ANALYTICAL MODELS OF LSS

Marcello Musso

Scope and motivation

Model the statistical properties of LSS ... analytically! :)
Most successfully used to predict abundance and correlation properties (mass function and bias) of DM halos. But can (and have been) used also for voids, filaments, ionized bubbles, primordial BH's ...
Similar models are used in very different contexts like quantitative finance (optimal selling time) and biology (birth-death processes, spread of disease...). Anything that involves the probability that a stochastic system reaches a state, or accumulates a given properties in a sufficient amount, ~~that~~ (threshold), that triggers some desired (or feared...) event.
In Cosmology, typically the probability of regions to be dense enough to collapse.

Analytically. Because:
*) easy and fast complementary approach
*) simulations are not always accurate enough (~~ver~~ e.g. very large masses or very high redshifts)
*) provide physically motivated fitting formulae
*) physical insight on the role of the various parameters
*) suggest what to look for in a simulation (e.g. halo finder)
*) semi-analytical models (e.g. for reionization)

Outcome Mass function and bias, of course, but also velocity distribution, alignments, spin, assembly history. Much ongoing effort in the field to predict the profile, and mass-concentration relations.

OUTLINE

1) Non-linear models of gravitational evolution ("The threshold")
   - beyond PT – spherical collapse
   - triaxial models
   - angular momentum, vorticity, velocity dispersion
   - an "agnostic" fit

2) Mass fraction in critically overdense regions ("Excursion sets")
   - cloud-in-cloud problem, first crossing, upcrossing
   - correlated vs uncorrelated steps
   - moving barriers, stochastic barriers
   - methods of solution

3) From mass fractions to number densities ("Peak theory")
   - statistics of extrema of random fields
   - peaks vs upcrossing
   - Excursion sets of peaks (ESP)
   - the good, the bad, the ugly

4) What's next?
   - Assembly bias, velocity distribution, accretion history / concentration, Profile?

1) The threshold (modelling the physics)

1.1) Spherical collapse

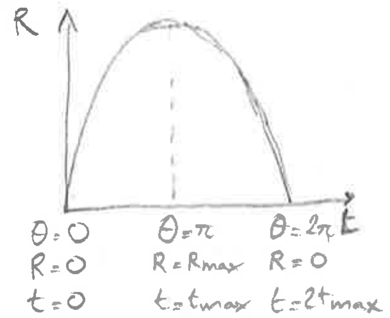
Exact solution with spherical symmetry, extending pert. theory into the non-linear regime. In EdS ($\Omega = 1$)

$$\ddot{R} = -\frac{GM}{R^2} \Rightarrow \frac{\dot{R}^2}{2} - \frac{GM}{R} = E \qquad \left(M = \frac{4\pi}{3} \bar{\rho}_{in} (1+\delta_{in}) R_{in}^3 \right)$$

For $E < 0$, same as Universe with positive curvature.

Parametrical solution (cycloid):

$$\frac{R}{R_{max}} = \frac{1-\cos\theta}{2} \qquad \frac{t}{t_{max}} = \frac{\theta - \sin\theta}{\pi}, \quad \text{with } \left(\frac{R_{max}}{2}\right)^3 = GM \left(\frac{t_{max}}{\pi}\right)^2$$



for given M, family of solutions parametrized by t_{max}. Where is the second free parameter?

$$E = -\frac{GM}{R_{max}} = \frac{\dot{R}_{in}^2}{2} - \frac{GM}{R_{in}}, \quad \text{so } R_{max} \text{ depends on } R_{in} \text{ and } \dot{R}_{in}$$

Expanding the cycloid for $\theta \ll 1$, one eventually gets

$$\rho = \frac{3M}{4\pi R^3} = \frac{3\pi}{32 G t_{max}^2} \left(\frac{R_{max}}{R}\right) \simeq \frac{1}{6\pi G t^2} \left[1 + \frac{3}{20} \left(\frac{6\pi t}{t_{max}}\right)^{2/3} \right]$$

But $\bar{\rho}_{EdS} = \frac{1}{6\pi G t^2}$, so $\delta_{ein} \equiv \frac{3}{20} \left(\frac{6\pi t}{t_{max}}\right)^{2/3}$

Zeldovich initial conditions:

$$R_{in} = a_{in}(x_{in} - \psi) \qquad \frac{\dot{\psi}}{\psi} = H \quad (\Omega = 1)$$

$$\dot{R}_{in} = H_{in}(R_{in} - a_{in}\psi)$$

$$\delta_{in} = \left(\frac{a_{in} x_{in}}{R_{in}}\right)^3 - 1 \simeq 3\frac{\psi}{x_{in}} \simeq \frac{3 a_{in}\psi}{R_{in}}$$

