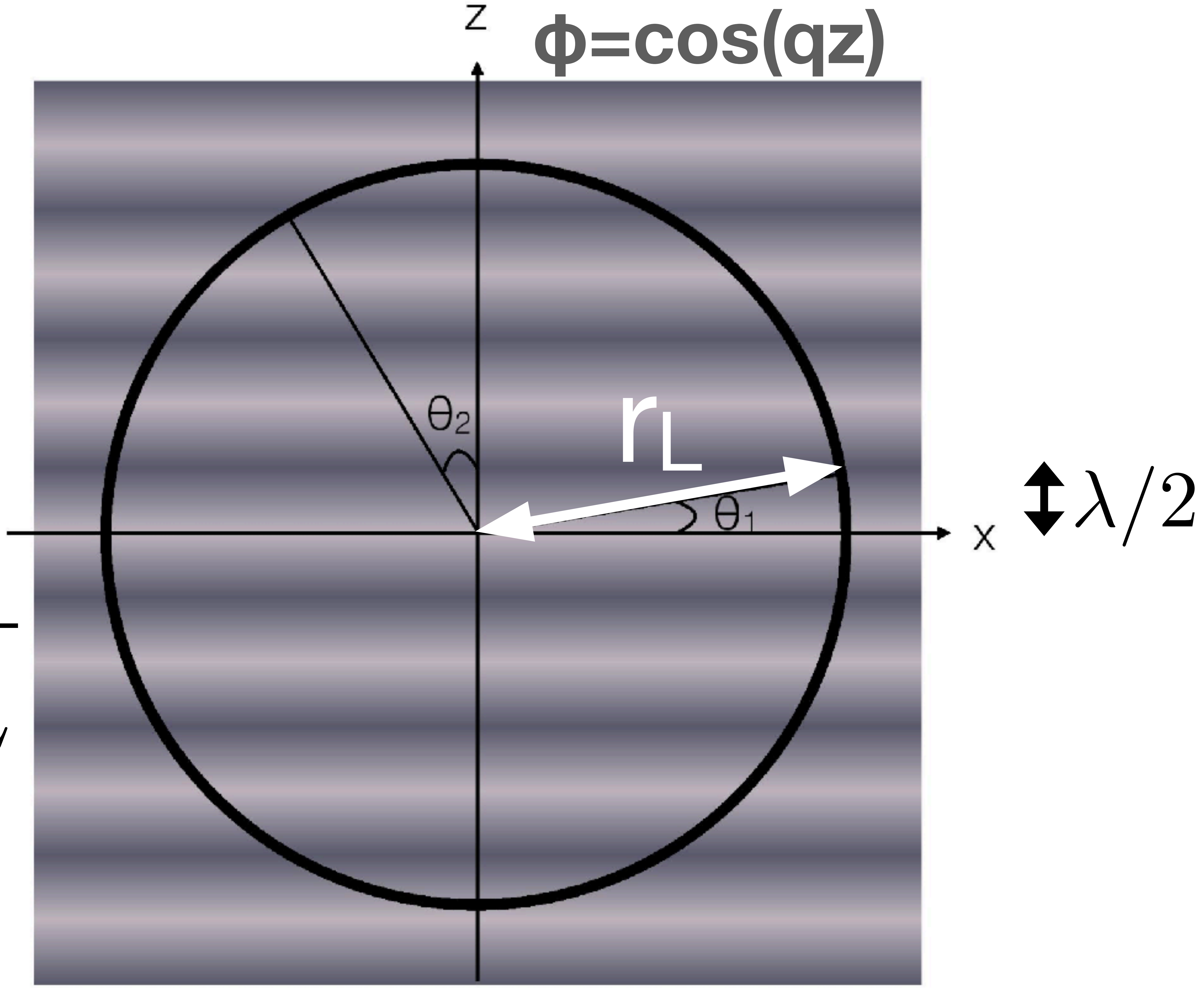
In the x-axis, the angle θ_1 subtends the half wavelength $\lambda/2$, with

$$\lambda = 2\pi/q$$

With trigonometry, we find

$$\tan\theta_1 \simeq \theta_1 = \frac{\lambda/2}{r_L} = \frac{\pi}{q r_L}$$

$$\ell_1 \approx \frac{\pi}{\theta_1} = q r_L$$



Gravitational Potential in 3D to Temperature in 2D

More generally: *How is a plane wave in 3D projected on the sky?*

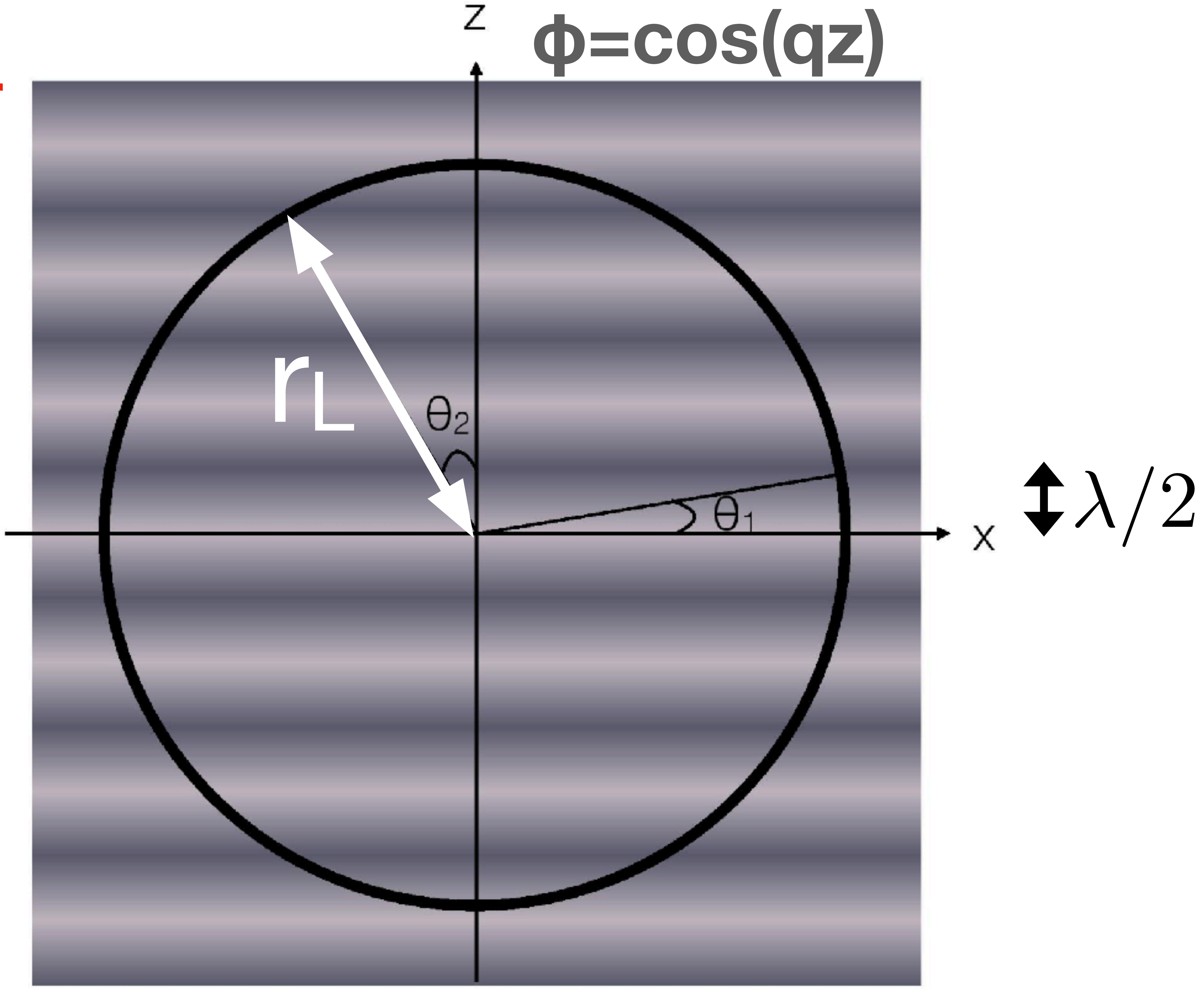
In the z-axis, the angle θ_2 is subtends **bigger than** the half wavelength $\lambda/2$, with

$$\lambda = 2\pi/q$$

With trigonometry, we find

$$\tan\theta_2 \simeq \theta_2 > \frac{\lambda/2}{r_L} = \frac{\pi}{q r_L}$$

$$\ell_2 \approx \frac{\pi}{\theta_2} < q r_L$$



How do we understand the relationship between the 3D wavenumber of the gravitational potential, Φ, and the 2D wavenumber of the temperature anisotropy, l?

$$\Phi(t_L, \mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} \Phi_{\mathbf{q}}(t_L) \exp(i\mathbf{q} \cdot \mathbf{x})$$

?

$$\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n})$$

t_L: the time at the last scattering surface

Part II: Flat-sky (Small-angle) Approximation

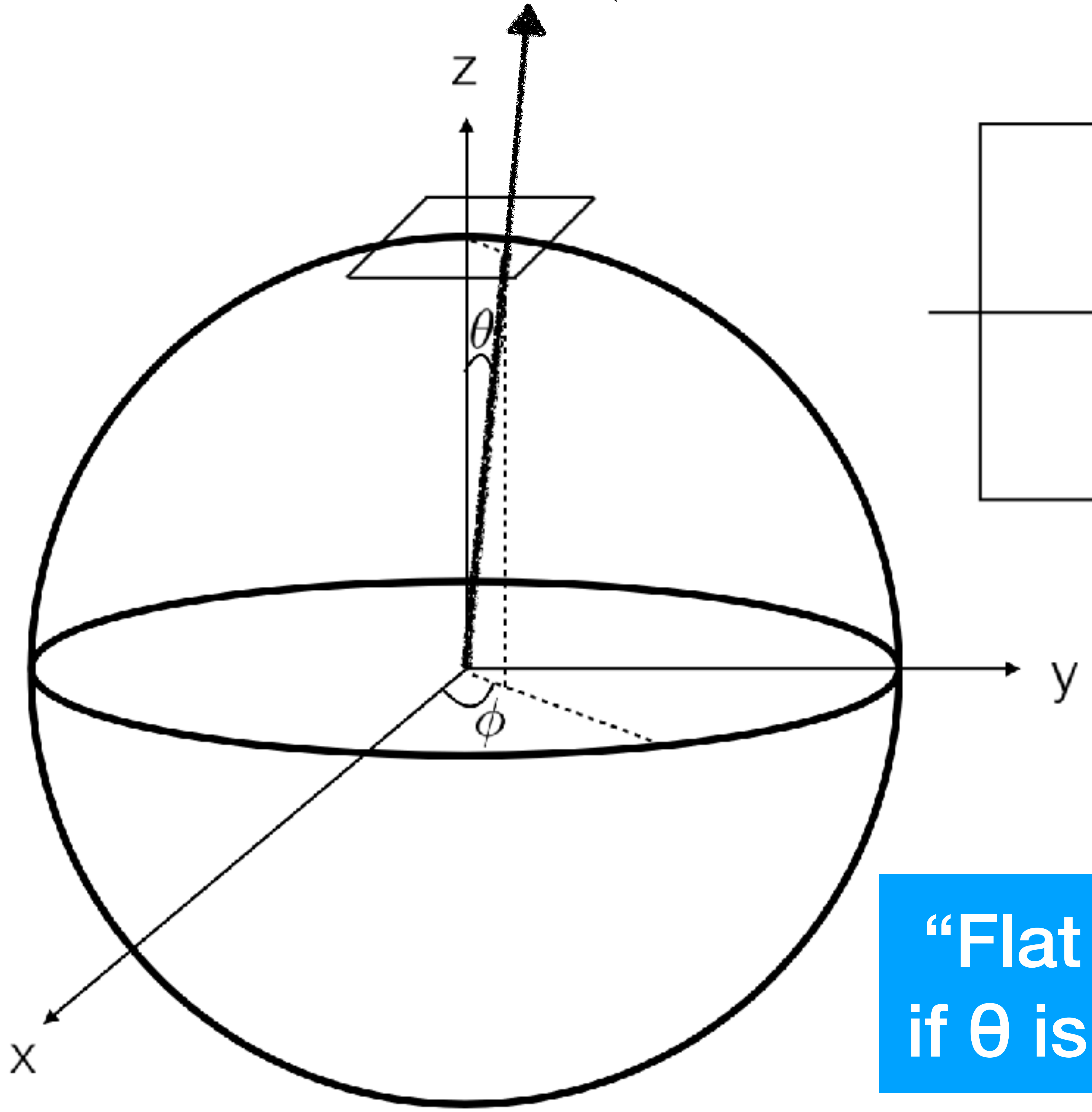
Fourier transform?

