Information Theory
Summer term 2020

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Exercise sheet 1

Exercise 1 - 1: Boolean Algebra

a) The NAND and NOR operators are defined as $A \uparrow B = \overline{AB}$ and $A \downarrow B = \overline{A+B}$, respectively. Construct the following expressions using only the statements A and B and the $\uparrow$- and $\downarrow$-operators (3 points):

$$\overline{A}, \qquad A+B, \qquad AB$$

b) What is the minimum number of operations that need to be defined on logical statements so that the negation, conjunction, and disjunction can be expressed? (1 point)

c) If 1 (True) and 0 (False) are a representation of logical values, how do we need to modify the rules of arithmetics to represent the Boolean algebra? (1 point)

Exercise 1 - 2: Product Rule

It was argued in the lecture that the plausibility of A and B must be a continuous and monotonic function f of the plausibility of B and of the plausibility of A given B, i.e.,

$$(AB|C) = f\left((B|C), (A|BC)\right). \tag{1}$$

Furthermore, this function needs to fulfill the equation

$$f\left(f(x,y), z\right) = f\left(x, f(y,z)\right). \tag{2}$$

It can be shown (Cox, 1946) that such a function can be written as $f(x,y) = \omega^{-1}\left(\omega(x)\omega(y)\right)$.

a) Verify that any such function does indeed fulfill Equation (2) (2 points).

b) Given three statements A, B, and C, consider the cases $C \Rightarrow A$ and $C \Rightarrow \overline{A}$. Derive the possible values of $\omega(A|C)$ in these two cases (2 points).

Hint: Use $\omega(AB|C) = \omega(B|C)$ in the first case, and $\omega(AB|C) = \omega(A|C) = \omega(A|BC)$ in the second case.

c) Use the product rule for plausibilities to derive Bayes' Theorem (1 point).

Exercise 1 - 3

Given two statements A and B, label the statement $A \Rightarrow B$, i.e., "If A is true, then B is true.", with C. Show the strong syllogisms $P(B|AC) = 1$ and $P(A|\overline{B}C) = 0$ (2 points).

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Exercise 1 - 4: Weather in Markovia

You are traveling to the beautiful country of Markovia. Your travel guide tells you that the weather w_i in Markovia on a particular day i is sunny, $w_i = \mathrm{s}$, for 80% of all days or it is cloudy, $w_i = \mathrm{c}$, for sudo apt-get upgrade 20% of all days. There are no other weather conditions in Markovia and the weather changes only during nights. The probability for a weather change is 10% if it is sunny,

$$P(w_{i+1} = \mathrm{c}|w_i = \mathrm{s}) = 0.1, \tag{3}$$

and 40% if it is cloudy,

$$P(w_{i+1} = \mathrm{s}|w_i = \mathrm{c}) = 0.4, \tag{4}$$

irrespective of what it has been on earlier days, $P(w_{i+1}|w_i, w_{i-1}, w_{i-2}, \ldots) = P(w_{i+1}|w_i)$.

a) You arrive on a sunny day, $w_i = \mathrm{s}$, in Markovia. Calculate the probability that it was cloudy there the day before, $P(w_{i-1} = \mathrm{c}|w_i = \mathrm{s})$ Hint: Use Bayes-Theorem. (2 points).

b) What is the total probability for a weather change $P(w_{i+1} \neq w_i)$ in Markovia during an arbitrary night (1 point)?

c) The Markovian weather forecast for some day i predicts a sunshine probability of

$$p_i = P(w_i = \mathrm{s}\,|\text{forecast}). \tag{5}$$

What is the sunshine probability there for the following day,

$$p_{i+1} = P(w_{i+1} = \mathrm{s}\,|\text{forecast})? \tag{6}$$

(1 point)

d) Verify or correct the travel guide's statement on the frequency of 80% sunny and 20% cloudy days in Markovia (1 point).

<u>Hint</u>: Your result of question **c)** might be useful for this.

e) Implement an algorithm to simulate the weather in Markovia and verify your results numerically (optional).

This exercise sheet will be discussed during the exercises.
group A, Wednesday 16:00 - 18:00 Theresienstr. 37 - room A 248,
group B, Thursday, 10:00 - 12:00, Theresienstr. 37 - room A 249
group C, Thursday, 16:00 - 18:00, Theresienstr. 37 - room A 449
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