Lecture 3: Gravitational Effects on Temperature Anisotropy

Part I: Sachs-Wolfe Effect(s)

Evolution of photon's energy

Sachs & Wolfe (1967)

γ^i is a unit vector of the direction of photon's momentum:

$$\sum_i (\gamma^i)^2 = 1$$

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\dot{a}}{a} + \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i - \frac{1}{2}\sum_{ij}\dot{D}_{ij}\gamma^i\gamma^j$$

Scalar curvature perturbation (Ψ)

Newtonian gravitational potential (Φ)

Tensor perturbation = Gravitational wave (D_{ij})

- Let's find a (formal) solution for p by integrating this equation over time.

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Evolution of photon's energy

Sachs & Wolfe (1967)

$$\frac{1}{ap}\frac{d(ap)}{dt} = \dot{\Psi} - \frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i - \frac{1}{2}\sum_{ij}\dot{D}_{ij}\gamma^i\gamma^j$$

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Evolution of photon's energy

Sachs & Wolfe (1967)

$$\frac{1}{ap}\frac{d(ap)}{dt} = \dot{\Psi} - \frac{d\Phi}{dt} + \dot{\Phi} - \frac{1}{2}\sum_{ij} \dot{D}_{ij}\gamma^i\gamma^j$$

because

$$\frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i = \frac{d\Phi}{dt} - \dot{\Phi}$$

- Let's find a (formal) solution for p by integrating this equation over time.

Formal Solution (Scalar)

Present-day time

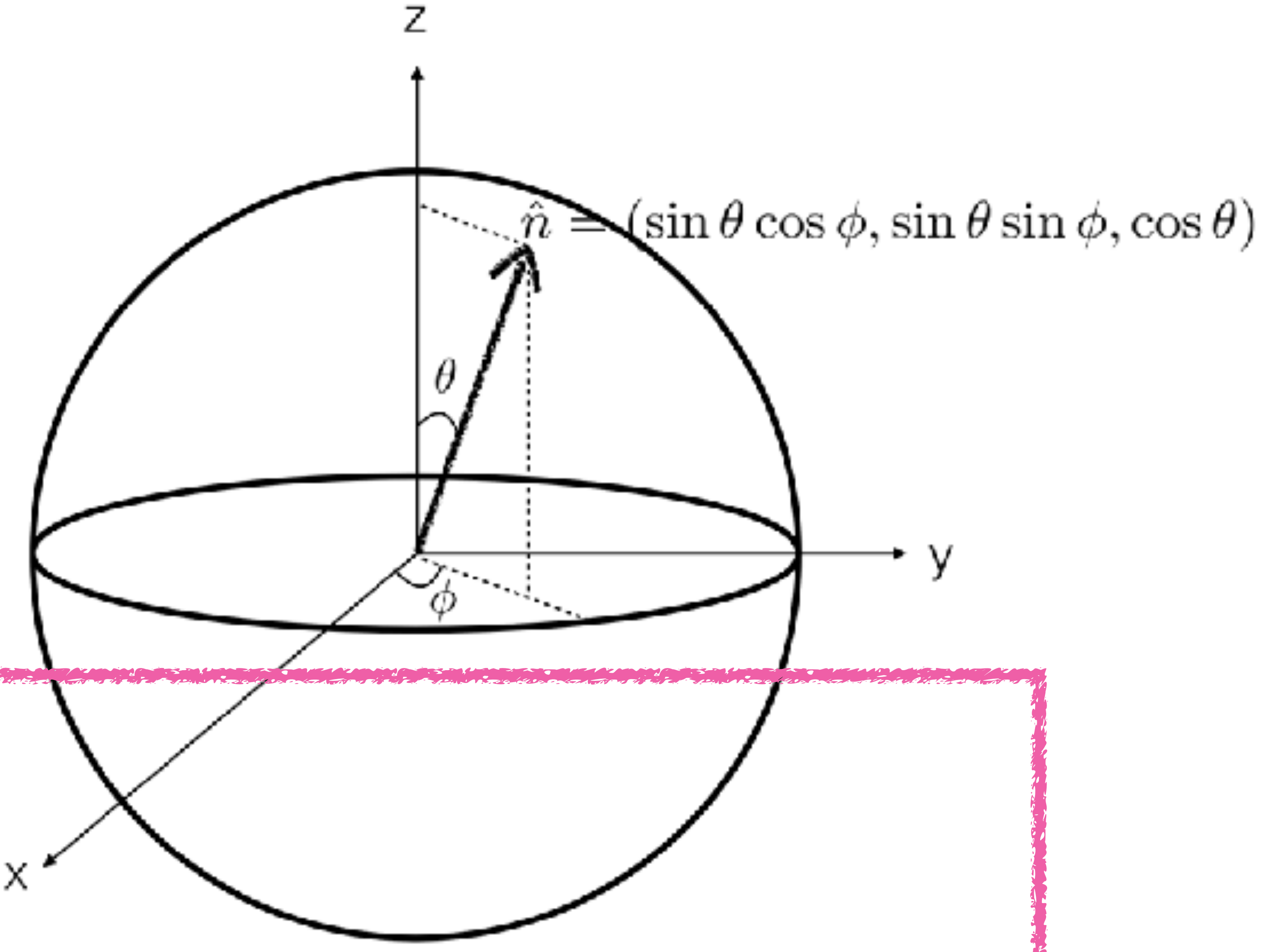
"L" for "Last scattering surface"

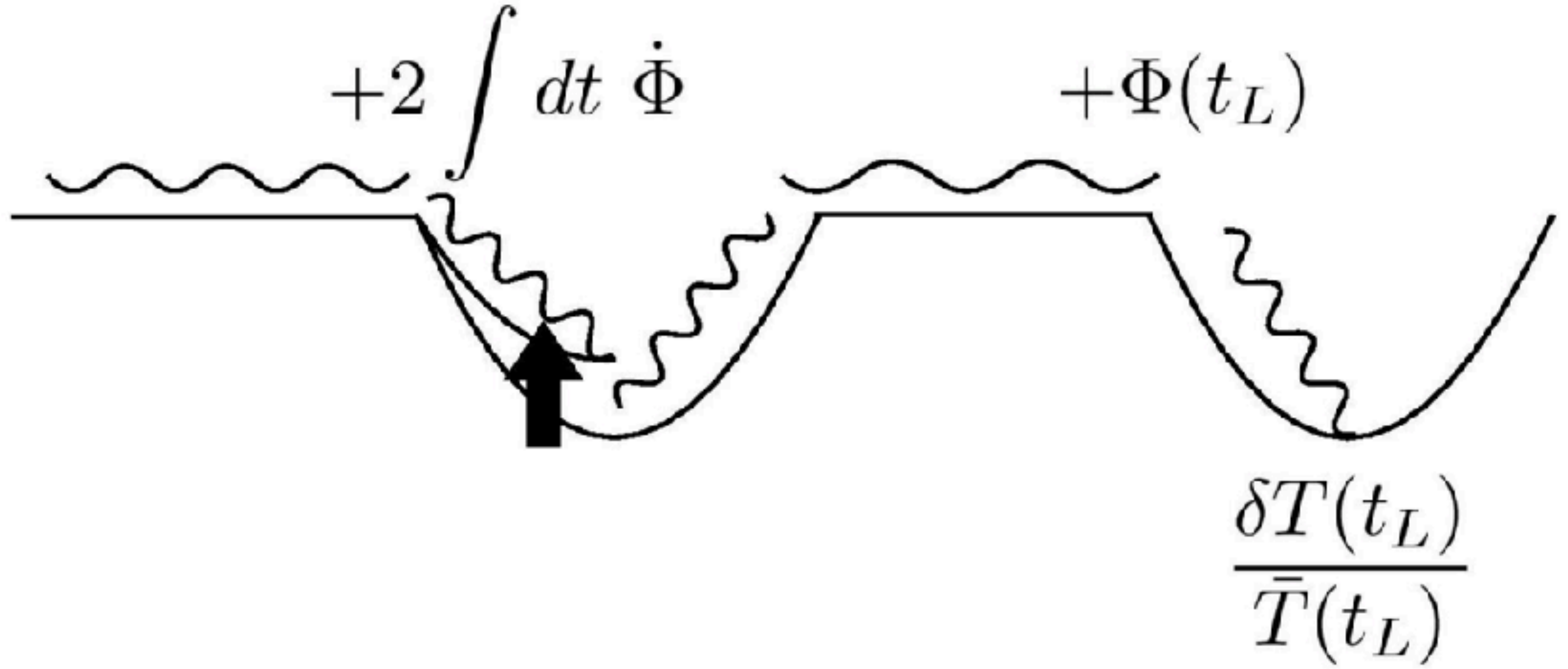
$$\ln(ap)(t_0) = \ln(ap)(t_L) + \Phi(t_L) - \Phi(t_0) + \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})$$

