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Exercise sheet 3

Exercise 3 - 1: Bit flip

Information is transmitted digitally as a binary sequence know as bits. However, noise on the channel corrupts the signal, in that a digit transmitted as *digit* is received as *as its complement* with probability $(1-\alpha)$. It has been observed that, across a large number of signals to be transmitted, the 0s and 1s are transmitted in the ratio $3:4$
Given that one can only measure sequences of length three and the received data is 101, what is the probability distribution over transmitted signals? Derive an expression for the probability that the transmitted signal is also the received one. Assume that the signal selection and the noise are independent of each other (4 points).

Exercise 3 - 2

A sports game in which two players play for a point in each round, the game is won by the first player who leads by two points. Player A always has the probability θ to win a round. What are his chances?

a) State the three probabilities (as a function of θ) that after two rounds player has A won, has lost, or has to continue the game. Call them in the following w, l, and c, respectively (1 points).

b) State now the three probabilities (as a function of w, l, and c) that after (at most) four rounds player A has won, has lost, or that the game still continues (1 points).

c) State now the three probabilities that after (at most) an infinite numbers of rounds player A has won, lost, or that the game still continues (2 points).

d) Now you hear that player A has won the game. What is the probability $P(n|W)$ (as a function of θ) that the game ended with round n? If you were not able to solve c) use the following:

$$P(W|n \leq \infty) = \frac{\theta^2}{1-2\theta(1-\theta)} \qquad P(L|n \leq \infty) = \frac{(1-\theta)^2}{1-2\theta(1-\theta)} \qquad P(C|n \leq \infty) = 0$$

(3 points)

Exercise 3 - 3

During World War II, Allied intelligence bureaus made sustained efforts to estimate the amount of German tank production. The statistical Ansatz to this problem is known as the German Tank Problem.

An intelligence bureau wants to estimate the number of opposing tanks n. Intelligence reports contain the serial numbers of spotted tanks. The data available to the data analyst are the number of spotted tanks k and the highest spotted serial number m. The intelligence bureau assumes that the serial numbers range from 1 to n and that they are randomly and uniformly distributed among the tanks in action.

Hint: You are allowed to use both, binomial coefficients and factorials, as you prefer.

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a) What is the likelihood of the highest observed serial number being m, assuming n tanks are in action and k have been spotted?

If you could not solve exercise **a)** you may use $P(m|n,k) \propto \frac{\binom{m-2}{k-2}+\binom{m-3}{k-3}}{\binom{n}{k}}$ (note, that this is not the correct result) for $1 \leq m \leq a$ and $P(m|n,k) = 0$ elsewhere. (3 points).

b) Assuming a flat prior between the number of observed tanks k and some maximum number Ω,

$$P(n) = \begin{cases} \frac{1}{\Omega} & \text{for } 1 \leq n < \Omega \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

derive the posterior $P(n|m,k)$ and take the limit $\Omega \to \infty$ (2 points) .

<u>Hint</u>: $\sum_{j=i}^{M} \frac{1}{\binom{j}{k}} = \frac{k}{k-1}\left(\frac{1}{\binom{i-1}{k-1}} - \frac{1}{\binom{M}{k-1}}\right) \iff \sum_{j=i}^{M} \frac{(j-k)!}{j!} = \frac{1}{k-1}\left(\frac{(i-k)!}{(i-1)!} - \frac{(M-k+1)!}{M!}\right)$ for $k \geq 2$

c) Calculate the posterior mean. What is the minimal amount of spotted tanks in order for the posterior mean to be finite (2 points)?

d) Plot the posterior for the number n of tanks given $k = 30$, $m = 250$, and $\Omega = 10000$ (optional).

This exercise sheet will be discussed during the exercises.
group A, Wednesday 16:00 - 18:00 Theresienstr. 39 - room B 101,
group B, Thursday, 10:00 - 12:00, Geschw.-Scholl-Pl. 1 (B) - room B 015
group C, Thursday, 16:00 - 18:00, Theresienstr. 37 - room A 449
backup slot, Wednesday, 8:00 - 10:00, Theresienstr. 37 - A 450
(to be used whenever Thursday is a vacation day)
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