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Exercise sheet 9

Exercise 9 - 1

Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous process that follows the following stochastic differential equation: $\mathrm{d}s_t/\mathrm{d}t = a\,\xi_t - b\,s_t$. Here, a and b are non-negative constants and ξ is a stationary, white random field of unit variance, i.e., $\langle \xi_t \xi_r \rangle_{(\xi)} = \delta(t-r)$. Let $S = \left\langle s s^\dagger \right\rangle_{(s)}$ be the signal covariance.

a) Assume for the moment $s_0 = 0$, $b = 0$, and $t, t' \geq 0$. Calculate $S_{tt'}$. Use S_{tt} to argue why this so-called Wiener process is a frequently used model for diffusive motion of a particle (2 points).

b) Calculate the signal power spectrum for any a and b being non-negative. Try to explain with words why the spectral normalization and the appearing characteristic frequency depend on a and b the way they do (2 points).

Hint: Transform the differential equation to Fourier space.

Exercise 9 - 2

Consider a field $\varphi \equiv \varphi_{x,t}$ with a time domain and a one-dimensional spatial domain, following the stochastic differential equation

$$\partial_t \varphi = \kappa \Delta_x \varphi + \xi \tag{1}$$

with independent Gaussian noise contribution ξ of unit variance and constant κ.

a) Calculate the auto-correlation $\langle \varphi^*_{(\omega,k)} \varphi_{(\omega',k')} \rangle$ in its full harmonic domain (temporal and spatial Fourier basis) (2 points).

b) Perform the inverse Fourier transformation in the time domain and give the expression of this auto-correlation in time - spatial frequency domain $\langle \varphi^*_{(t,k)} \varphi_{(t',k')} \rangle$ using the Residue theorem. (2 points)

Exercise 9 - 3

Be s_t a stock price as a function of time t, which is believed to follow the Black-Scholes model

$$\frac{1}{s_t}\frac{\mathrm{d}s_t}{\mathrm{d}t} = \mu + \sigma\,\xi_t \tag{2}$$

with known and temporary constant drift rate μ and volatility σ. The stochastic field ξ is a white noise Gaussian random field with unit dispersion.

a) Find a variable $x_t = x(s_t)$ which follows a drift free Wiener process ($\dot{x}_t = \sigma\,\xi_t$). Give its probability distribution $\mathcal{P}(x)$ (1 point).

b) Use $\mathcal{P}(x)$ to calculate the expected (actually the average) future stock price $\bar{s}_t = \langle s_t \rangle_{(s_t|s_{t_0})}$. Why does this depend on σ^2 (2 points)?

c) At time $t_0 = 0$ a bank has given the guarantee to a customer (for a fee, of course) that she can buy one of these stocks at a specified later time t for at most the fixed price K, independent of its actual price s_t then. Calculate the expected (average) amount of money L the bank will need to invest at "stroke" time t to fulfill its obligation from this "European call option" (2 points).

d) Calculate now the minimal price C the bank has to ask for this call option in order not to loose any money on average. Consider that the bank has access to a risk-free interest rate r, which amplifies any amont of money C received at time $t_0 = 0$ to $C\,\mathrm{e}^{r\,t}$ at a later time t. Find the most compact notation for this Black-Scholes option pricing formula by (i) using the property $1-\mathrm{erf}(x) = \mathrm{erf}(-x)$ for $\mathrm{erf}(x) = \int_{-\infty}^{x} \mathcal{G}(x,1)$ and (ii) expressing the current stock price through $S = \bar{s}_t\,\mathrm{e}^{-r\,t}$, the expected future value of the stock for the bank at present time (a good brain twister, isn't it?). Compare your result to the famous Black-Scholes formula

$$C = S\,\mathrm{erf}(d_1) - K\,\mathrm{e}^{-rt}\,\mathrm{erf}(d_2). \qquad (3)$$

Argue why in the Black-Scholes world $S = s_{t_0}$ is assumed (2 points).

e) **[Only for future stock traders]** Calculate the so-called "Greeks" (e.g., see Wikipedia), the derivatives of the option price C with respect to the various model parameters (μ, σ^2, t, r, K, and S).

This exercise sheet will be discussed in the tutorials.
group A, Wednesday 16:00 - 18:00 Theresienstr. 39 - room B 101,
group B, Thursday, 10:00 - 12:00, Geschw.-Scholl-Pl. 1 (B) - room B 015
group C, Thursday, 16:00 - 18:00, Theresienstr. 37 - room A 449
backup slot, Wednesday, 8:00 - 10:00, Theresienstr. 37 - A 450
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