or

$$\frac{1}{a}\sum_i \frac{\partial \Phi}{\partial x^i}\gamma^i = \frac{d\Phi}{dt} - \dot{\Phi}$$

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0) + \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$





Line-of-sight direction

$$\hat{n}^i = -\gamma^i$$

Comoving distance (r)

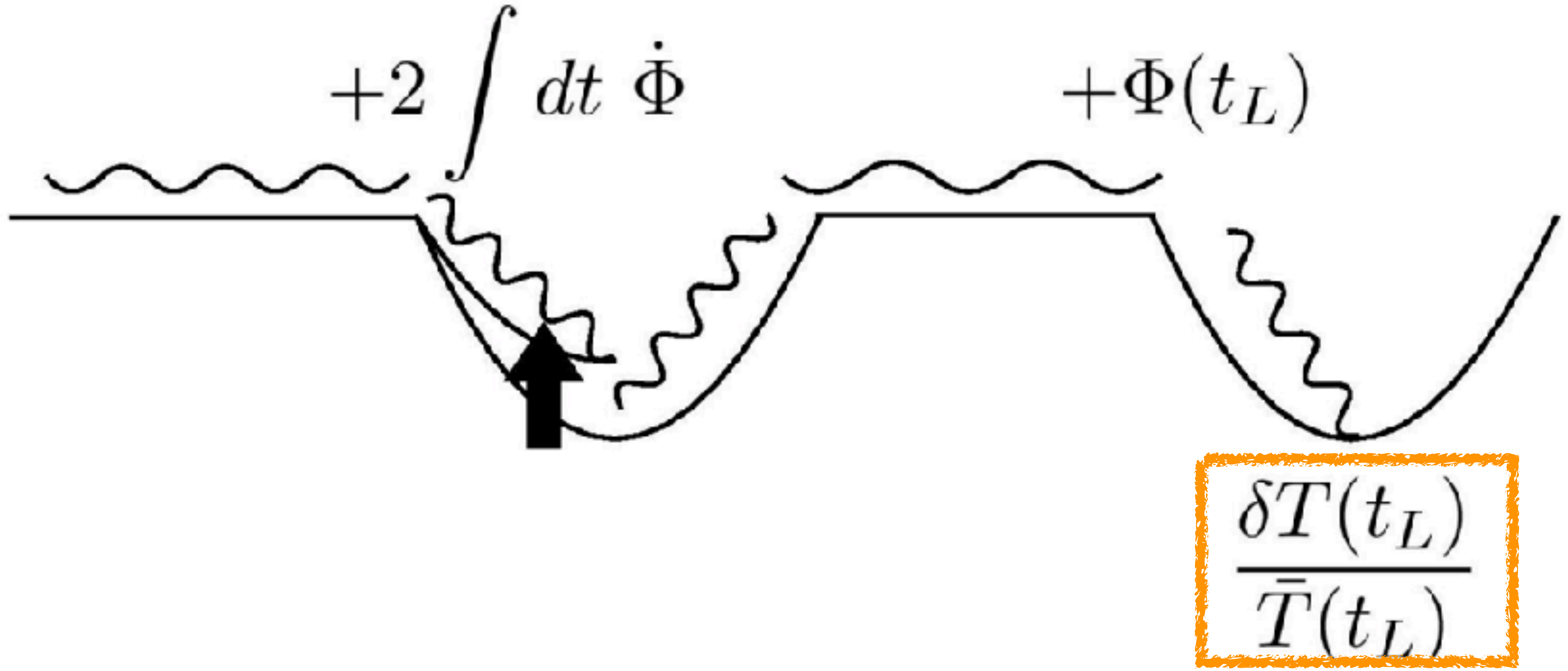
$$x^i = \hat{n}^i r \qquad r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

Sachs & Wolfe (1967)

Formal Solution (Scalar)

Initial Condition

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0) + \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$



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Formal Solution (Scalar)

Gravitational Redshit

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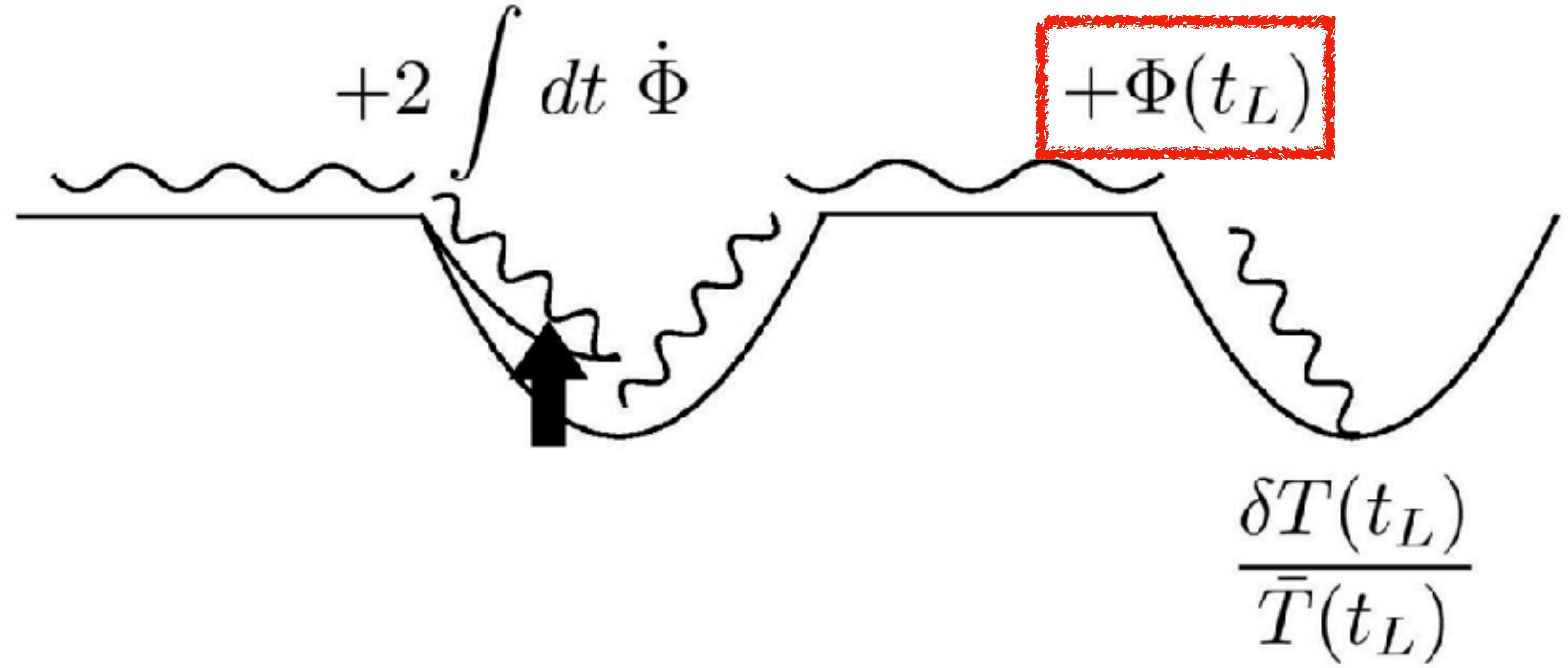