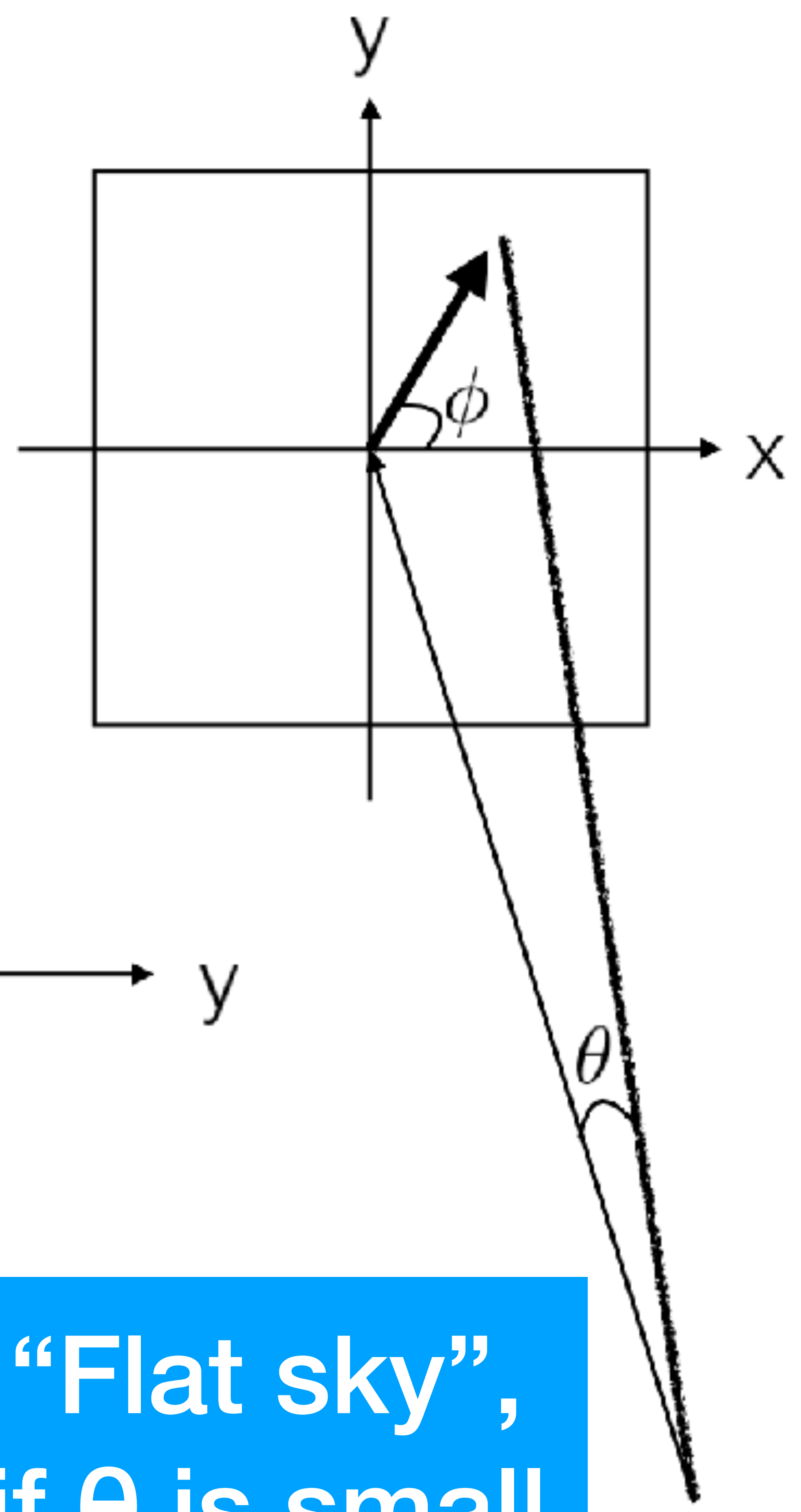
- The simplest way to decompose fluctuations into waves is Fourier transform.
  - However, Fourier transform works only for plane waves in flat space.
- The sky is a sphere. How do we decompose fluctuations on a sphere into waves?
  - The answer: **Spherical Harmonics**.
- But, this seems too complicated for understanding the relationship between the gravitational potential in 3D and the temperature anisotropy in 2D (i.e., sky).
- Alternative (approximate) approach?

Fourier transform!

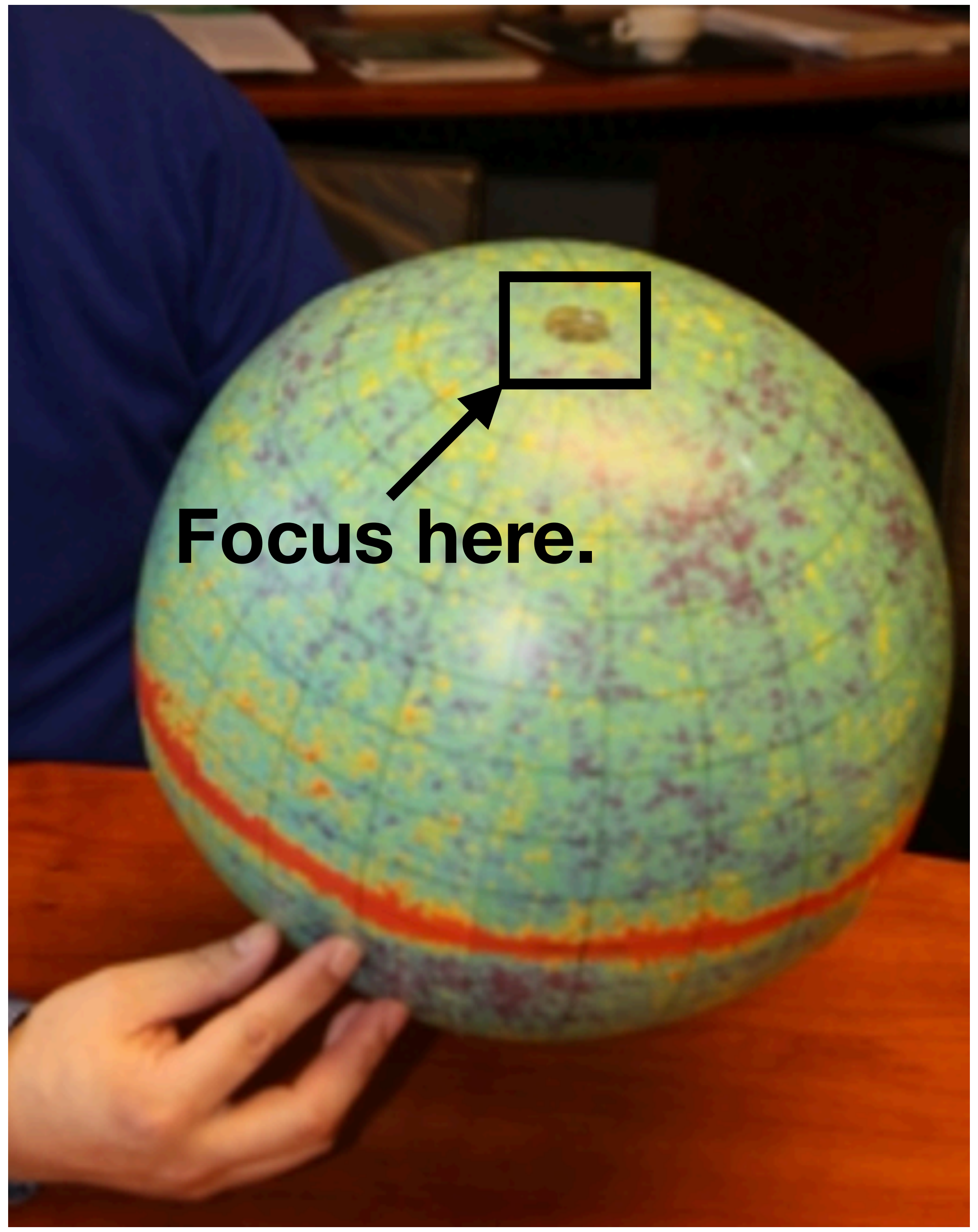
Approximately correct in a small region in the sky

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$





“Flat sky”, if θ is small



- Take z-axis to anywhere we want in the sky. Then, treat a small area around the z-axis as a “flat sky”.
- We then apply the usual 2D Fourier transform to analyse temperature fluctuations, and relate it to the 3D Fourier transform of the potential Φ.