Cons. of energy: $\frac{R_{in}}{R_{max}} = 1 - \frac{R_{in}\dot{R}_{in}^2}{2GM} = 1 - \frac{(R_{in} - a_{in})^2}{R_{in}^2 (1+\delta_{in})} \approx \frac{5}{3} \delta_{in}$

and $\delta_{ein} = \frac{3}{20} \left(\frac{6\pi t}{t_{max}}\right)^{2/3} = \frac{3}{5} (1+\delta_{in})^{1/3} \frac{R_{in}}{R_{max}} \underbrace{\left(\frac{t}{t_{in}}\right)^{2/3}}_{a/a_{in}} \approx \delta_{in} \frac{a}{a_{in}}$

matches perturbation theory

At turnaround: $\rho_{max} = \dfrac{3M}{4\pi R_{max}^3} = \dfrac{3}{32\pi G}\left(\dfrac{\pi}{t_{max}}\right)^2 = \dfrac{9\pi^2}{16}\,\bar{\rho}_{EdS}(t_{max})$

$$\Rightarrow \quad 1+\delta_{max} = \frac{9\pi^2}{16} \approx 5.55$$

while $\delta_{lin}(t_{max}) = \dfrac{3}{20}(6\pi)^{2/3} \approx 1.062$

Virialization At $t = 2t_{max}$, formally $\delta = \infty$. In practice, from the virial theorem $U = -2K$, one gets $\dfrac{GM}{R_{vir}} = \dot{R}_{vir}^2$

$$E = -\frac{GM}{R_{max}} = \frac{\dot{R}_{vir}^2}{2} - \frac{GM}{R_{vir}} = -\frac{GM}{2R_{vir}} \quad \Rightarrow \quad \boxed{R_{vir} = \frac{R_{max}}{2}}$$

Therefore $1+\delta_{vir} = \dfrac{\rho_{vir}}{\bar{\rho}_{EdS}(t_{vir})} = \underbrace{\dfrac{\rho_{max}}{\bar{\rho}_{EdS}(t_{max})}}_{9\pi^2/16}\underbrace{\left(\dfrac{R_{max}}{R_{vir}}\right)^3}_{8}\underbrace{\left(\dfrac{t_{vir}}{t_{max}}\right)^2}_{4} \approx 178$

while $\delta_{lin}(2t_{max}) = \dfrac{3}{20}(12\pi)^{3/2} \approx 1.686 \equiv \delta_c$

Including Λ, one gets a higher virial density

Spherical shells

Same equations also for an eterogeneos spherically symm. distribution of matter.

Each shell with $R_{in,i}$ and $\delta_{in,i}$ virialises at

$$R_{v,i} = \frac{R_{max,i}}{2} = \frac{3}{10}\frac{R_{in,i}}{\delta_{in,i}}, \quad \text{when } \delta_{in,i}\left(\frac{t}{t_{in}}\right)^{2/3} = \delta_c$$

that is for $t_{vir,i} = \left(\dfrac{\delta_c}{\delta_{in,i}}\right)^{3/2} t_{in}$.

Internal denser shells virialise first, and reach

$$\rho_i = \frac{3M_i}{4\pi R_{vir,i}^3} = \bar{\rho}_{in}(1+\delta_{in,i})\left(\frac{2R_{in,i}}{R_{max,i}}\right)^3 \approx \bar{\rho}_{in}\left(\frac{10\,\delta_{in,i}}{3}\right)^3$$

The slope of the final profile is

$$\frac{d\rho}{dR} = \frac{d\rho/dR_{in}}{dR/dR_{in}} = \frac{(d\rho/d\delta_{in})(d\delta_{in}/dR_{in})}{\frac{3}{10}\left(\frac{1}{\delta_{in}} - \frac{R_{in}}{\delta_{in}^2}\frac{d\delta_{in}}{dR_{in}}\right)} = -\frac{3}{R}\rho X$$

where $X \equiv \dfrac{-(R_{in}/\delta_{in})\,d\delta_{in}/dR_{in}}{1-(R_{in}/\delta_{in})\,d\delta_{in}/dR_{in}} = \dfrac{-(R/\delta_c)\,d\delta_R/dR}{1-(R/\delta_c)\,d\delta_R/dR}$ (linearly evolved)

For steep initial profiles ($X \to 1$) reproduces the outer NFW slope

1.2) Triaxial models (Bond & Myers 96, Chandrasekhar '60)