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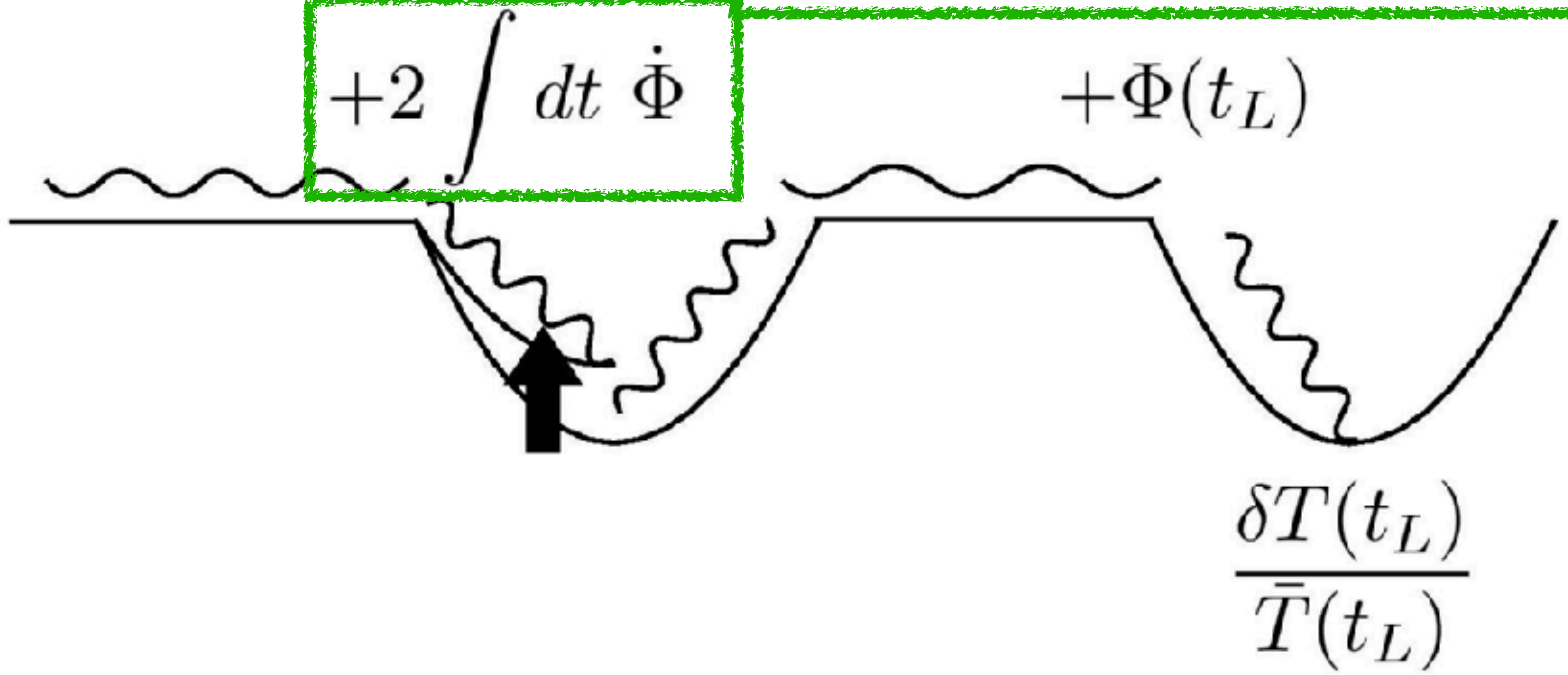
Sachs & Wolfe (1967)

Formal Solution (Scalar)

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0)$$

"integrated Sachs-Wolfe" (ISW) effect

$$+ \int_{t_L}^{t_0} dt \; (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$



Line-of-sight direction

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Comoving distance (r)