2D Fourier Transform

$$\Delta T(\hat{n}) = \int \frac{d^2\ell}{(2\pi)^2}\; a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

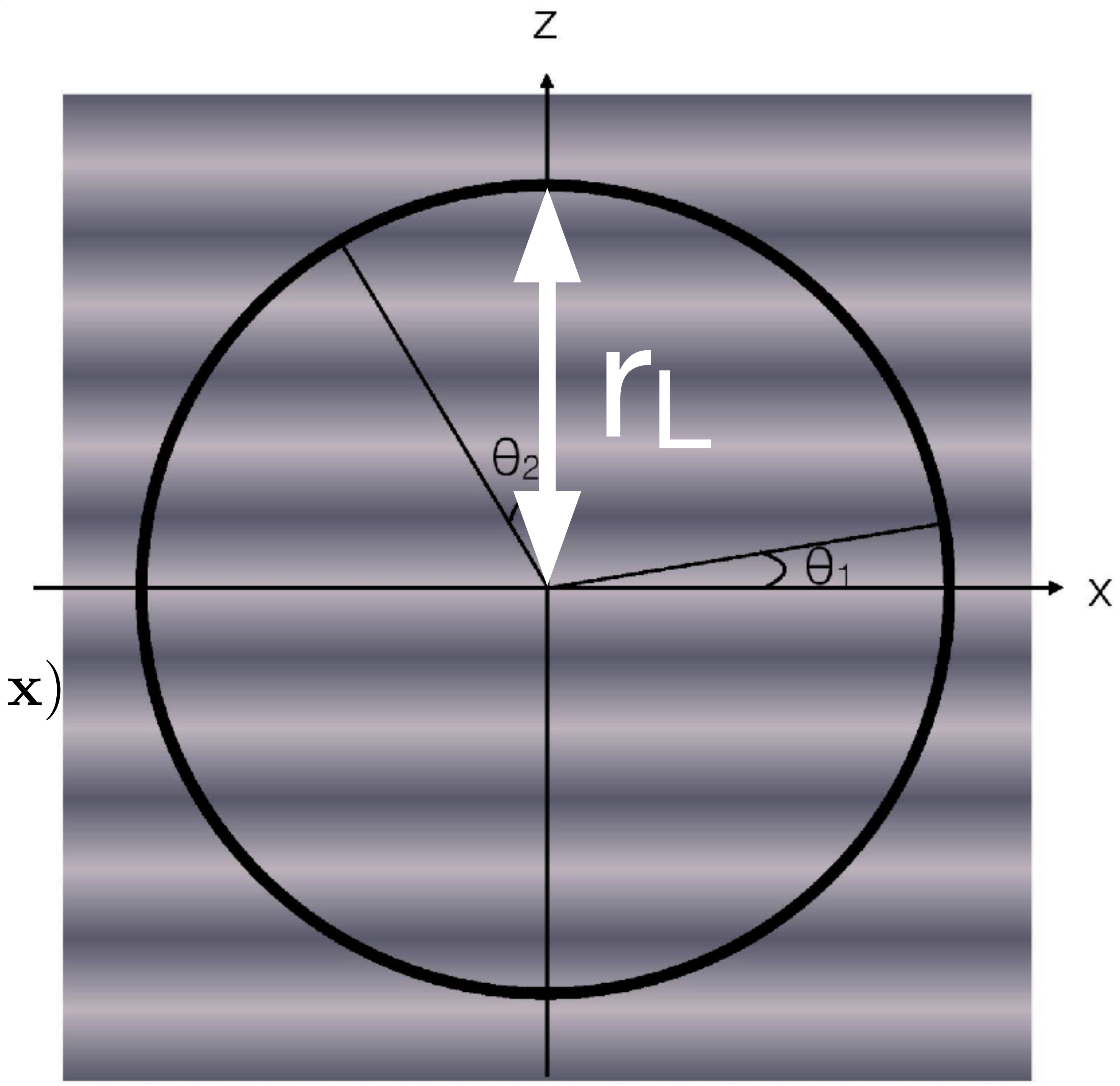
$$= \int_0^\infty \frac{\ell d\ell}{2\pi} \int_0^{2\pi} \frac{d\phi_\ell}{2\pi}\; a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

C.f.,

$$\left(\Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n}) \right)$$

a(l) of the Sachs-Wolfe effect

- Take the inverse 2D Fourier transform of the Sachs-Wolfe formula for the adiabatic initial condition:



$$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{n} r_L)$$

- And Fourier transform Φ in 3D: $\Phi(t_L, \mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} \Phi_{\mathbf{q}}(t_L) \exp(i\mathbf{q}\cdot\mathbf{x})$

$$a_{\boldsymbol{\ell}}^{\mathrm{SW}} = \frac{T_0}{3} \int d^2\theta \; \exp(-i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

$$\times \int \frac{d^3q}{(2\pi)^3} \, \Phi_{\boldsymbol{q}} \; \exp(i\boldsymbol{q}_\perp r_L \cdot \boldsymbol{\theta} + i q_{||} r_L \cos\theta)$$

***q is the 3D Fourier wavenumber**

→ 1 **[flat-sky approximation]**

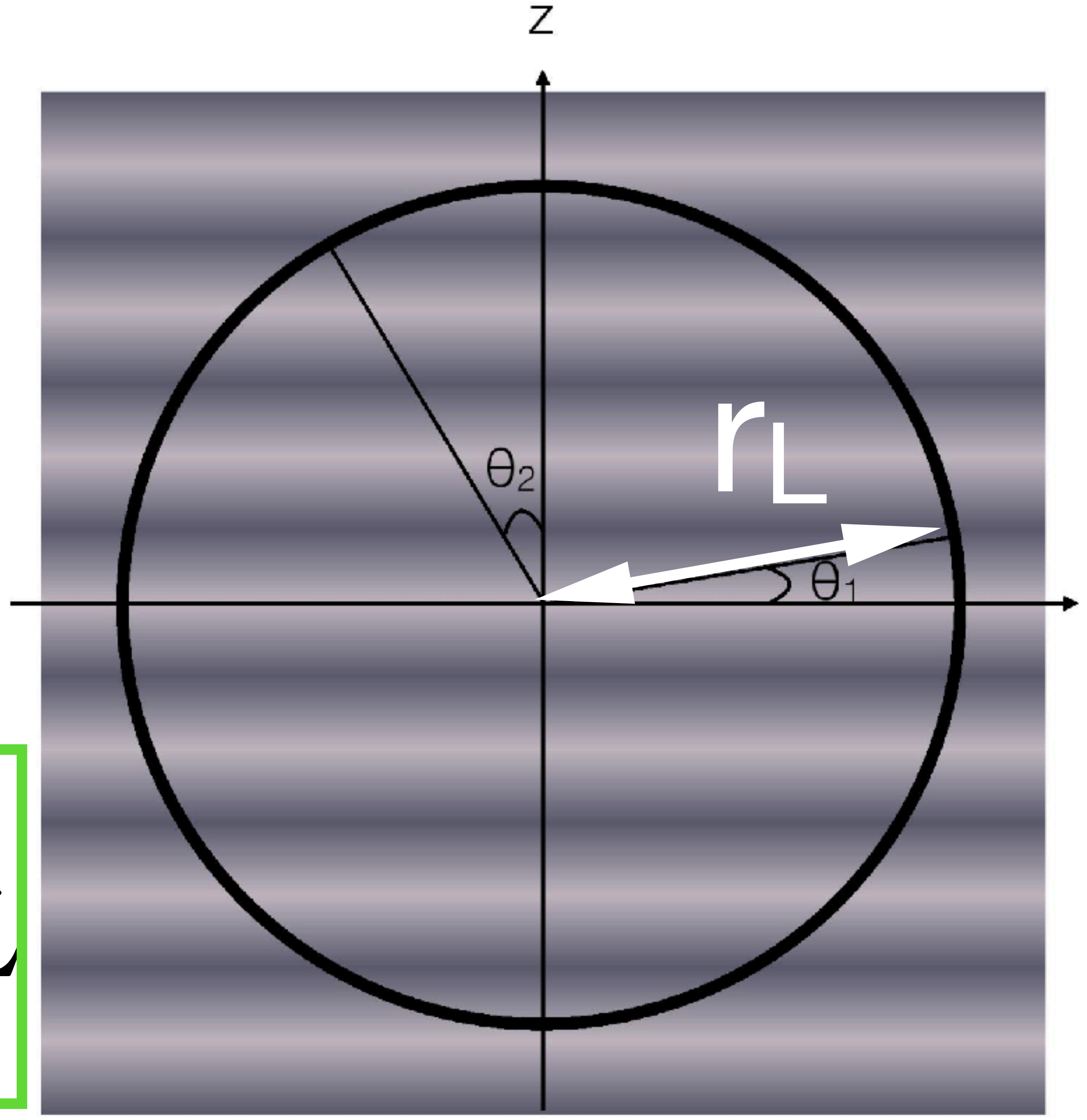
Flat-sky Result

$$a_{\boldsymbol{\ell}}^{\rm SW} = \frac{T_0}{3r_L^2}\int_{-\infty}^{\infty}\frac{dq_{\parallel}}{2\pi}\,\Phi_{\boldsymbol{q}}\left(\boldsymbol{q}_{\perp} = \frac{\boldsymbol{\ell}}{r_L}, q_{\parallel}\right)\exp(iq_{\parallel}r_L)$$

$q = \sqrt{\ell^2/r_L^2 + q_{\parallel}^2}$ **i.e.,** $q \geq \ell/r_L$

- It is **now manifest** that only the perpendicular wavenumber contributes to l, i.e., **$l=q_{perp}r_L$**, giving $l<qr_L$

$$\ell_1 \approx \frac{\pi}{\theta_1} = qr_L$$



Flat-sky Result

$$a_{\boldsymbol{\ell}}^{\rm SW} = \frac{T_0}{3r_L^2}\int_{-\infty}^{\infty}\frac{dq_\parallel}{2\pi}\,\Phi_{\boldsymbol{q}}\left(\boldsymbol{q}_\perp = \frac{\boldsymbol{\ell}}{r_L}, q_\parallel\right)\exp(iq_\parallel r_L)$$

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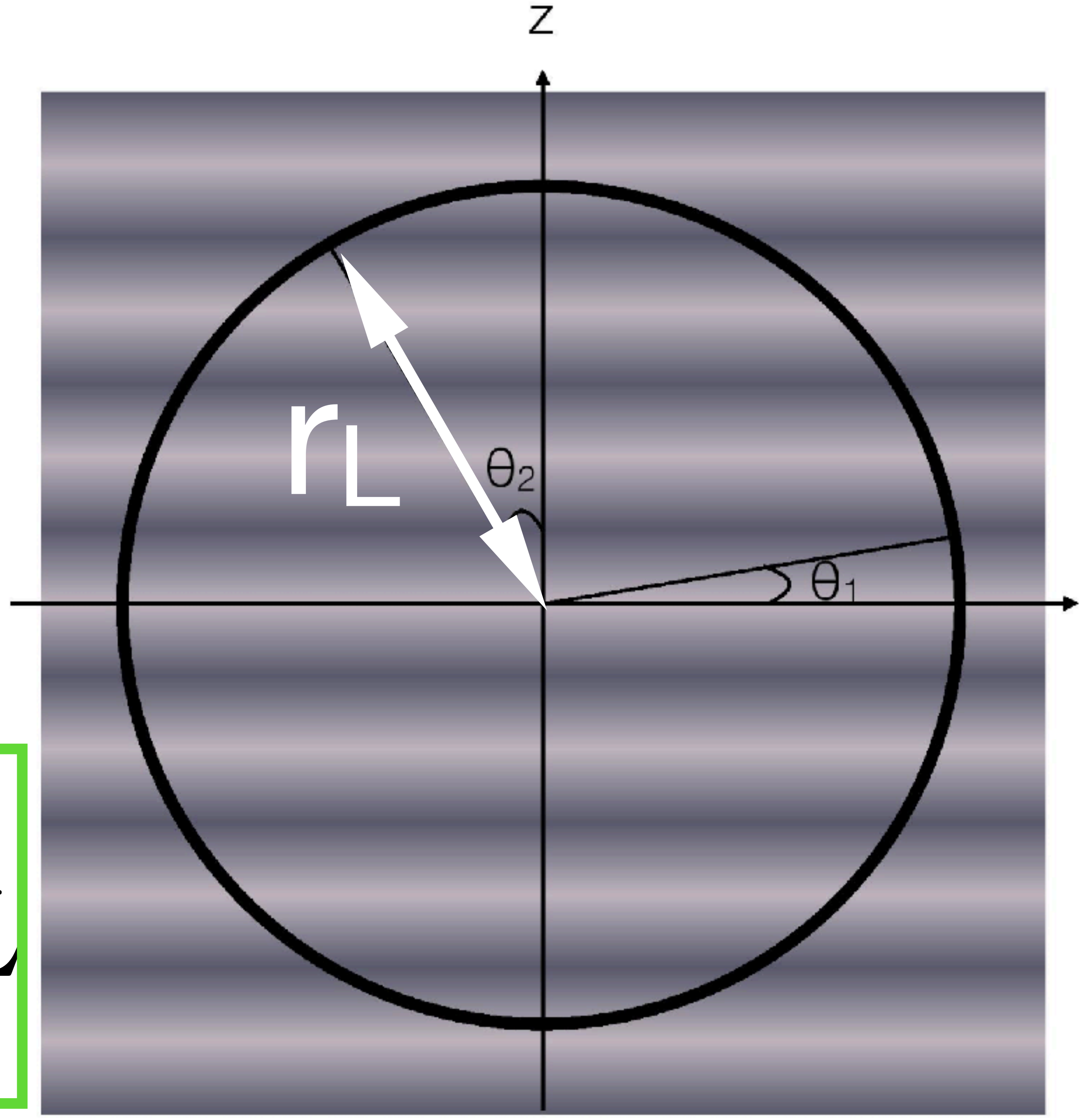
- It is **now manifest** that only the perpendicular wavenumber contributes to l, i.e., **$l = q_{perp} r_L$**, giving $l < qr_L$