Spherical collapse is the monopole term of the multipole expansion of the potential inside a sphere.
If the sphere is deformed, ellipsoidal harmonics ~~shel~~ (rather than spherical) should be considered.
The first 4 of them give homogeneous ellipsoidal collapse.
The acceleration is still linear in the distance:

$$\ddot{R} = -\frac{4\pi G}{3}\bar{\rho}(1+\delta)R \longrightarrow \ddot{a}_i = -4\pi G\bar{\rho}\left[\frac{1+\delta}{3} + \frac{\beta_i\delta}{2} + \lambda_i^{ext}\right]a_i.$$

$$\text{where } \beta_i \doteq \left(\int_0^\infty d\tau \frac{1}{a_i^2+\tau}\frac{1}{[(a_1^2+\tau)(a_2^2+\tau)(a_3^2+\tau)]^{1/2}}\right) - \frac{2}{3}$$

are the eigenvalues of the shear due to the ellipticity, and λ_i^{ext} those of the ~~shea~~ external shear. $(\sum_i \beta_i = \sum_i \lambda_i^{ext} = 0)$

$$\boxed{\begin{array}{c} a_1 > a_2 > a_3 \\ \Updownarrow \\ \beta_1 < \beta_2 < \beta_3 \\ \text{(opposite order)} \end{array}} \qquad \boxed{\lambda_1^{ext} > \lambda_2^{ext} > \lambda_3^{ext}}$$

Initially spherical perturbation: (BM'96, ST'99)
λ_1^{ext} (max. compression) produces a_3 (shortest axis) and aligns with β_3. Internal and external shear add up. Threshold grows.
Need larger δ_{in} to compensate.

Initial ellipsoid in external sheare ( )
If the initial shape aligns a_1 with λ_1^{ext} and a_3 with λ_3^{ext}, then β_i compensate the λ_i^{ext}. If $\frac{\beta_i\delta}{2} = -\lambda_i^{ext}$, linear evolution preserves the axis ratio (like spherical collapse).
Non-linearly, β_i still dominates, but the effect is weaker.
Threshold grows less. It is lowest when the initial ellipticity is such that the 3 axes virialize at the same time (according to the tensor virial theorem).
The calculation has not been done yet.

1.3) Angular momentum, vorticity, velocity dispersion.

· Spherical models $\frac{\dot{R}^2}{2} - \frac{GM}{R} + \frac{L^2}{2R^2} = E$

Still admits a cycloidal solution. But $R \to 0$ as $t \to 0$ implies $L = 0$, or ~~viol~~ non-conservation of L.
Not self-~~consistent~~ in spherical systems.
(Nusser 01, Mo et al 15)

Triaxial models

The kinetic energy is

$$K = \frac{1}{2}\sum_i \dot{a}_i^2 + \frac{1}{2}\sum_i \left[(\lambda_i^2 + \omega_i^2)(a_j^2 + a_k^2) - 2 a_j a_k \lambda_i \omega_i \right]$$

λ_i and ω_i represent rotation of the entire ellipsoid and of particles within the shell.
Induce angular momentum (created by external torques) and circulation (conserved)
Typically this kinetic energy slows down the shortest axis most. Counters the non-linear shear due to ellipticity.
Unfortunately, non comprehensive description has been found

1.4) Simulations show a scatter in δ_{in} of protohalos (e.g. Robertson et. al 2008).
The scatter is roughly log. normal, with a mean compatible with the mean ~~of~~ ellipsoidal barrier of Sheth et al (01), or even with a linear fit.

$$\langle \delta_c \rangle = \alpha + \beta \sigma$$

2) EXCURSION SETS (Bond et al. 1991)

Statistics of halos $\iff$ stats of "dense enough" regions

ANSATZ "Halos form out of the LARGEST patches whose initial mean density (linearly evolved to today) exceeds δ_c

That is, spherical regions in a given volume bin that:

*) have critical mean density

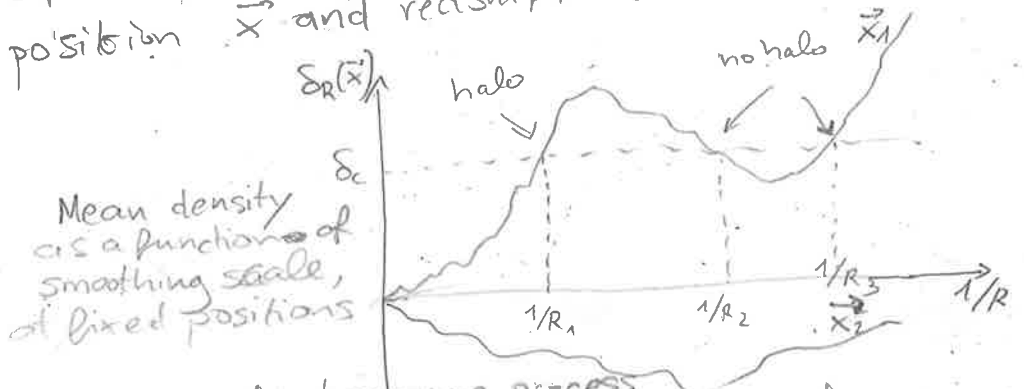
$$\delta_R(\vec{x}) \equiv \frac{1}{V}\int_{y<R} d^3y \, \delta_{in}(\vec{x}+\vec{y}) = \delta_c$$

*) are not included in larger such regions (cloud-in-cloud)

$$\delta_{R'}(\vec{x}) \leq \delta_c \quad \text{for} \quad R' > R$$

Here, $\delta_c = 1.686$ can be replaced by $b(R,z) = \delta_c(R)/D_+(z)$ to identify halos with a different barrier at another redshift.