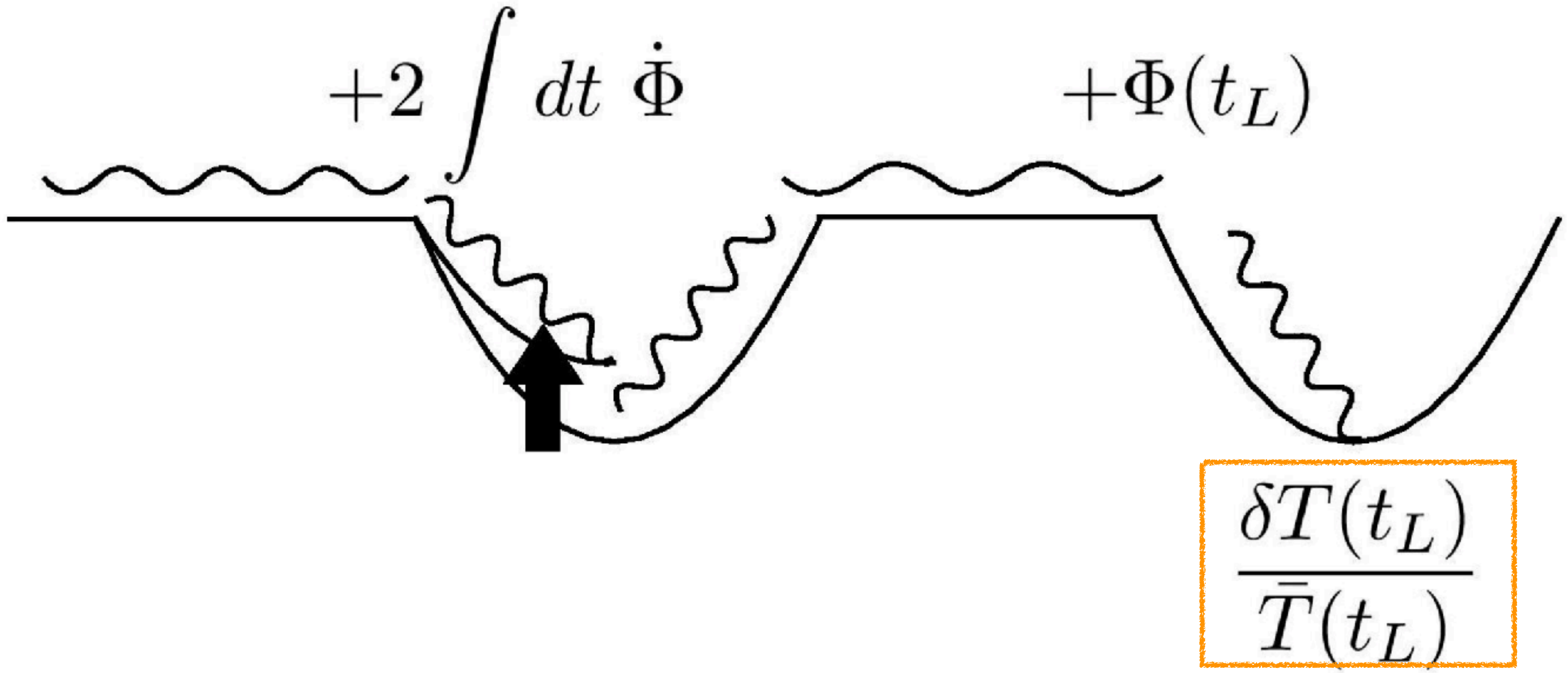
$$x^i = \hat{n}^i r$$

$$r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

Part II: Initial Condition

Initial Condition

Only the data can tell us!



- Were photons hot, or cold, at the bottom of the potential well at the last scattering surface?

"Adiabatic Initial Condition"

The initial condition that fits the current data best

- Definition: "*Ratios of the number densities of all species are equal everywhere initially*"
  - For i^{th} and j^{th} species, **$n_i(x)/n_j(x)$ = constant**
- For a quantity X(t,x), let us define the **fluctuation, *δX*,** as

$$\delta X(t, \boldsymbol{x}) \equiv X(t, \boldsymbol{x}) - \bar{X}(t)$$

- Then, the adiabatic initial condition is

$$\frac{\delta n_i(t_{\rm initial}, \mathbf{x})}{\bar{n}_i(t_{\rm initial})} = \frac{\delta n_j(t_{\rm initial}, \mathbf{x})}{\bar{n}_j(t_{\rm initial})}$$

Example of the adiabatic initial condition

Thermal equilibrium

- When photons and baryons were in thermal equilibrium in the past, then
  - $n_{photon} \sim T^3$ and $n_{baryon} \sim T^3$
  - That is to say, **thermal equilibrium naturally gives rise to the adiabatic initial condition**, because n_{photon} / n_{baryon} = constant
  - This gives

$$3\frac{\delta T(t_i, \boldsymbol{x})}{\bar{T}(t_i)} = \frac{\delta \rho_B(t_i, \boldsymbol{x})}{\bar{\rho}_B(t_i)}$$

- "B" for "Baryons"
- ρ is the mass density

A Big Question

- ***How about dark matter?***

- If dark matter and photons were in thermal equilibrium in the past, then they should also obey the adiabatic initial condition

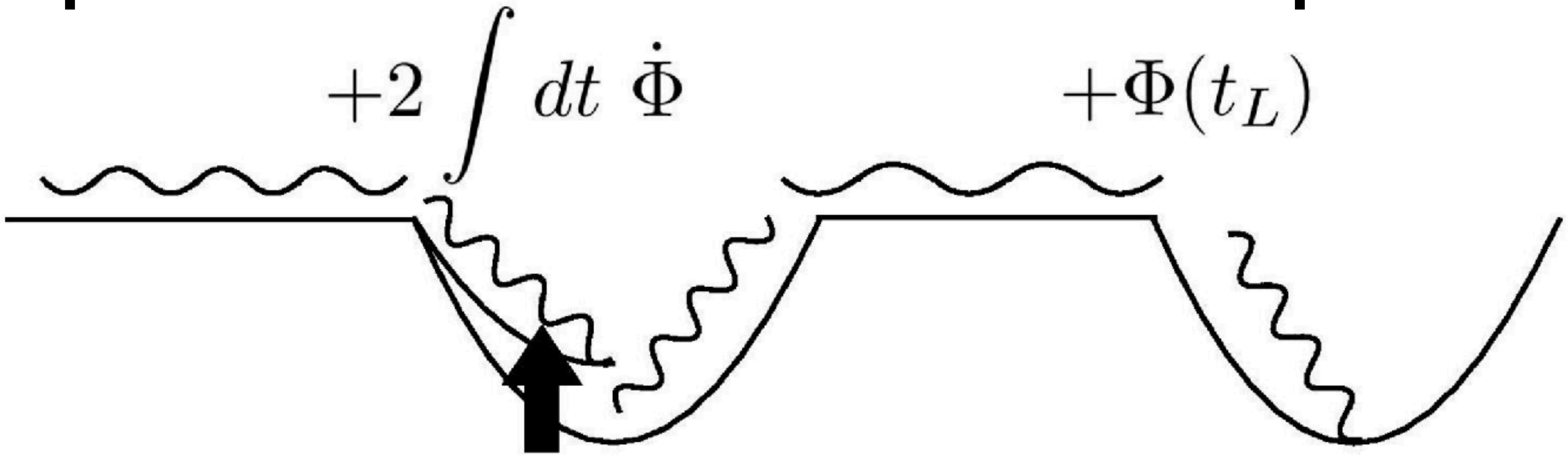
    - If not, *there is no a priori reason to expect the adiabatic initial condition!*

- The current data are consistent with the adiabatic initial condition. This means something important for the nature of dark matter!

We shall assume the adiabatic initial condition throughout the lectures

Adiabatic solution

Was the temperature hot or cold at the bottom of potential?

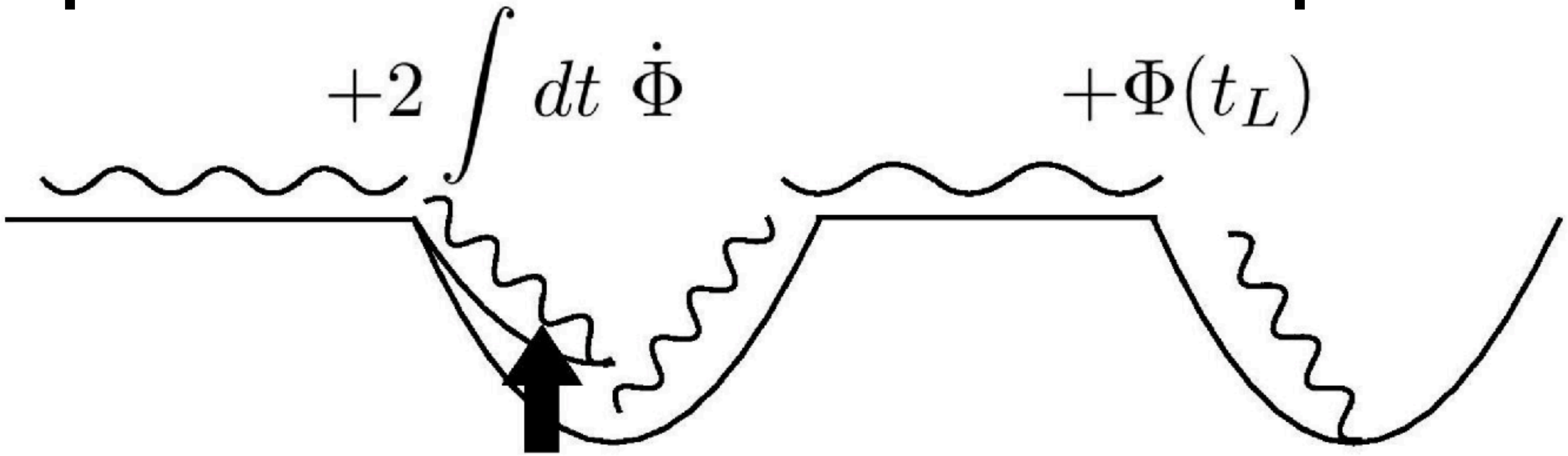


- At the last scattering surface, the temperature fluctuation is given by the matter density fluctuation as $\frac{\delta T(t_L)}{\bar{T}(t_L)}$

$$\frac{\delta T(t_L, \mathbf{x})}{\bar{T}(t_L)} = \frac{1}{3}\frac{\delta\rho_M(t_L, \mathbf{x})}{\bar{\rho}_M(t_L)}$$

