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Exercise sheet 10

Exercise 10 - 1

Assume that you are measuring a field ψ with symmetric statistics, i.e.

$$\mathcal{P}(\psi) = \mathcal{P}(-\psi) \ \forall \psi, \tag{1}$$

with a perfect instrument, i.e.

$$d = \psi. \tag{2}$$

You are interested in the power of the field, i.e.

$$s = \psi^2. \tag{3}$$

a) Calculate the signal response and the noise using the definition

$$R(s) = \langle d \rangle_{(d|s=\psi_0^2)} \tag{4}$$

of the signal response (1 point).

b) Do the same for a new data set d' that is the square of the old data set, $d' = d^2$ (1 point).

Exercise 10 - 2

Assume a linear measurement of some field. Assume further a log-normal model for this field and an additive Gaussian noise term, i.e.

$$d = Re^s + n, \ s \hookleftarrow \mathcal{G}(s, S), \ n \hookleftarrow \mathcal{G}(n, N). \tag{5}$$

a) Derive the information Hamiltonian $H(s, d)$ for this problem (2 points).

b) Give a recursion relation of the type

$$m_{\text{MAP}} = f(m_{\text{MAP}}) \tag{6}$$

for the *maximum a posteriori* solution m_{MAP} of the signal field s (1 point).

Exercise 10 - 3

A signal $s : \mathbb{R}^u \to \mathbb{R}$ with Gaussian statistics and known covariance $S = \langle ss^\dagger \rangle_{(s)}$ is measured via $d_x = s_x + n_x$. The noise follows Gaussian statistics and is homogeneous except for a slight enhancement in an area Ω, i.e., $N_{xy} = \langle n_x n_y \rangle_{(n)} = \delta(x - y)\,(1 + \epsilon\,\Theta_\Omega(x))\,\sigma^2$. Here, $\Theta_\Omega(x) = 1$ for $x \in \Omega$ and $\Theta_\Omega(x) = 0$ for $x \notin \Omega$. Consider the Wiener filter for this inference problem.

a) Calculate perturbatively to first order in ϵ the effect of the noise inhomogeneity on the real-space structure of the propagator (3 points).

b) Calculate N in its Fourier representation for general Ω and for $\Omega = [-L, L]$ in the one-dimensional case (3 points).

Exercise 10 - 4

Consider a real-valued signal field s with a Gaussian prior,

$$\mathcal{P}(s) = \mathcal{G}(s, S), \tag{7}$$

that is observed with an instrument that exhibits an almost linear response,

$$d = R\left(s + rs^2\right) + n. \tag{8}$$

Here, R is a linear operator, $r \in \mathbb{R}$ with $|r| \ll 1$ is a small parameter that determines the strength of the nonlinearity in the instrumental response, s^2 denotes the local squaring of the signal field, i.e., $\left(s^2\right)_x = \left(s_x\right)^2$, and n is additive Gaussian noise, i.e.,

$$\mathcal{P}(n) = \mathcal{G}(n, N). \tag{9}$$

a) Consider first the case of an exactly linear response, i.e., $r = 0$. Derive the Hamiltonian

$$H(d, s) = -\log\left(\mathcal{P}(d, s)\right) \tag{10}$$

for this problem. You may drop all terms that do not depend on s (1 point) .

b) Show that the posterior probability density in the case with $r = 0$ is of Gaussian form, i.e., $\mathcal{P}(s|d) = \mathcal{G}(s - m_0, D)$, and derive expressions for its mean and covariance,

$$m_0 = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad D = \left\langle (s - m_0)(s - m_0)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{11}$$

as a function of d, S, N, and R (2 points).

c) Now consider the case with small but non-zero r. Calculate the Hamiltonian in this case and write it in the form

$$H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{\infty} \frac{1}{k!} \Lambda^{(k)}_{x_1 x_2 \cdots x_k} s_{x_1} s_{x_2} \cdots s_{x_k}, \tag{12}$$

where only the coefficients $\Lambda^{(k)}$ depend on r and we use the convention that repeated indices are integrated over. Give expressions for j, D, and all non-zero $\Lambda^{(k)}$. You do not need to calculate H_0 (3 points).

This exercise sheet will be discussed during the exercises.
group A, Wednesday 16:00 - 18:00 Theresienstr. 39 - room B 101,
group B, Thursday, 10:00 - 12:00, Geschw.-Scholl-Pl. 1 (B) - room B 015
group C, Thursday, 16:00 - 18:00, Theresienstr. 37 - room A 449
backup slot, Wednesday, 8:00 - 10:00, Theresienstr. 37 - A 450
(to be used whenever Thursday is a vacation day)
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