If satisfied, a halo of mass $M = \frac{4\pi}{3}\bar{\rho}_{in} R^3$ forms at position $\vec{x}$ and redshift z.



One halo of mass $\propto R_1^3$ at $\vec{x}_1$

No halo at $\vec{x}_2$

2.1) Halos as a first passage process

Each $\vec{x}$ sees a different realization of

$$\delta_R(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} W(kR)\, \delta_{\vec{k}} \longrightarrow \text{Fourier modes}$$

a random trajectory for $\delta_R(\vec{x})$ as a function of R, with variance

$$s(R) \equiv \langle \delta_R^2(\vec{x}) \rangle = \int dk \frac{k^2 P(k)}{2\pi^2} W^2(kR)$$

Compute the PDF of first passage through δ_c between R and $R+\Delta R$ [$f(R)$, or equivalently $f(s)$ or $f(M)$]

$$\boxed{f(s) \longleftrightarrow \frac{dn}{dM} = \frac{\bar{\rho}}{M}\frac{ds}{dM} f(s)}$$

mass function 1st X-ing PDF

2.2) Correlated steps

Because in general $\langle \frac{d\delta_R}{dR} \frac{d\delta_{R'}}{dR'} \rangle \neq 0$, the steps are CORRELATED.
The first passage problem is difficult. Formally

$$f(s) = \frac{1}{\Delta s} \langle \vartheta(\delta_N - \delta_c)\,\vartheta(\delta_c - \delta_{N-1}) \dots \vartheta(\delta_c - \delta_1) \rangle$$

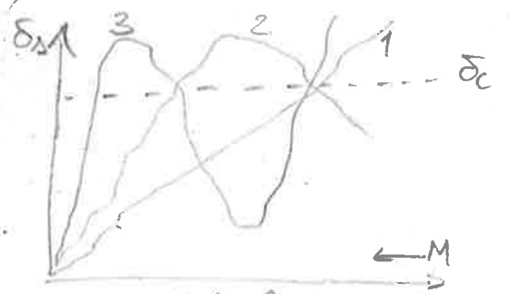
in the limit $\Delta s \to 0$, $N \to \infty$ and $s = N\Delta s$ = constant. Steps are uncorrelated if e.g.
But the limit is difficult.
$W(kR) = \vartheta(1 - kR)$. But for a Top-Hat smoothing, $W(kR) = 3\frac{j_1(kR)}{kR}$

2.3) Crossing, first crossing, upcrossing

Press and Schechter (74) did $f_{PS} = \langle \frac{d}{ds}\vartheta(\delta_s - \delta_c) \rangle = \frac{d}{ds}\int_{\delta_c}^{+\infty} d\delta\, p(\delta_s)$.
Easy, but wrong. It gives the probability of ANY crossing, not of the first. Also, $\int_0^\infty ds\, f_{PS}(s) = \frac{1}{2}$. They multiplied by 2 by hand to normalize the mass function.
Exact solution found by Chandrasekhar 43

$$f(s) = 2\frac{d}{ds}\int_{\delta_c}^{+\infty} d\delta\, p(\delta_s) = -\frac{d}{d\delta_c}\frac{e^{-\delta_c^2/2s}}{\sqrt{2\pi s}}$$

for constant δ_c and uncorrelated steps.
Bond et al. 91 used this to "justify" PS74.



1: most likely
2: less likely
3: very unlikely

Yet, steps are correlated.
Bond et al already noticed that
$f_{PS}(s)$ gets the correct $s \to 0$ limit.
A better approximation is

$$f_{up}(s) = \langle \underbrace{\frac{d}{ds}\vartheta(\delta_s - \delta_c)}_{(d\delta/ds)\delta_D(\delta_s - \delta_c)}\, \vartheta\!\left(\frac{d\delta}{ds}\right) \rangle$$

$$= p(\delta_s = \delta_c)\int_0^{+\infty} d\!\left(\frac{d\delta}{ds}\right)\left(\frac{d\delta}{ds}\right) p\!\left(\frac{d\delta}{ds}\,\Big|\,\delta_s = \delta_c\right)$$

enforcing upcrossing of δ_c at s (Musso and Sheth 12).
It fails to remove walks with earlier crossings but these are very unlikely at small s (large M)

$$f_{up}(s) = \underbrace{\frac{\delta_c}{2s}\frac{e^{-\delta_c^2/2s}}{\sqrt{2\pi s}}}_{f_{PS}(s)}\left[\frac{\mathrm{erf}(\Gamma\delta_c/\sqrt{2s}) + 1}{2} + \frac{e^{-\frac{\Gamma^2\delta_c^2}{2s}}}{\Gamma\delta_c/\sqrt{s}\,\sqrt{2\pi}}\right]$$

$$\Gamma^2 = \frac{\gamma^2}{1-\gamma^2} \qquad \gamma^2 = \frac{\langle \delta\, d\delta/ds \rangle^2}{\langle \delta^2 \rangle \langle (d\delta/ds)^2 \rangle}$$

Typically, $\Gamma \sim \frac{1}{2} \div 1$, and f_{PS} is recovered as $s \to 0$.