Adiabatic solution

Was the temperature hot or cold at the bottom of potential?



$$\frac{\delta T(t_L)}{\bar{T}(t_L)}$$

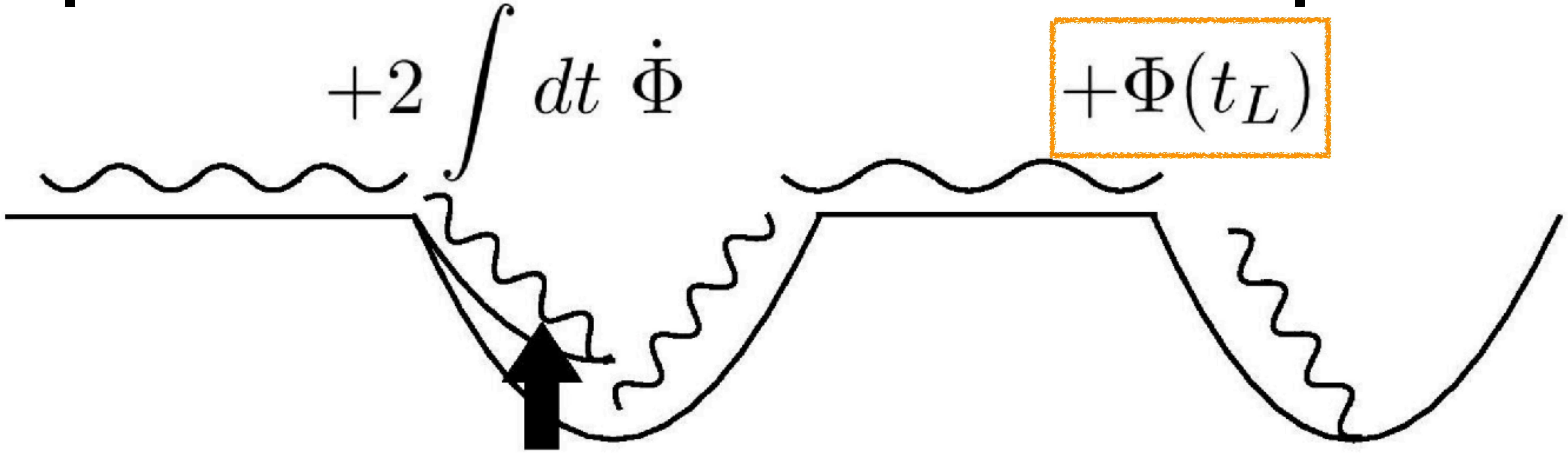
- On large scales, the matter density fluctuation during the matter-dominated era is given by $\delta\rho_M/\bar{\rho}_M = -2\Phi$

$$\frac{\delta T(t_L,\mathbf{x})}{\bar{T}(t_L)} = \frac{1}{3}\frac{\delta\rho_M(t_L,\mathbf{x})}{\bar{\rho}_M(t_L)} = -\frac{2}{3}\Phi(t_L,\mathbf{x})$$

Hot **at the bottom of the potential well, but…**

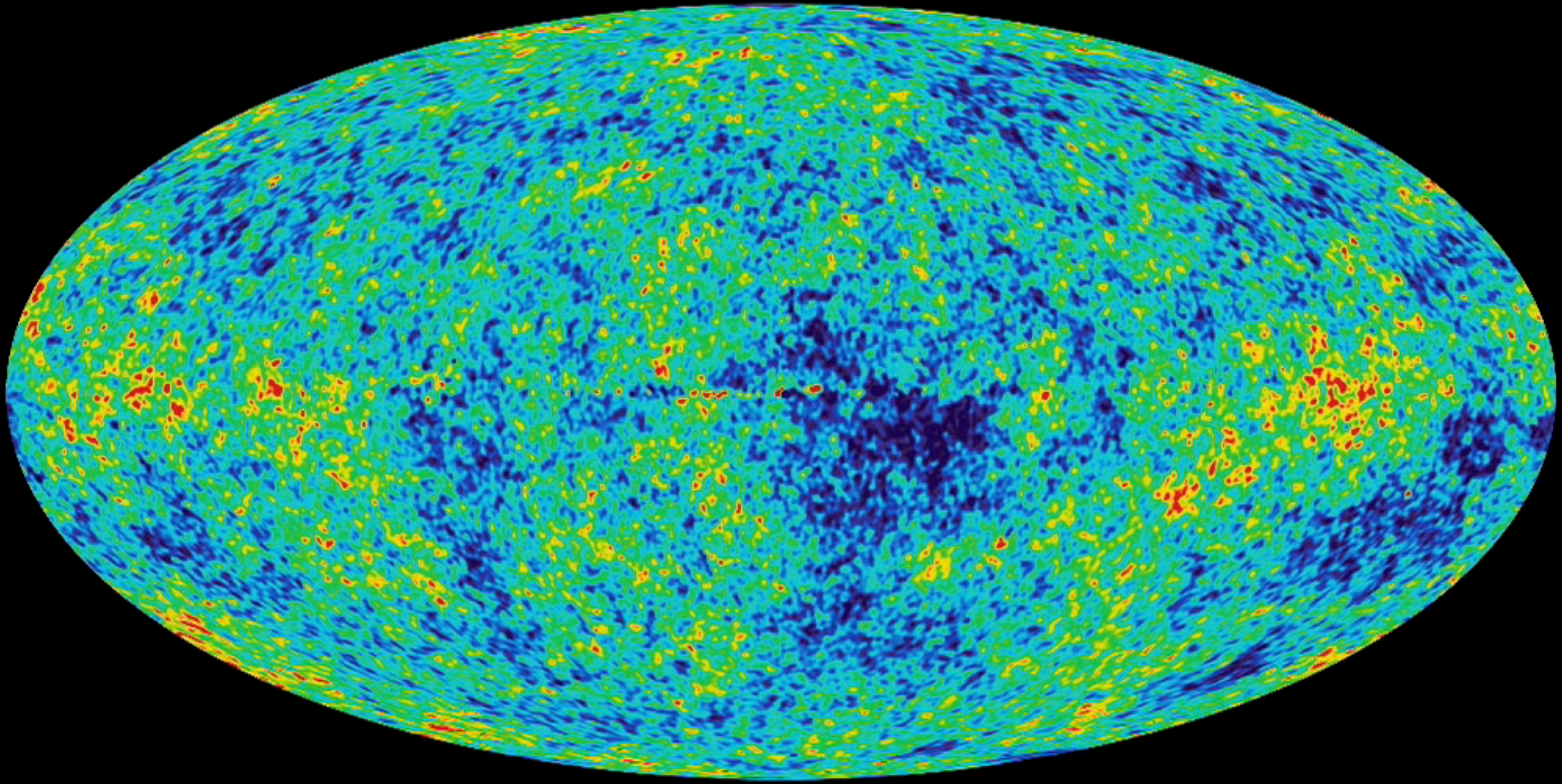
Adiabatic solution

Was the temperature hot or cold at the bottom of potential?