$$\ell_2 \approx \frac{\pi}{\theta_2} < qr_L$$



The relationship between q and l Understood? Let's go to the full sky treatment.

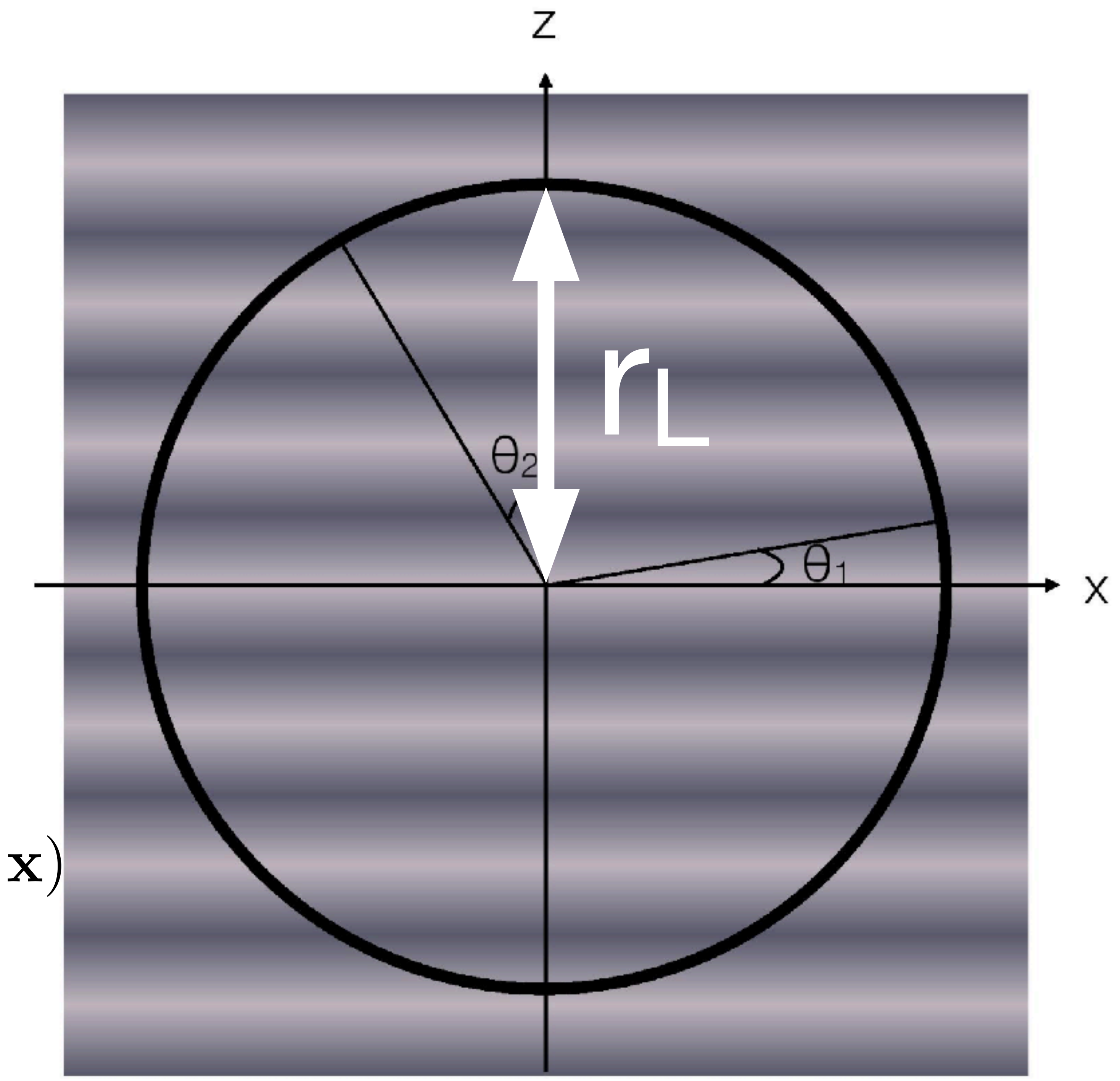
$$\Delta T(\hat{n}) = \int \frac{d^2\ell}{(2\pi)^2} \, a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell} \cdot \boldsymbol{\theta}) \quad \Rightarrow \quad \Delta T(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\hat{n})$$

a_{lm} of the Sachs-Wolfe effect

- Take the inverse spherical harmonics transform

$$a_{\ell m} = \int d\Omega \Delta T(\hat{n}) Y_\ell^{m*}(\hat{n})$$

of the Sachs-Wolfe formula for the adiabatic initial condition: $\dfrac{\Delta T(\hat{n})}{T_0} = \dfrac{1}{3}\Phi(t_L, \hat{n} r_L)$



- And Fourier transform Φ in 3D: $\Phi(t_L, \mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} \Phi_\mathbf{q}(t_L) \exp(i\mathbf{q}\cdot\mathbf{x})$

$$a_{\ell m}^{\rm SW} = \frac{T_0}{3} \int d\Omega\, Y_\ell^{m*}(\hat{n}) \int \frac{d^3q}{(2\pi)^3}\, \Phi_{\boldsymbol{q}} \exp(i\boldsymbol{q}\cdot\hat{n} r_L)$$

***q is the 3D Fourier wavenumber**

Spherical wave decomposition of a plane wave

- How to obtain a plane wave by combining spherical waves? The answer is

$$\exp(i\boldsymbol{q}\cdot\hat{n}r_L) = 4\pi\sum_{\ell=0}^{\infty} i^\ell j_\ell(qr_L)\sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{n})Y_\ell^{m*}(\hat{q})$$

- which is called the “partial wave decomposition” or “Rayleigh’s formula”. Then we obtain

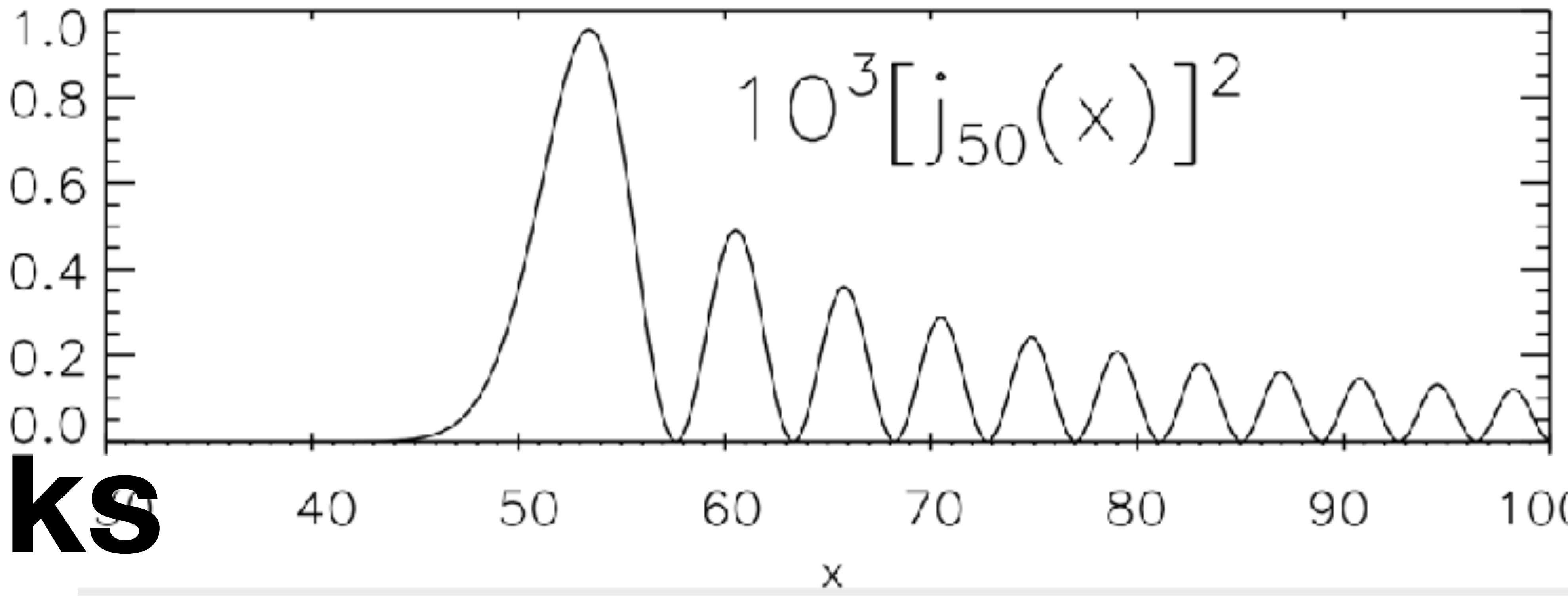
$$a_{\ell m}^{\mathrm{SW}} = \frac{4\pi T_0 i^\ell}{3}\int\frac{d^3q}{(2\pi)^3}\,\Phi_{\boldsymbol{q}} j_\ell(qr_L)Y_\ell^{m*}(\hat{q})$$

- This is the *exact* formula relating Φ in 3D at the last scattering surface to a_{lm}. **How do we understand this?**

q -> l projection

$$a_{\ell m}^{\rm SW} = \frac{4\pi T_0 i^\ell}{3} \int \frac{d^3q}{(2\pi)^3}\, \Phi_{\mathbf q}\, j_\ell(qr_L) Y_\ell^{m*}(\hat q)$$

- A half wavelength, λ/2, at the last scattering surface subtends an angle of $\lambda/2r_L$. Since $q=2\pi/\lambda$, the angle is given by $\delta\theta=\pi/qr_L$. Comparing this with the relation $\delta\theta=\pi/l$, we obtain $l=qr_L$. How can we see this?