The result however still holds for moving barriers and non-Gaussian processes. (Musso and Sheth 2014)

Generic case: (Volterra equation for $f(s)$)

$$f_{PS}(s) = \alpha f(s) + \int_0^s dS\, f(S) \underbrace{\frac{d}{ds} \langle \vartheta(\delta_s - \delta_c) \mid 1st\ S \rangle}_{\text{conditional mean at } s \text{ given first crossing at } S}$$

where $\alpha = 1$ for correlated steps and $1/2$ for uncorrelated.

The conditional mean can be approximated with

$$\delta_s' = \frac{d\delta_s}{ds}$$

$$\frac{d}{ds} \langle \vartheta(\delta_s - \delta_c) \mid up\ S \rangle = \frac{1}{f_{up}(S)} \int_{\delta_c}^{+\infty} d\delta_s \int_0^{+\infty} d\delta_S' \frac{d}{ds} p(\delta_s, \delta_S = \delta_c, \delta_S')$$

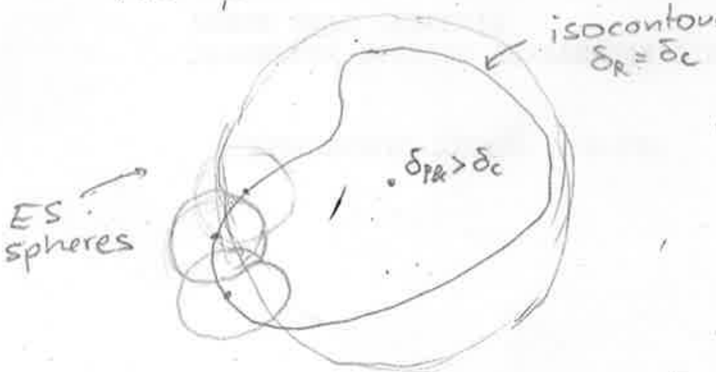
$$= \frac{1}{f_{up}(S)} \int_0^{+\infty} d\delta_S' \left[\frac{1}{2}\frac{\partial}{\partial \delta_s} + \langle \delta_s' \delta_S \rangle \frac{\partial}{\partial \delta_S} + \langle \delta_s' \delta_S' \rangle \frac{\partial}{\partial \delta_S'} \right] p(\delta_s = \delta_c, \delta_S = \delta_c, \delta_S')$$

which is analytical, ~~and~~ correctly normalized and extremely good at all scales (Musso and Sheth 2013)

2.4) Applications to Cosmology. Moving / stochastic barrier

The above gives very good first-crossing pdf, but not a good mass function. There are flaws in the ES ansatz. Using a stochastic barrier to mimic the scatter of δ_c (Maggiore and Riotto '10, Corasaniti and Achitouv '11) gives good results. It amounts to study the process $\delta_s - b$ with variance $\langle \delta_s^2 \rangle + \langle b^2 \rangle = (1+d)s$, that is to replace $\delta_c \to \delta_c/\sqrt{1+d}$ and try to recover the large mass tail $\sim \sqrt{0.7}\,\delta_c$ of the ST mass function.

This works on average, but does not halo by halo. Halos do not form at generic locations, but near peaks.



ES assigns the mass ~~of~~ in the small spheres to halos of mass $\propto R^3$, rather than ~~as~~ the mass in the large sphere to halos of larger mass

3) From mass fraction to number density
PEAK THEORY (BBKS'86)

ANSATZ Halos form at peaks of critical heigt in the density field (smoothed on the corresponding mass scale)

At the peak, $\vec{\nabla}\delta_R = 0$. In the vicinity, the physical volume is

$$d^3x = \underbrace{|\vec{\nabla}\vec{\nabla}\delta_R|}_{\text{Jacobian of the change of coordinate } \vec{x} \to \vec{\nabla}\delta_R} d^3(\nabla\delta_R)$$