- Therefore, $$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{3}\Phi(t_L, \hat{r}_L)$$

This is negative in an over-density region!



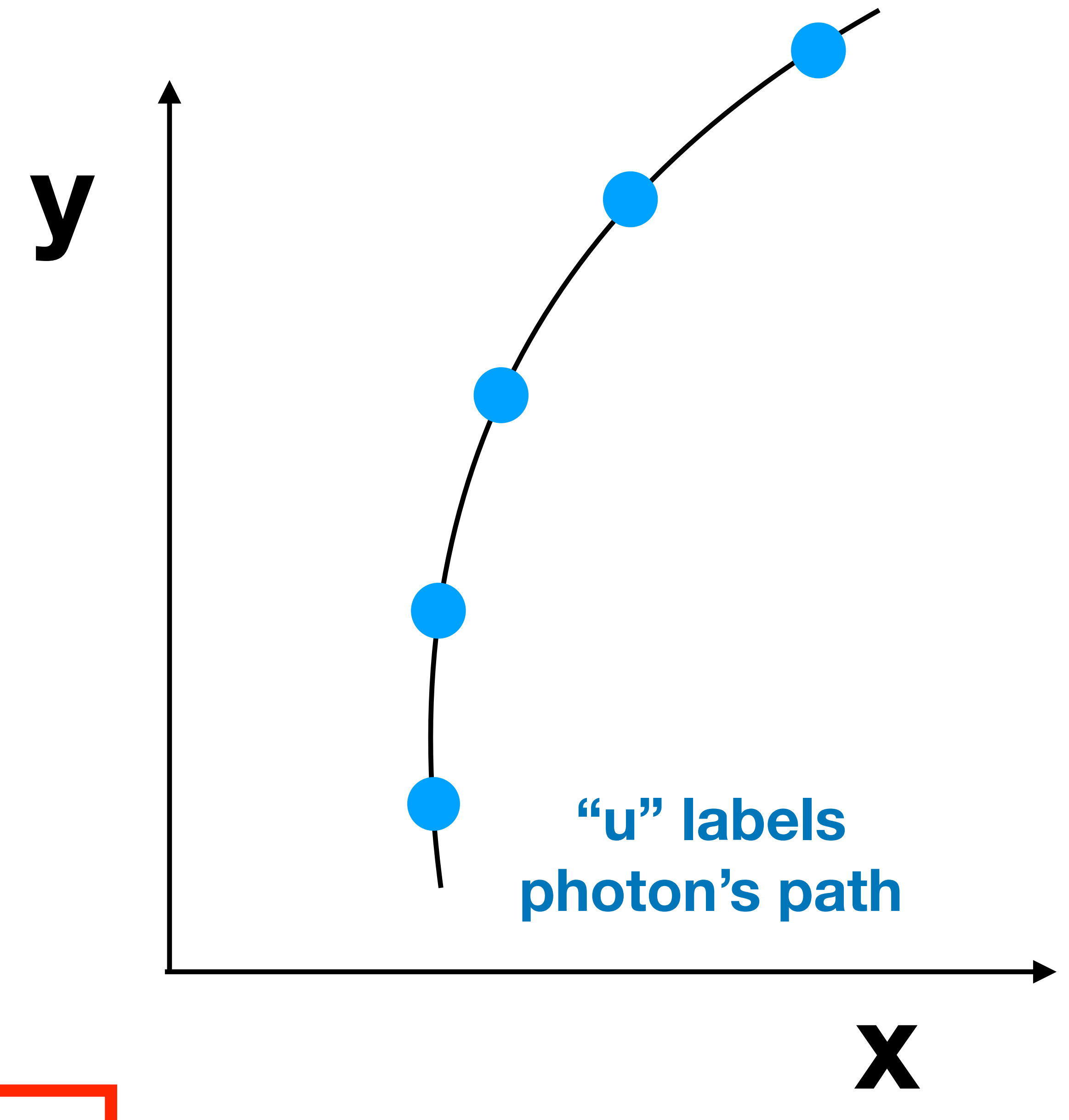
Part III: Gravitational Lensing

Equation of motion for photons

Evolution of the *direction* of photon's momentum

- Instead of the magnitude of photon's momentum, write the equation of motion for photon's momentum

$$\frac{dp^\lambda}{dt} + \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \Gamma^\lambda_{\mu\nu}\frac{p^\mu p^\nu}{p^0} = 0$$



in terms of the unit vector of the direction of photon's momentum, γ^i:

$$\frac{d\gamma^i}{dt} = \frac{1}{a}\sum_{j=1}^{3}(\gamma^i\gamma^j - \delta^{ij})\frac{\partial}{\partial x^j}(\Phi + \Psi)$$

The sum of two potentials!

Einstein's what could-have-been the biggest blunder

Φ or Φ+Ψ?

- In 1911, Einstein calculated the deflection of light by Sun, and concluded that it would be 0.87 arcsec.
    - At that time, Einstein had not realised yet the role of spatial curvature (Ψ). Thus, his metric was still $ds_4^2 = -(1+2\Phi)dt^2 + dx^2$. As a result, his prediction was a factor of two too small: **the correct value is 1.75 arcsec.**
- In 1914, the expedition organised by Erwin Freundlich (Berliner Sternwarte) to detect the deflection of light by Sun during the total solar eclipse **failed.**
- In 1916, Einstein predicted 1.75 arcsec by incorporating Ψ, which is equal to Φ.
- In 1919, the expedition organised by Arthur Eddington (Cambridge Observatory) confirmed Einstein's prediction.

What if Freundlich's expedition was successful?

Getting 1.75 arcsec

Let's calculate!

$$\frac{d\gamma^i}{dt} = \frac{1}{a}\sum_{j=1}^{3}(\gamma^i\gamma^j - \delta^{ij})\frac{\partial}{\partial x^j}(\Phi + \Psi)$$

Look at i=2:

$$\frac{d\gamma^2}{dt} = 2\gamma^2\sum_j \gamma^j\frac{\partial\Phi}{\partial x^j} - 2\frac{\partial\Phi}{\partial x^2}$$