- For $l\gg1$, the spherical Bessel function, **$j_l(qr_L)$, peaks at $l\sim qr_L$** and falls gradually toward $qr_L>l$. Thus, a given q mode contributes to large angular scales too.

We learned this already from the flat-sky approximation!

Part III: Power Spectrum of the Sachs-Wolfe Effect

Let's compute the temperature power spectrum

Temperature Cl

- We use $$a_{\ell m}^{\rm SW} = \frac{4\pi T_0 i^\ell}{3} \int \frac{d^3q}{(2\pi)^3}\, \Phi_{\boldsymbol{q}} j_\ell(qr_L) Y_\ell^{m*}(\hat{q})$$

  to compute $$C_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^*$$

Result

The power spectrum of the Sachs-Wolfe effect?

- We use $$a_{\ell m}^{\mathrm{SW}} = \frac{4\pi T_0 i^\ell}{3} \int \frac{d^3 q}{(2\pi)^3} \, \Phi_{\boldsymbol{q}} j_\ell(q r_L) Y_\ell^{m*}(\hat{q})$$

to compute $$C_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell m}^*$$

$$C_{\ell,\mathrm{SW}} = \frac{4\pi T_0^2}{9} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 q'}{(2\pi)^3} \, \Phi_{\boldsymbol{q}} \Phi_{\boldsymbol{q}'}^* j_\ell(q r_L) j_\ell(q' r_L) P_\ell(\hat{q} \cdot \hat{q}')$$

- But this is not exactly what we want. We want the **statistical average** of this quantity.

Power Spectrum of ϕ

- Statistical average of the right hand side contains

$$\langle \Phi_{\boldsymbol{q}} \Phi^*_{\boldsymbol{q}'} \rangle = \int d^3x \int d^3r \, \langle \Phi(\boldsymbol{x}) \Phi(\boldsymbol{x}+\boldsymbol{r}) \rangle \exp\left[i(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{x} - i\boldsymbol{q}'\cdot\boldsymbol{r} \right]$$

two-point correlation function

If $\langle \Phi(\boldsymbol{x}) \Phi(\boldsymbol{x}+\boldsymbol{r}) \rangle$ **does not depend on locations (x) but only on separations between two points (r), then**

$$\langle \Phi_{\boldsymbol{q}} \Phi^*_{\boldsymbol{q}'} \rangle = (2\pi)^3 \delta_D^{(3)}(\boldsymbol{q}-\boldsymbol{q}') \int d^3r \, \xi_\phi(\boldsymbol{r}) \exp(-i\boldsymbol{q}\cdot\boldsymbol{r})$$

consequence of “statistical homogeneity”

where we defined $\xi_\phi(\boldsymbol{r}) \equiv \langle \Phi(\boldsymbol{x}) \Phi(\boldsymbol{x}+\boldsymbol{r}) \rangle$

and used $\int d^3x \; \exp(i\boldsymbol{q}\cdot\boldsymbol{x}) = (2\pi)^3 \delta_D^{(3)}(\boldsymbol{q})$

Power Spectrum of ϕ

- In addition, if $\xi_\phi(\boldsymbol{r}) \equiv \langle \Phi(\boldsymbol{x})\Phi(\boldsymbol{x}+\boldsymbol{r})\rangle$ depends only on the magnitude of the separation r and not on the directions, then

$$\langle \Phi_{\boldsymbol{q}}\Phi^*_{\boldsymbol{q}'}\rangle = (2\pi)^3\delta_D^{(3)}(\boldsymbol{q}-\boldsymbol{q}')\int 4\pi r^2 dr\ \xi_\phi(r)\frac{\sin(qr)}{qr}$$

$$= (2\pi)^3\delta_D^{(3)}(\boldsymbol{q}-\boldsymbol{q}')\boxed{P_\phi(q)}$$

Power spectrum!

Generic definition of the power spectrum for statistically homogeneous and isotropic fluctuations

The Power Spectrum of the Sachs-Wolfe Effect

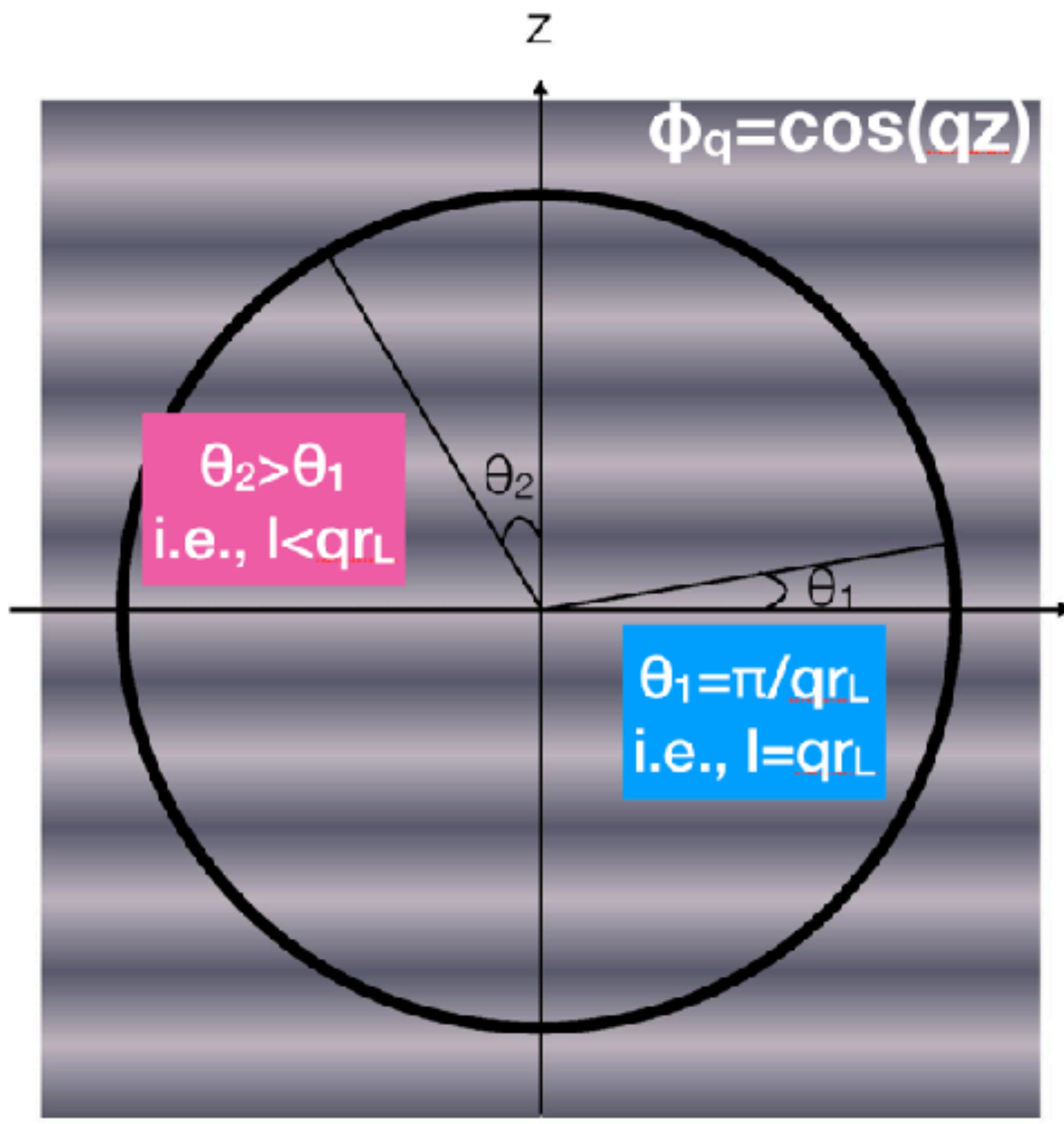
- Thus, the power spectrum of the CMB in the Sachs-Wolfe limit is

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{16\pi^2 T_0^2}{9} \int_0^\infty \frac{q^2 dq}{(2\pi)^3} P_\phi(q) j_\ell^2(q r_L)$$

- In the flat-sky approximation,

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{T_0^2}{9 r_L^2} \int_{-\infty}^\infty \frac{dq_\parallel}{2\pi} P_\phi\left(\sqrt{\frac{\ell^2}{r_L^2} + q_\parallel^2}\right)$$

Perpendicular wavenumber, $(q_{perp})^2$



The Power Spectrum of the Sachs-Wolfe Effect

- Thus, the power spectrum of the CMB in the SW limit is

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{16\pi^2 T_0^2}{9} \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \; P_\phi(q) j_\ell^2(q r_L)$$

- In the flat-sky approximation,

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{T_0^2}{9 r_L^2} \int_{-\infty}^{\infty} \frac{dq_\parallel}{2\pi} \; P_\phi \left(\sqrt{\frac{\ell^2}{r_L^2} + q_\parallel^2} \right)$$

For a power-law form, $P_\phi(q) = (2\pi)^3 N_\phi^2 q^{n-4}$ **, we get**

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{8\pi^2 N_\phi^2 T_0^2}{9\ell^2} \left(\frac{\ell}{r_L} \right)^{n-1} \frac{\sqrt{\pi}}{2} \frac{\Gamma[(3-n)/2]}{\Gamma[(4-n)/2]}$$

The Power Spectrum of the Sachs-Wolfe Effect

- Thus, the power spectrum of the CMB in the SW limit is

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{16\pi^2 T_0^2}{9} \int_0^\infty \frac{q^2 dq}{(2\pi)^3} P_\phi(q) j_\ell^2(q r_L)$$

- In the flat-sky approximation,

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{T_0^2}{9 r_L^2} \int_{-\infty}^\infty \frac{dq_\parallel}{2\pi} P_\phi\left(\sqrt{\frac{\ell^2}{r_L^2} + q_\parallel^2}\right)$$