The point process for peaks of height $\nu_c = \delta_c/\sqrt{S}$ is

$$\det(\zeta)\,\vartheta(\zeta_3)\,\delta_D(\nu-\nu_c)\,\delta_D(\vec{\eta})$$

where $\vec{\eta} \equiv \vec{\nabla}\delta_R$, $\nu \equiv \delta/\sqrt{S}$, $\zeta \equiv -\vec{\nabla}\vec{\nabla}\delta_R$ is the negative Hessian of δ_R ~~with~~ and $\zeta_1 > \zeta_2 > \zeta_3$ its eigenvalues.
The Hessian is negative definite at a peak, hence $\zeta_3 > 0$.
The number density is

$$\frac{dn_{pk}}{dM} = \frac{d\nu_c}{dM} \langle \det(\zeta)\,\vartheta(\zeta_3)\,\delta_D(\nu-\nu_c)\,\delta_D(\vec{\eta})\rangle$$

$$= \frac{d\nu_c}{dM}\, p(\nu=\nu_c)\, p(\vec{\eta}=0)\, \underbrace{\langle \det(\zeta)\,\vartheta(\zeta_3) \,|\, \nu=\nu_c\rangle}_{\text{conditional mean given } \nu=\nu_c \text{ since } \nu \text{ correlates with } \mathrm{tr}(\zeta)}$$

uncorrelated variables

$$= \underbrace{\frac{d\nu_c}{dM}\frac{e^{-\nu_c^2/2}}{\sqrt{2\pi}}}_{\frac{dS}{dM} f_{PS}(S)} \frac{1}{V_*} \langle F(x)\,\vartheta(x) \,|\, \nu=\nu_c\rangle$$

where $x = \frac{\mathrm{tr}(\zeta)}{\sqrt{\langle \mathrm{tr}(\zeta)^2\rangle}}$ normalized peak curvature,

$V_* = \left(6\pi \frac{\langle \vec{\eta}\cdot\vec{\eta}\rangle}{\langle(\mathrm{tr}(\zeta))^2\rangle}\right)^{3/2}$ characteristic peak volume

$F(x) \sim x^3 - 3x$ at $x \gg 1$, full expression in BBKS

$$\langle F(x)\,\vartheta(x) \,|\, \nu=\nu_c\rangle = \int_0^\infty dx\, F(x) \frac{e^{-\frac{(x-\gamma\nu_c)^2}{2(1-\gamma^2)}}}{\sqrt{2\pi(1-\gamma^2)}} \sim (\gamma\nu_c)^3 - 3\gamma\nu_c$$

and $\gamma = \langle x\nu\rangle = \frac{\langle \delta\, \mathrm{tr}(\zeta)\rangle}{\sqrt{S\langle(\mathrm{tr}(\zeta))^2\rangle}} = \frac{\langle\vec{\eta}\cdot\vec{\eta}\rangle}{\sqrt{S\langle(\mathrm{tr}(\zeta))^2\rangle}}$

that is $\frac{dn_{pk}}{dM} \approx \frac{d\nu_c}{dM}\left(\frac{\langle\vec{\eta}\cdot\vec{\eta}\rangle}{6\pi S}\right)^{3/2} \nu_c^3 \frac{e^{-\nu_c^2/2}}{\sqrt{2\pi}}$

Peak theory involves the spectral moments

$$\langle \vec{\eta}\cdot\vec{\eta}\rangle = \int dk \, \frac{k^2 P(k)}{2\pi^2} \, k^2 W^2(kR) \equiv \sigma_1^2$$

$$\langle [\mathrm{Tr}(\zeta)]^2\rangle = \int dk \, \frac{k^2 P(k)}{2\pi^2} \, k^4 W^2(kR) \equiv \sigma_2^2$$

besides $\sigma_0^2 \equiv s$. For TH filter, $\sigma_2 = \infty$, model is ill defined (*)
Usually, the Gaussian filter is used:

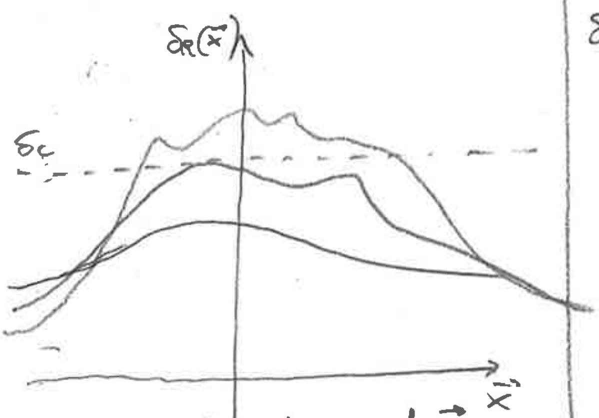
$$W_G(kR) = e^{-\frac{k^2R^2}{2}} \quad \longleftrightarrow \quad W_{TH}(kR) = \frac{3 j_1(kR)}{kR}$$

With W_G, $\frac{d\delta_R}{dR} = R\nabla^2\delta_R = -R\,\mathrm{Tr}(\zeta)$. Peak curvature and ES slope coincide