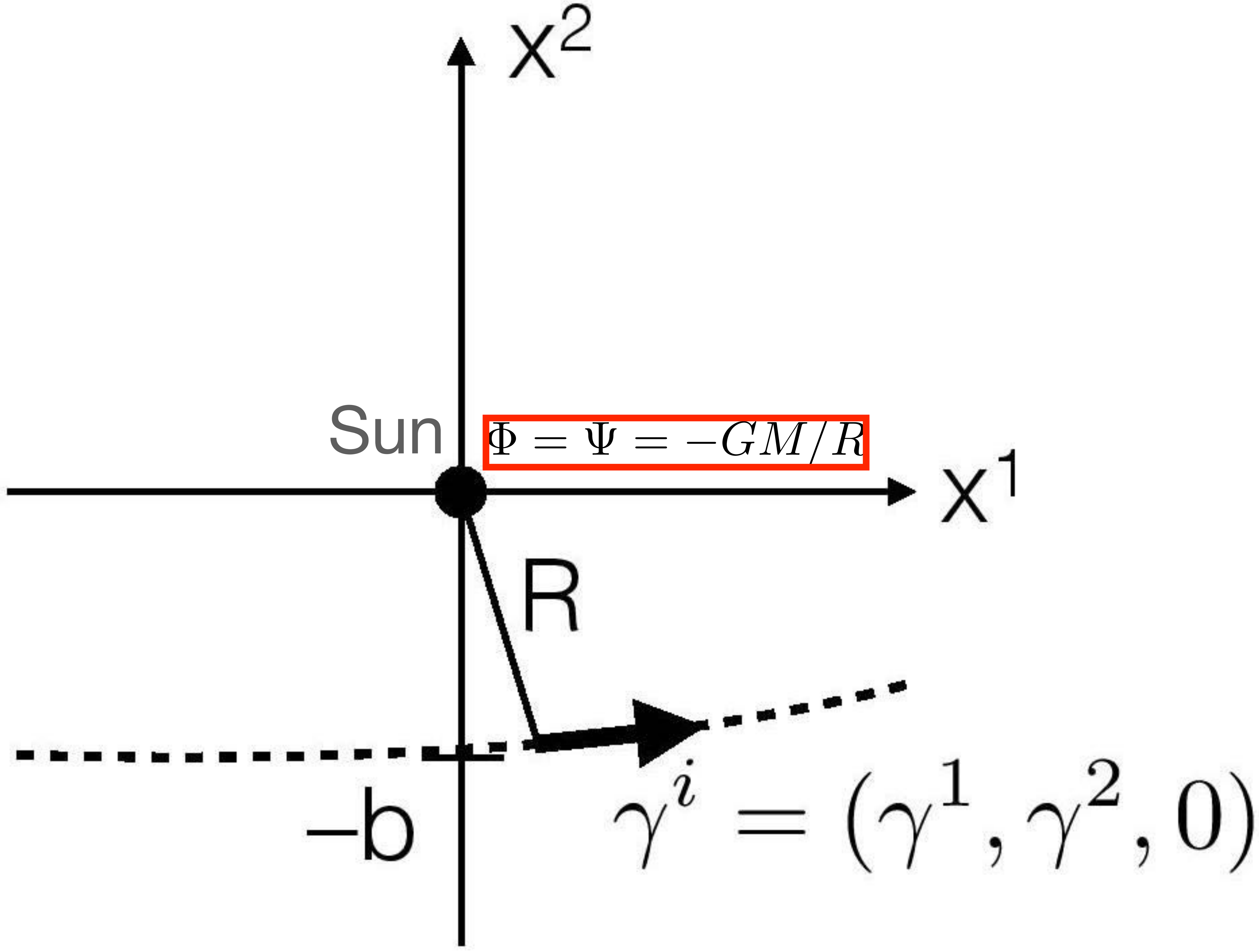
$$= (\text{2nd order}) + \frac{2GMb}{[(x^1)^2 + b^2]^{3/2}}$$

Integrating over dt = dx^1, we obtain

$$\gamma^2 = \frac{4GM}{b} = 8.49 \times 10^{-6}\ \text{rad} = 1.75\ \text{arcsec}$$

Yay!

$$\left(\begin{array}{l} R = 6.96 \times 10^8\ \text{m} \\ M = 1.99 \times 10^{30}\ \text{kg} \end{array}\right.$$