For a power-law form, $P_\phi(q) = (2\pi)^3 N_\phi^2 q^{n-4}$ **, we get**

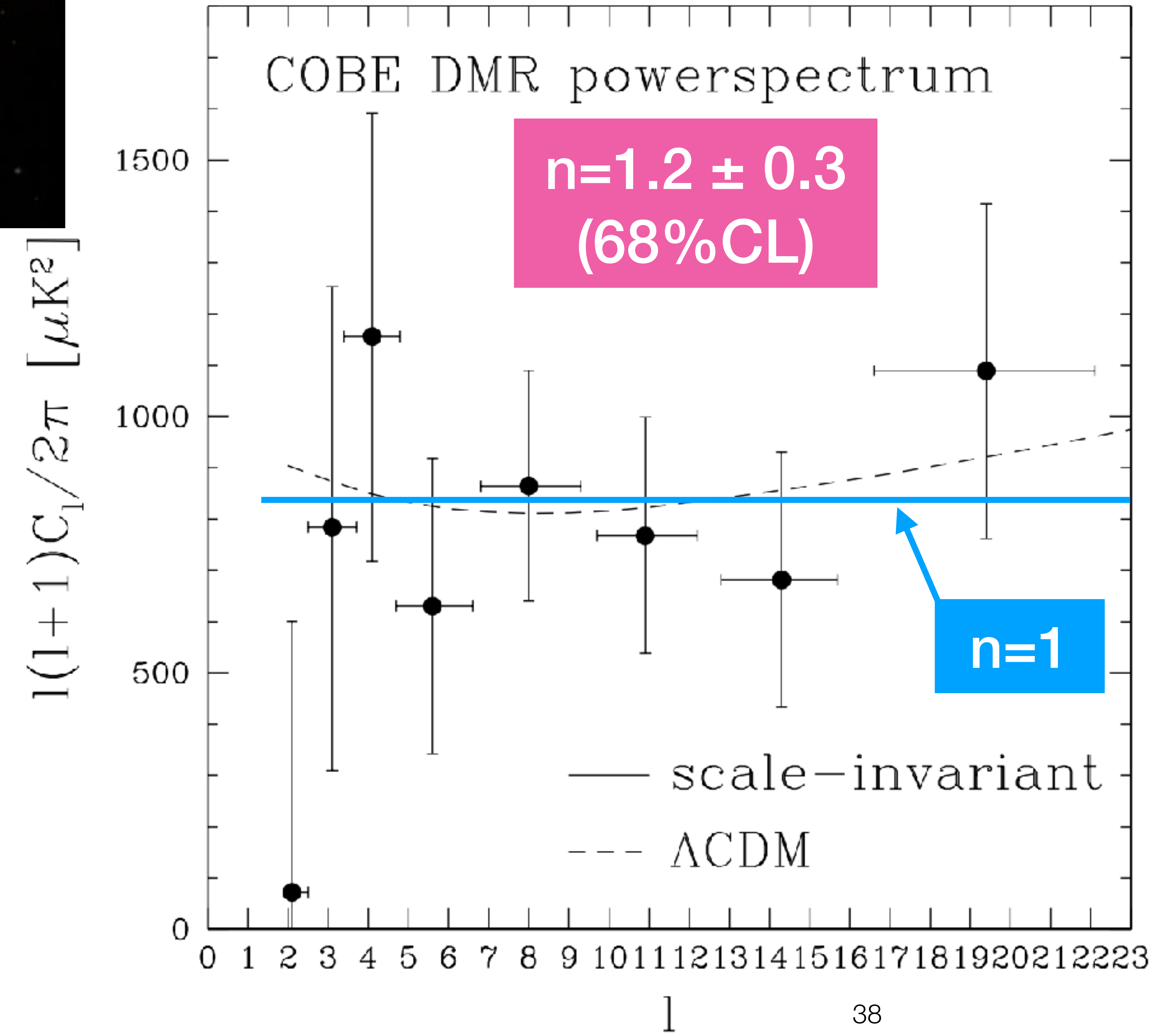
$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{8\pi^2 N_\phi^2 T_0^2}{9\ell^2} \left(\frac{\ell}{r_L}\right)^{n-1} \frac{\sqrt{\pi}}{2} \frac{\Gamma[(3-n)/2]}{\Gamma[(4-n)/2]}$$

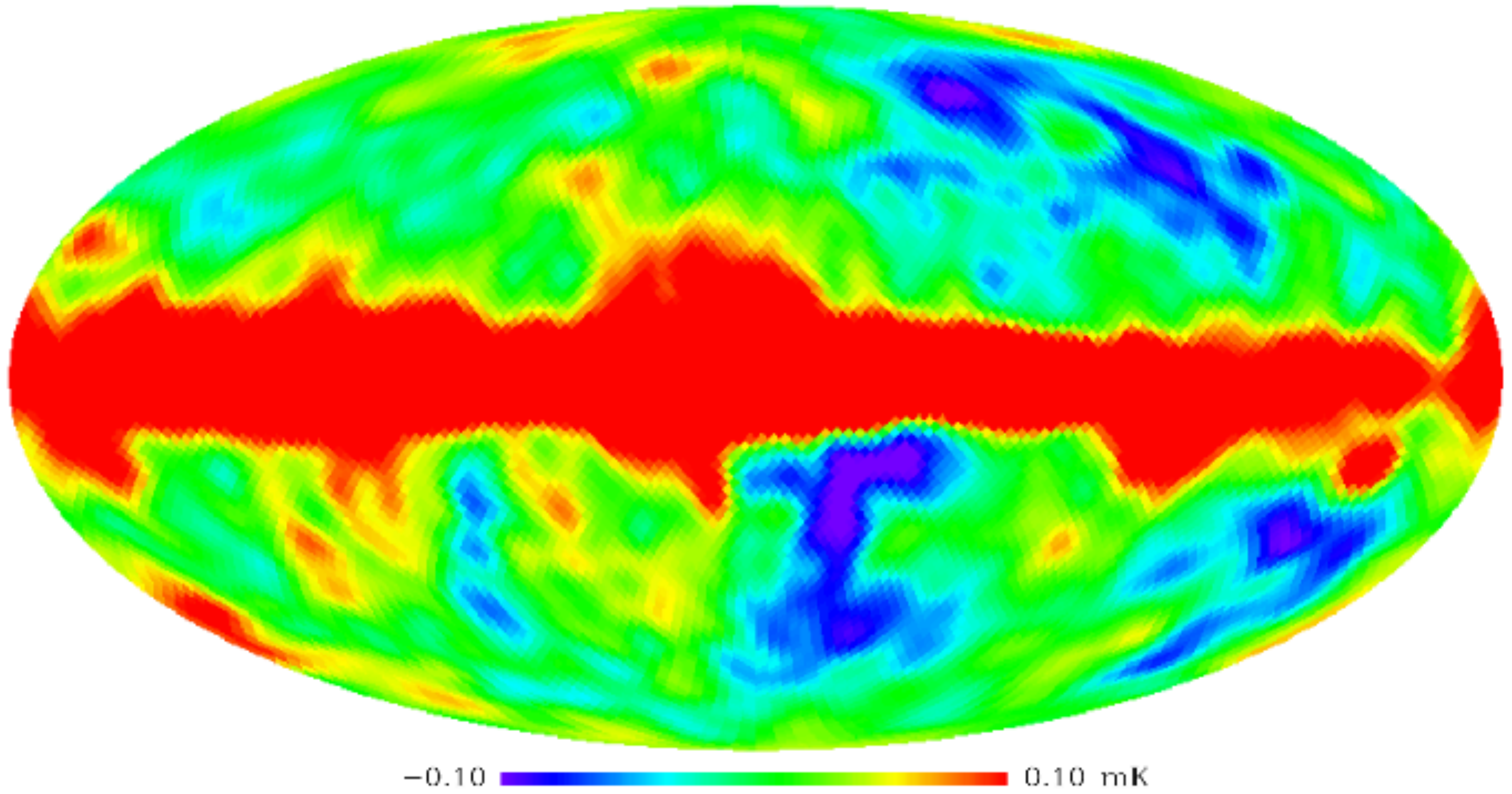
n=1 → $$\frac{8\pi^2 N_\phi^2 T_0^2}{9\ell(\ell+1)}$$

full-sky correction

Bennett et al. (1996)