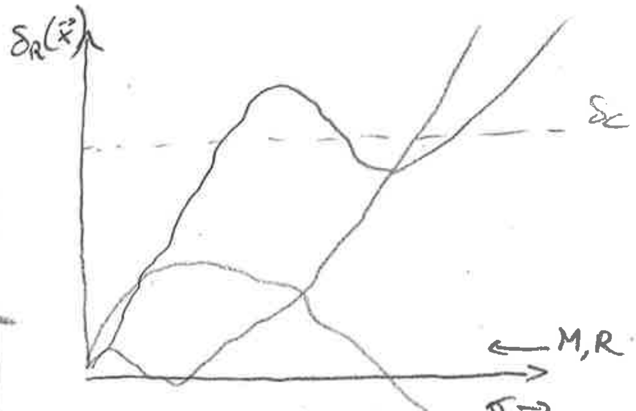
(*) But note that σ_2 drops out in the large mass limit

3.2) Peaks vs upcrossing

	Peaks	Upcrossing
Variables:	ν, $x \equiv -\nabla^2\delta_R/\sqrt{\langle(\nabla^2\delta_R)^2\rangle}$ (peak curvature)	ν, $x \equiv -\frac{d\delta_R/dR}{\sqrt{\langle(d\delta_R/dR)^2\rangle}}$ (slope)
Change of coordinates:	$\vec{x} \to \vec{\nabla}\delta_R$, $d^3(\nabla\delta) = \lvert\det(\vec{\nabla}\vec{\nabla}\delta)\rvert\, d^3x$	$dM \to ds$ $\quad d\delta_s = \frac{ds}{dM}\left(\frac{d\delta}{ds}\right) dM$
Constraint:	$-\vec{\nabla}\vec{\nabla}\delta_R$ is pos. def.	$\frac{d\delta}{ds} > 0$ $\quad \left(\frac{d\delta}{ds} > \frac{db}{ds}\right)$
Modulation factor:	$\langle F(x)\,\vartheta(x) \,\vert\, \nu = \nu_c\rangle$	$\frac{\langle x\,\vartheta(x) \,\vert\, \nu = \nu_c\rangle}{\gamma \nu_c}$
Characteristic volume:	$1/V_*$	$\bar{\rho}/M = 1/V$



$\delta_R(\vec{x})$ as a function of $\vec{x}$ for fixed R



$\delta_R(\vec{x})$ as a function of $\vec{R}$ for fixed $\vec{x}$

3.3) Peak theory gets the peaks right, but misses the cloud in cloud problem.
Combining the two : Excursion sets of Peaks (ESP)

ANSATZ Halos form at peaks that are upcrossing, that is peaks of critical height when smoothed on the scale of the halo mass, but below threshold on larger scales.
Linear stochastic ~~barrier~~ threshold $b = \delta_c + \beta\sigma$, where β is log-normal (Robertson et al.)
Gaussian filter. (with radius chosen to match TH)

$$\underbrace{\left(\frac{d\delta}{d\sigma} - \frac{d\beta}{d\sigma}\right)\vartheta\left(\frac{d\delta}{d\sigma} - \frac{d\beta}{d\sigma}\right)\delta_D(\delta - b)}_{ES}\quad \underbrace{|\det(\zeta)|\,\vartheta(\zeta_3)\,\delta_D(\vec{\eta})}_{P}$$

$$\left(x = \gamma\frac{d\delta}{d\sigma}\right) \qquad \frac{dn_{ESP}}{dM} = \underbrace{\frac{d\sigma}{dM}\frac{\nu_c}{\sigma}\frac{e^{-(\nu_c+\beta)^2}}{\sqrt{2\pi}}}_{f_{PS}(\sigma)}\ \underbrace{\frac{1}{V_*}\int_{\gamma\beta}^{+\infty} dx\, \frac{x-\gamma\beta}{\gamma\nu}\, F(x)\, p(x|\nu = \nu_c+\beta)}_{\frac{\langle (x-\gamma\beta)\vartheta(x-\gamma\beta)F(x)|\nu=\nu_c+\beta\rangle}{\gamma\nu}}$$

$$= \frac{\langle x\,\vartheta(x) F(x+\gamma\beta)|\nu = \nu_c\rangle}{\gamma\nu}$$

Paranjape and Sheth '12
Desjacques, Paranjape, Sheth '13

Mass function within 5% from e.g. Tinker et al. 08
Large scale bias well within error (Lazeyras and Schmidt 15)