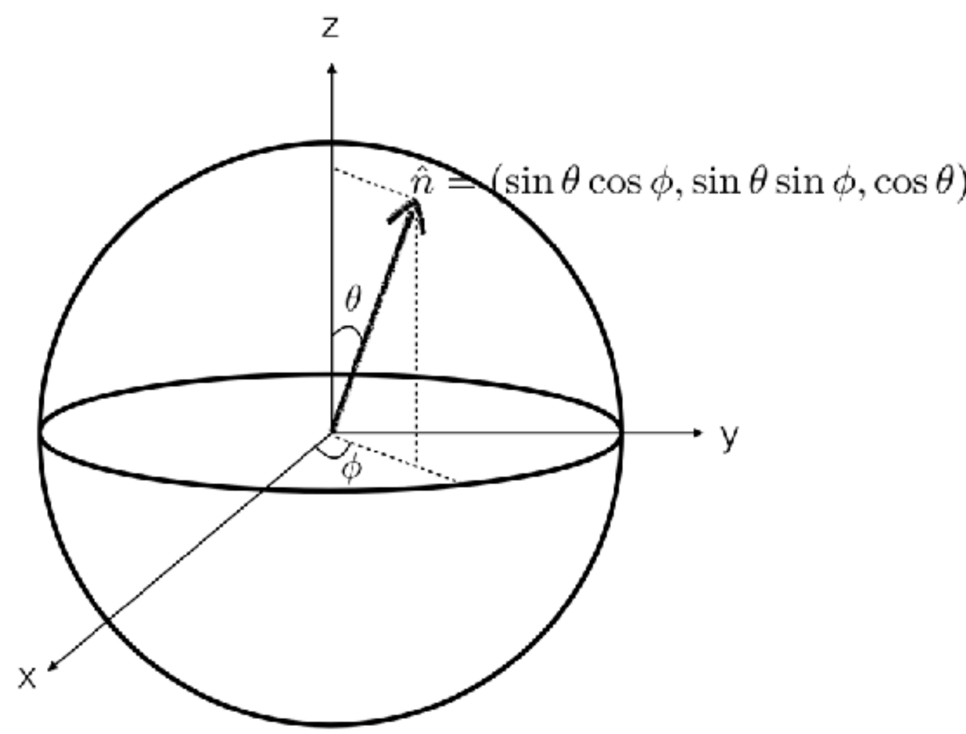


Gravitational lensing effect on the CMB

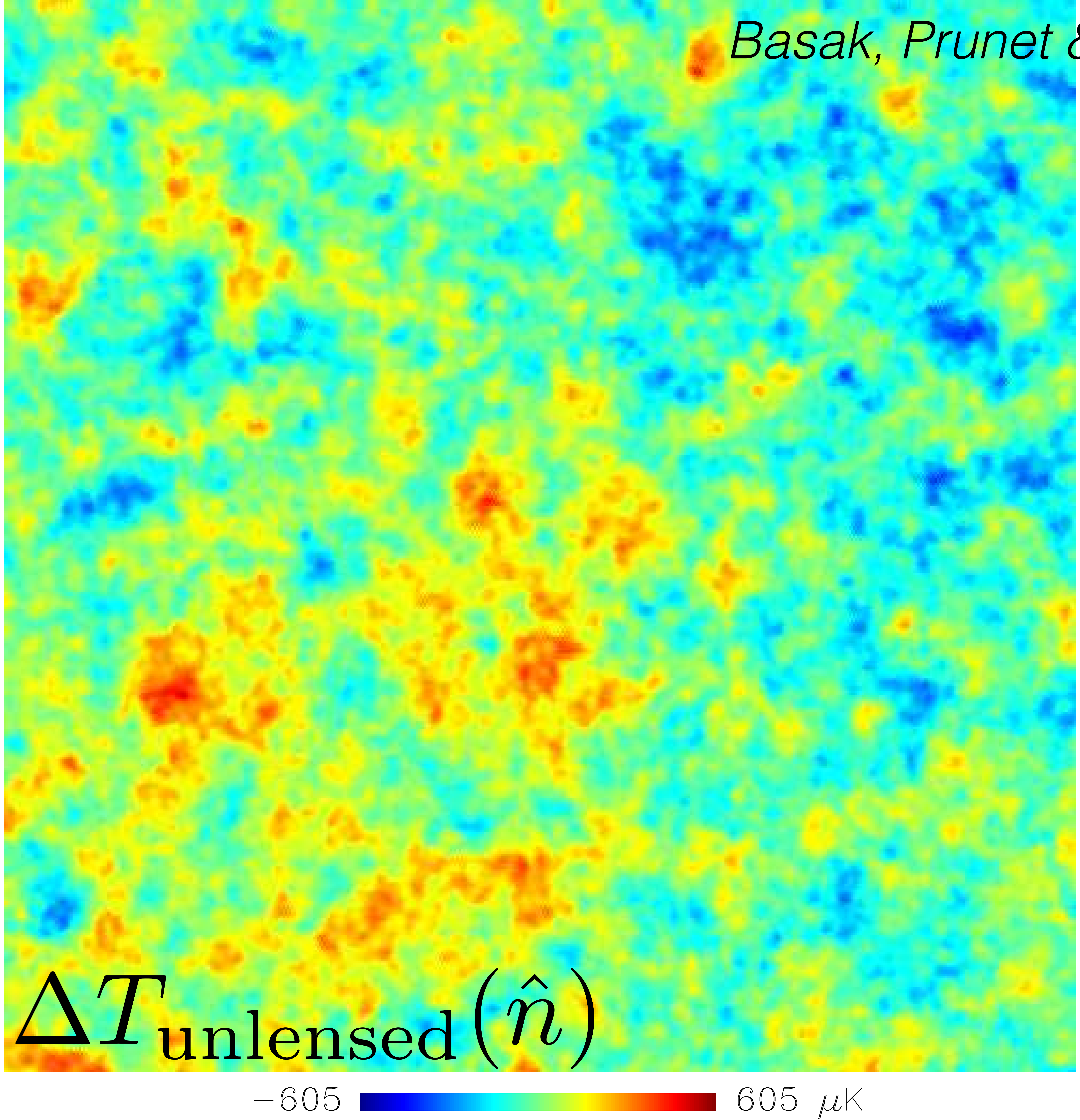
What does it do to CMB?

- The important fact: **the gravitational lensing effect does not change the surface brightness**.
- This means that the value of CMB temperature does not change by lensing; only the directions change.
  - You might be asked during your PhD exam: “*Is the uniform CMB temperature affected by lensing?*” The answer is no.
- Only the **anisotropy** (and polarisation; Lecture 8) is affected:

$$\Delta T_{\mathrm{lensed}}(\hat{n}) = \Delta T_{\mathrm{unlensed}}(\hat{n} + \mathbf{d})$$



Basak, Prunet & Benumbed (2008)



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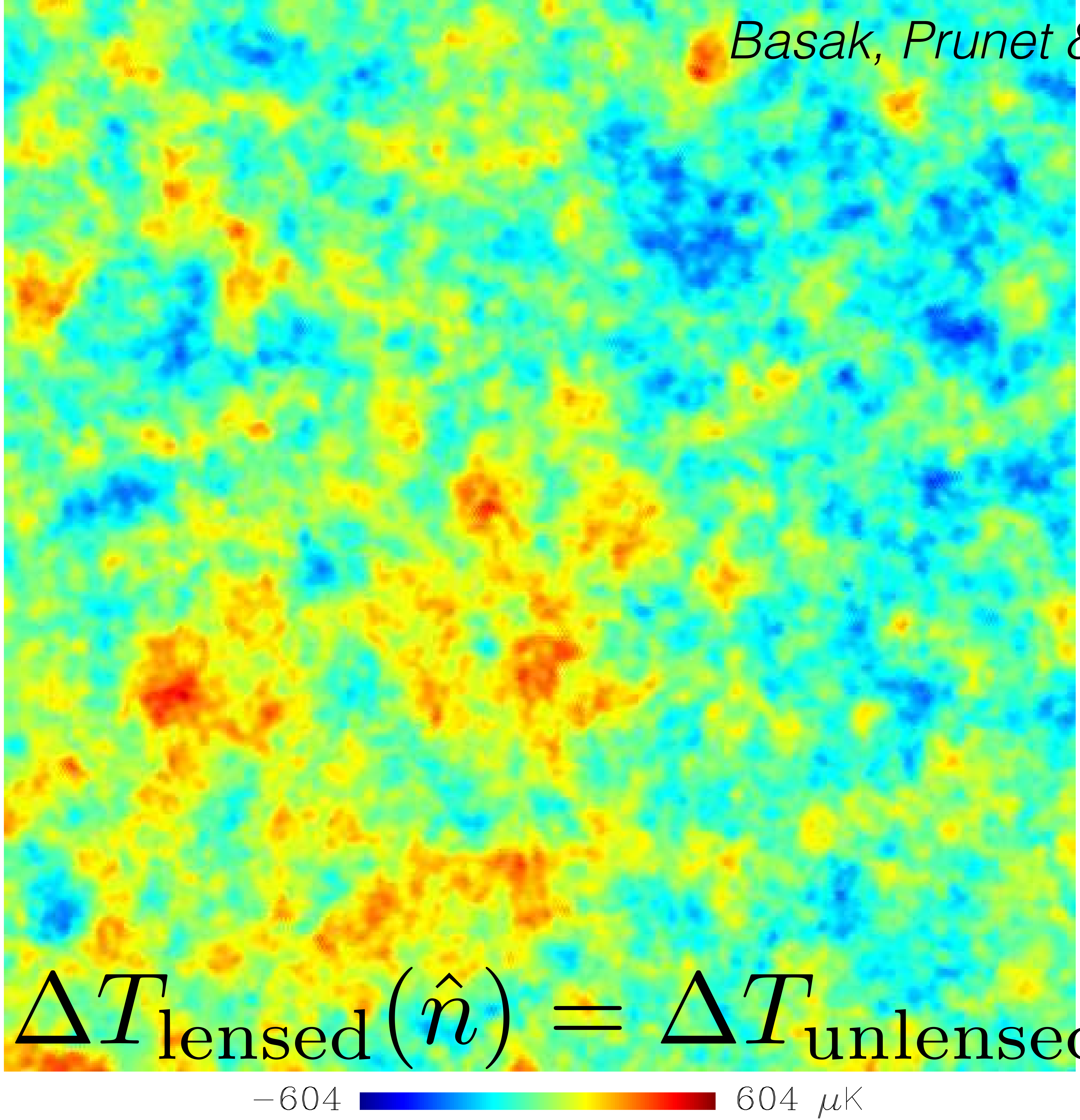

$$\Delta T_{\rm lensed}(\hat{n}) = \Delta T_{\rm unlensed}(\hat{n} + \mathbf{d})$$

Gravitational lensing effect on the CMB

Deflection angle and the “lens potential”

$$\Delta T_{\rm lensed}(\hat{n}) = \Delta T_{\rm unlensed}(\hat{n} + \mathbf{d})$$

- The vector “**d**” is called the *deflection angle*. For the scalar perturbation, we can write **d** as a gradient of a scalar potential (like the electric field):

$$\mathbf{d} = \frac{\partial \psi}{\partial \hat{n}}$$

with

$$\psi(\hat{n}) = -\int_0^{r_L} dr\ \frac{r_L - r}{r_L r}(\Phi + \Psi)(r, \hat{n}r)$$

r_L: the comoving distance from the observer to the last scattering surface