COBE 4-year Power Spectrum



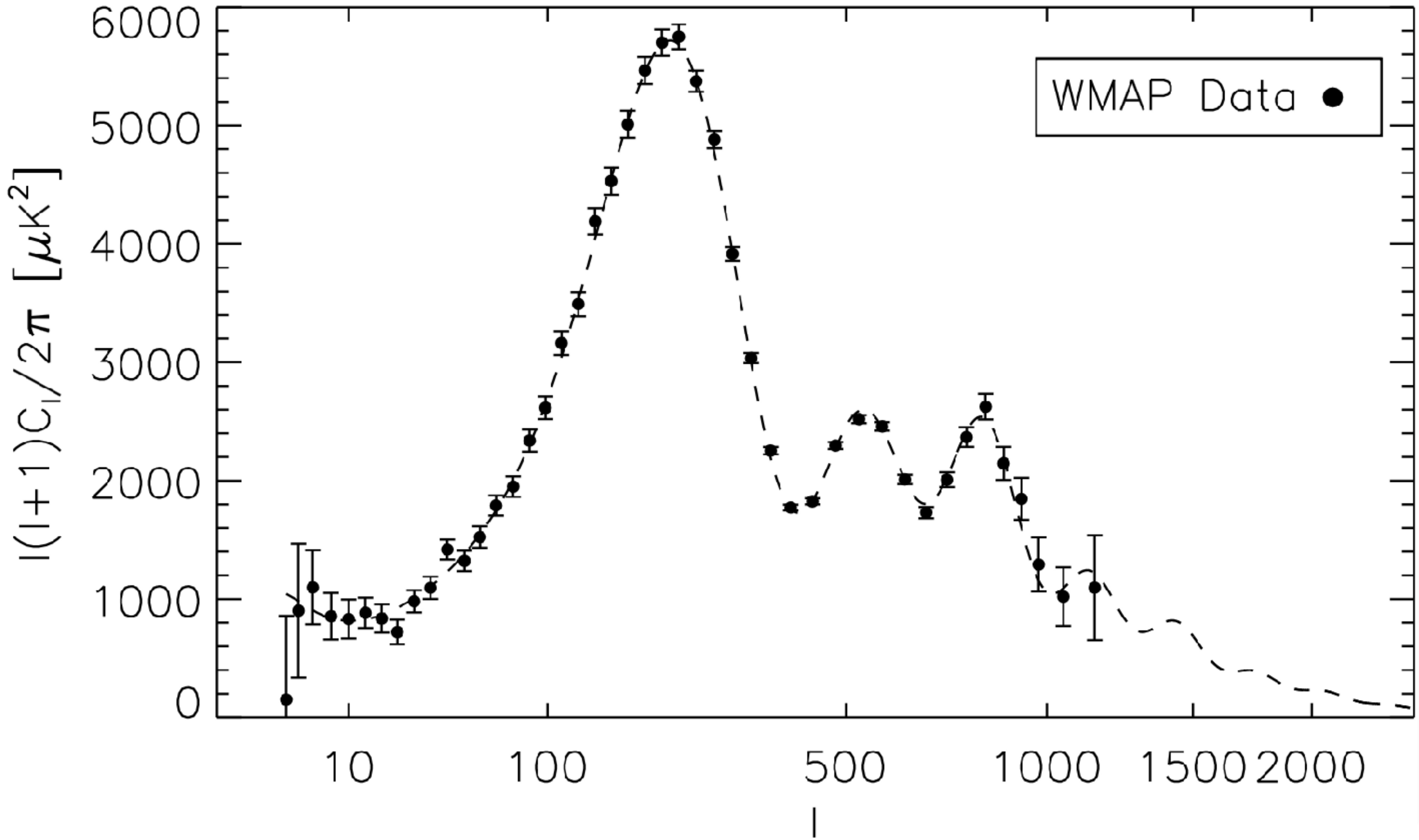


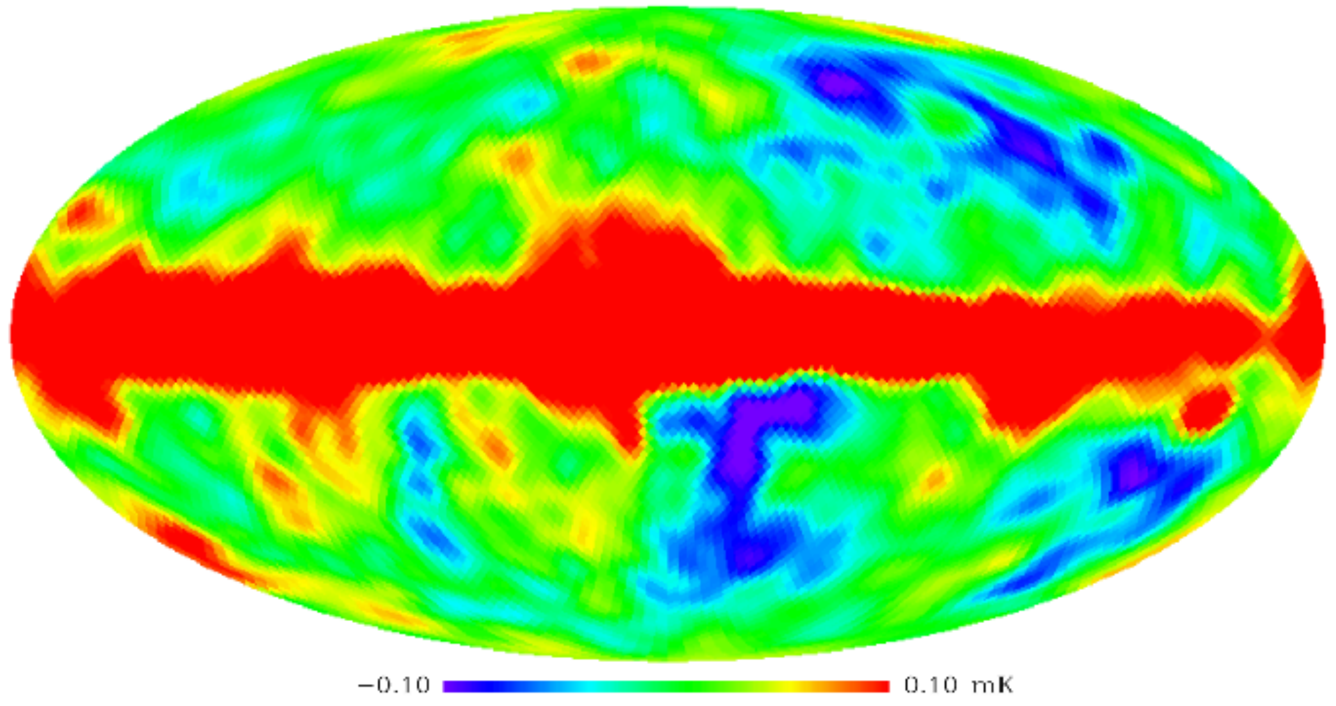
What physics can we learn from this measurement?

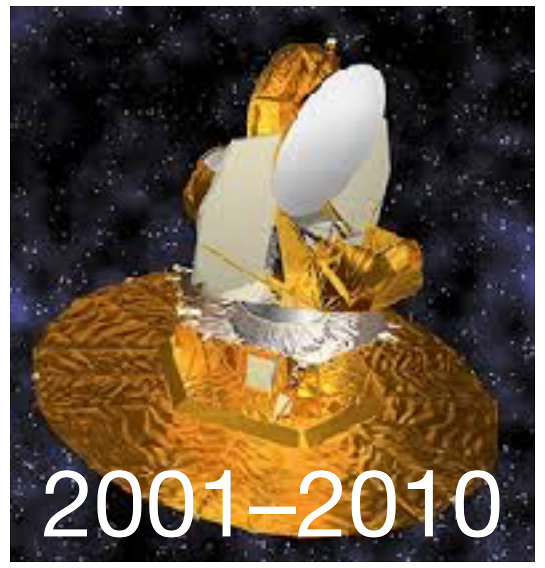
Φ!!

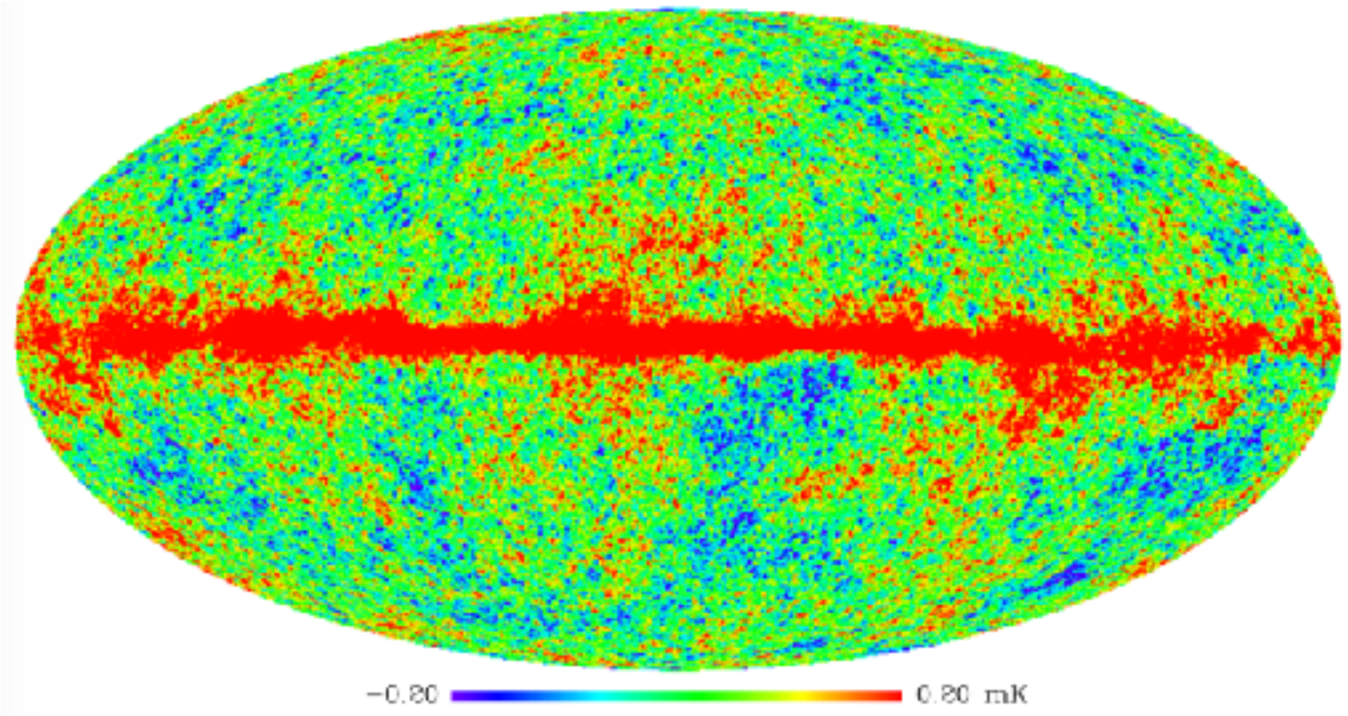
Bennett et al. (2013)