3.4) ESP : the good, the bad and the ugly

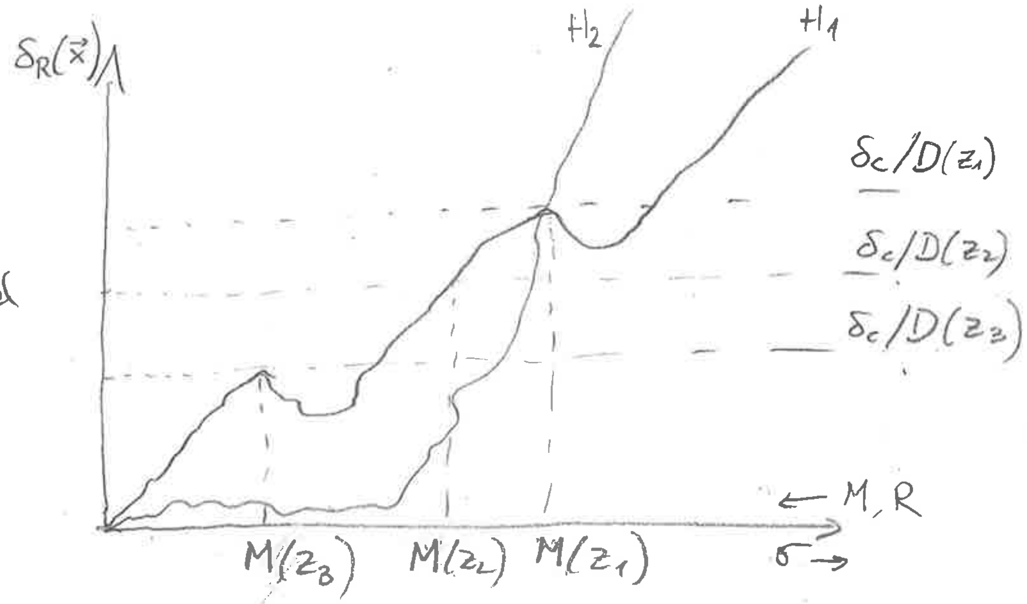
The good: it works! Not only mass function, but also correct halo bias

The bad: upcrossing only. Full first crossing may spoil the agreement. Shear but no ellipticities (misalignments)
The scatter of β is within 30% from measures
Not all halos form from peaks (at smaller masses)

The ugly: mixed filters, reason of β scatter is obscure, not clear how to improve at smaller masses

4) Accretion history and assembly bias

Lacey and Cole 93

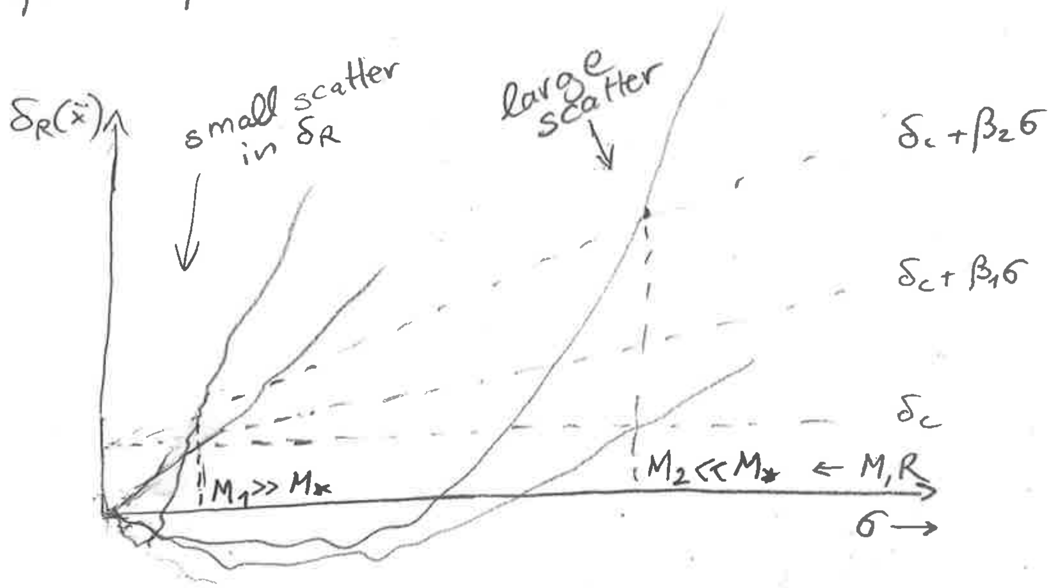


As $\delta_c/D(z)$ drops, the trajectory gives $M(z)$

$$\frac{d}{dt}\frac{\delta_c}{D} = - H_D \frac{\delta_c}{D} = \frac{d}{dt}\delta_R(x) = \frac{d\delta_R}{dR}\frac{dR}{dM}\dot{M}$$

$$\Rightarrow \dot{M} = - H_D \frac{4\pi \bar{\rho}_{in} R^2}{D} \frac{1}{d\delta_R/dR}$$

Steeper slope $\longleftrightarrow$ lower $\dot{M}$, higher concentration



β_2 selects steeper slopes than $\beta = 0$.
At large mass (small scatter in $\delta_R \approx \delta_c$) steeper slope at same mass means lower density environment.
At small mass (large scatter in δ_R) it's the opposite.
But what is exactly β??