WMAP 9-year Power Spectrum



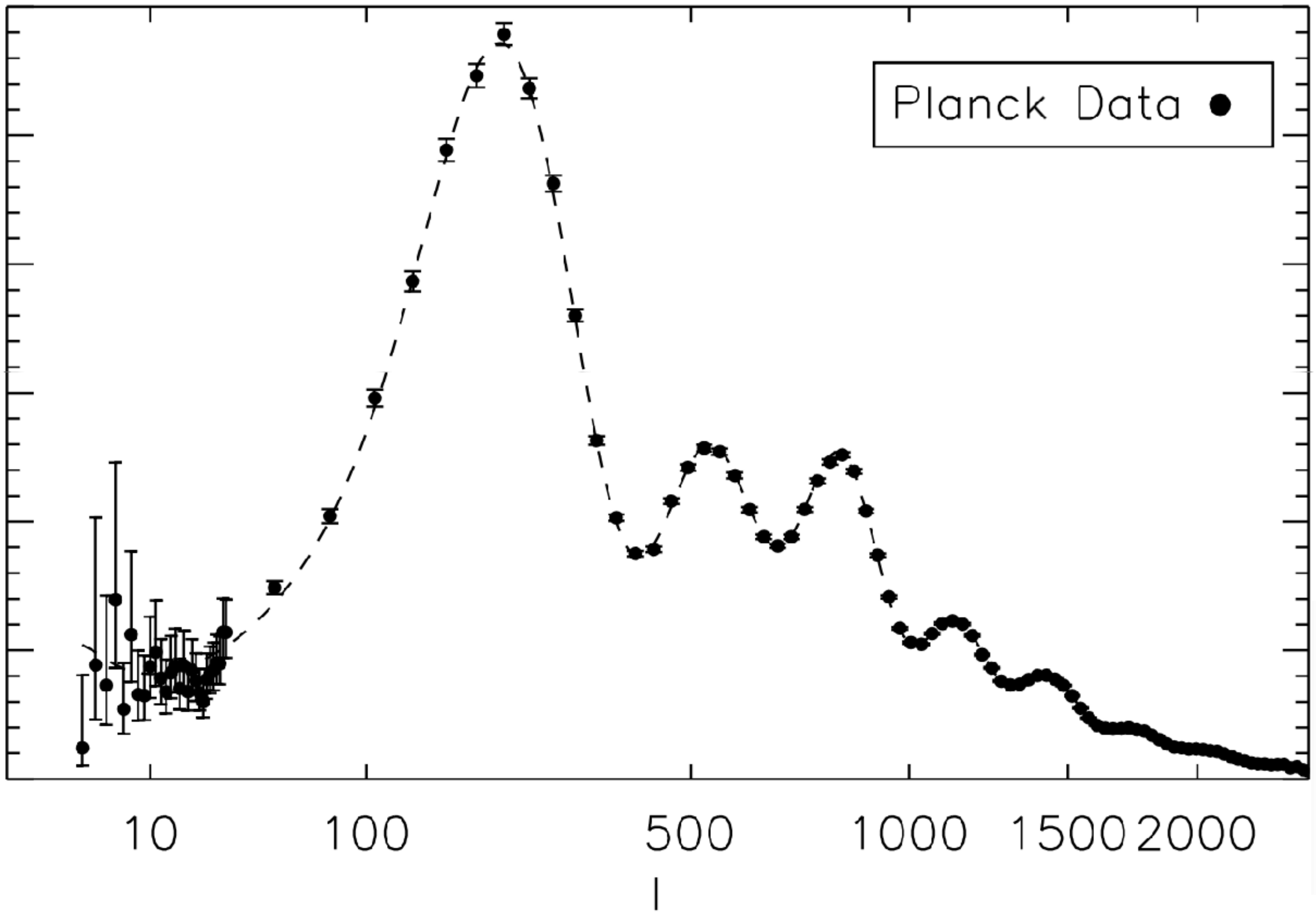


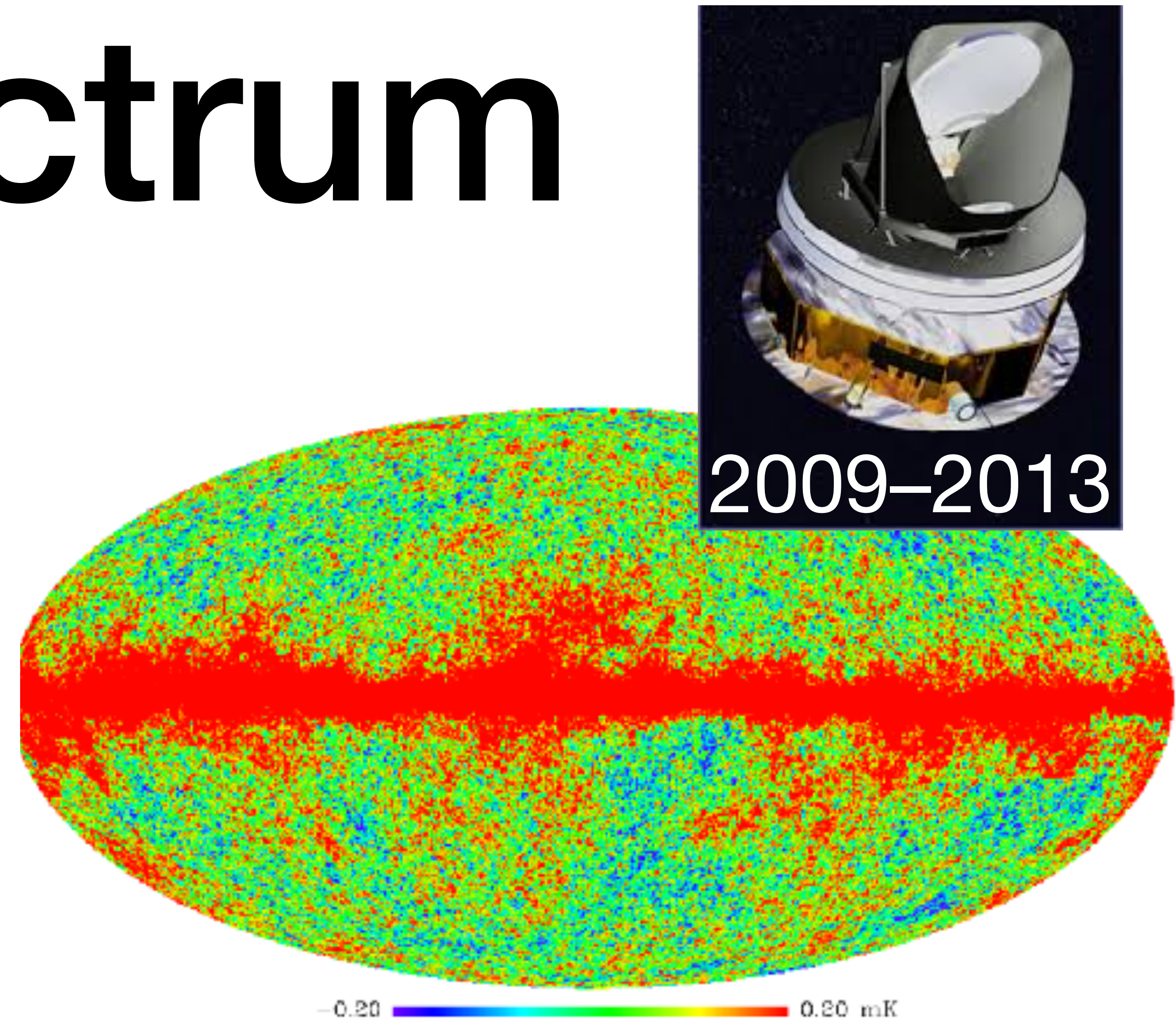


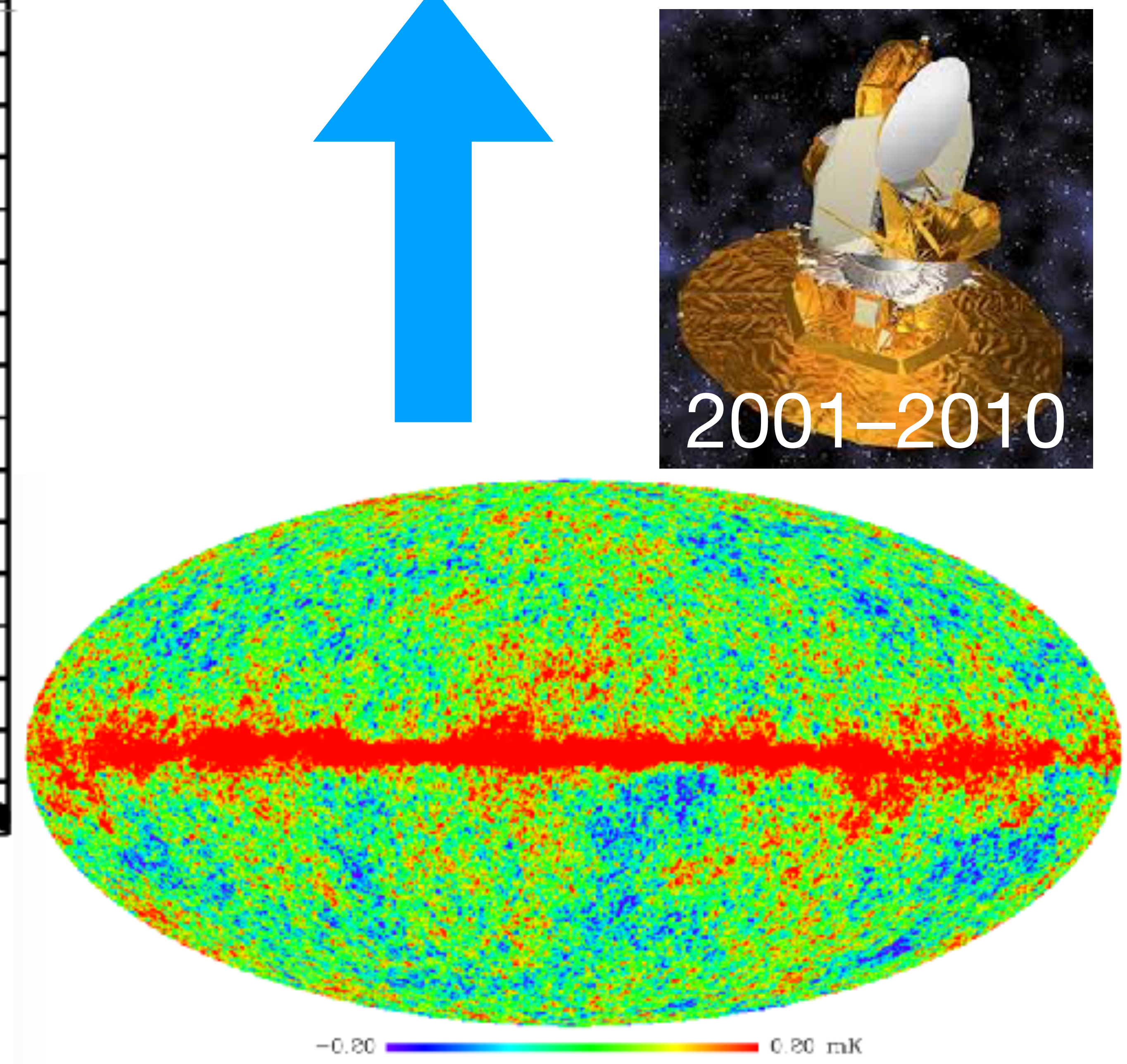


Planck Collaboration

Planck 29-mo Power Spectrum

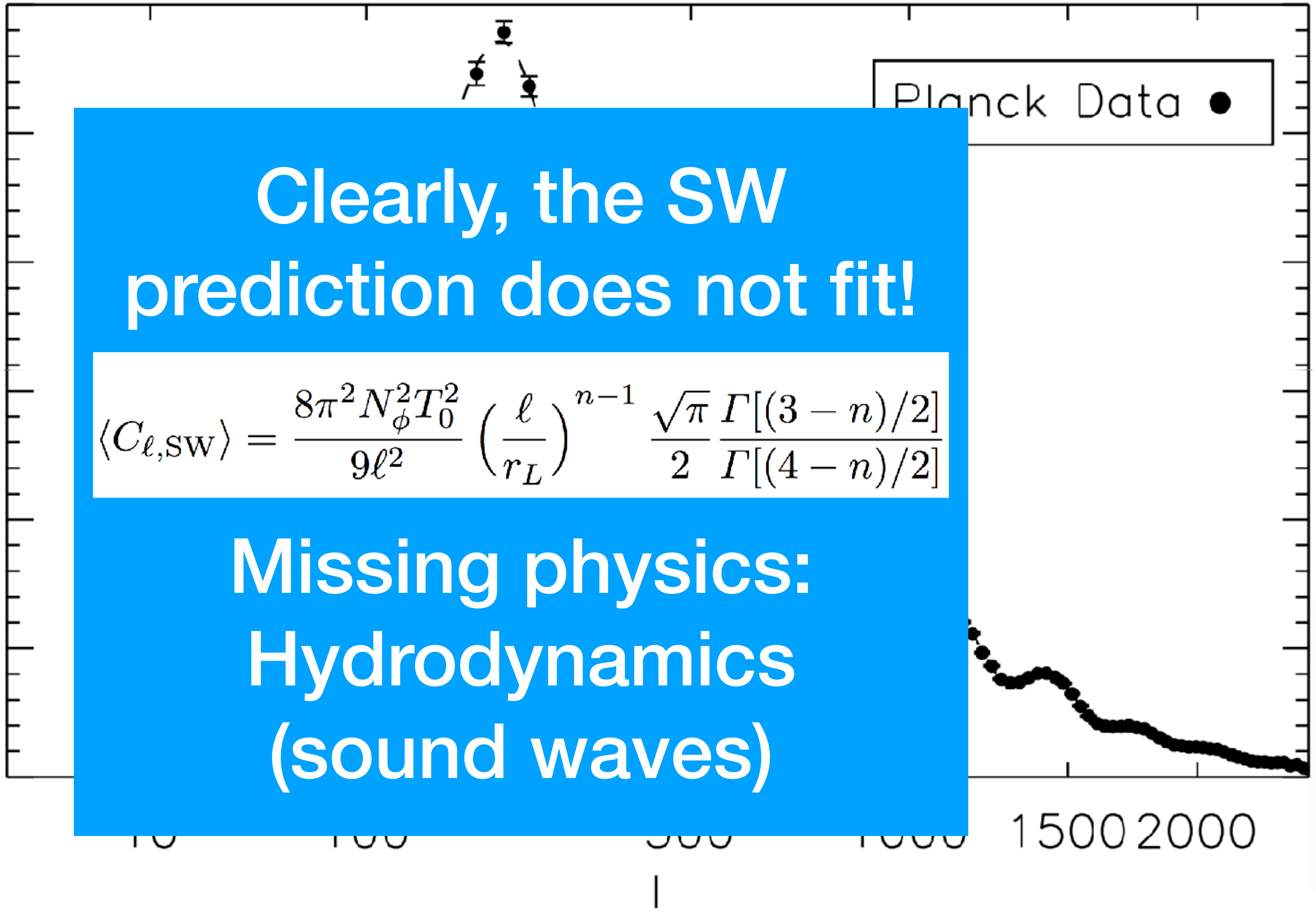






Planck Collaboration

Planck 29-mo Power Spectrum



Clearly, the SW prediction does not fit!

$$\langle C_{\ell,\mathrm{SW}} \rangle = \frac{8\pi^2 N_\phi^2 T_0^2}{9\ell^2} \left(\frac{\ell}{r_L}\right)^{n-1} \frac{\sqrt{\pi}}{2} \frac{\Gamma[(3-n)/2]}{\Gamma[(4-n)/2]}$$

Missing physics:
Hydrodynamics
(sound waves)