

The δN Formalism

Naonori S. Sugiyama*

Astronomical Institute, Graduate School of Science, Tohoku University, Sendai 980-8578, Japan

Eiichiro Komatsu

*Texas Cosmology Center and the Department of Astronomy,
The University of Texas at Austin, 1 University Station, C1400, Austin, TX 78712*

*Kavli Institute for the Physics and Mathematics of the Universe,
Todai Institutes for Advanced Study, the University of Tokyo,
Kashiwa, Japan 277-8583 (Kavli IPMU, WPI) and*

Max Planck Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 85741 Garching, Germany

Toshifumi Futamase

Astronomical Institute, Graduate School of Science, Tohoku University, Sendai 980-8578, Japan

Precise understanding of non-linear evolution of cosmological perturbations during inflation is necessary for the correct interpretation of measurements of non-Gaussian correlations in the cosmic microwave background and the large-scale structure of the universe. The “ δN formalism” is a popular and powerful technique for computing non-linear evolution of cosmological perturbations on large scales. In particular, it enables us to compute the curvature perturbation, ζ , on large scales without actually solving perturbed field equations. However, people often wonder why this is the case. In order for this approach to be valid, the perturbed Hamiltonian constraint and matter-field equations on large scales must, with a suitable choice of coordinates, take on the same forms as the corresponding unperturbed equations. We find that this is possible when (1) the unperturbed metric is given by a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker metric; and (2) on large scales and with a suitable choice of coordinates, one can ignore the shift vector (g_{0i}) as well as time-dependence of tensor perturbations to $g_{ij}/a^2(t)$ of the perturbed metric. While the first condition has to be assumed *a priori*, the second condition can be met when (3) the anisotropic stress becomes negligible on large scales. However, in order to explicitly show that the second condition follows from the third condition, one has to use gravitational field equations, and thus this statement may depend on the details of theory of gravitation. Finally, as the δN formalism uses only the Hamiltonian constraint and matter-field equations, it does not *a priori* respect the momentum constraint. We show that the violation of the momentum constraint only yields a decaying mode solution for ζ , and the violation vanishes when the slow-roll conditions are satisfied.

I. INTRODUCTION

Given the success of cosmological linear perturbation theory, the focus has shifted to *non-linear* evolution of cosmological perturbations. As the magnitude of the primordial curvature perturbation is of order 10^{-5} , any non-linearities are expected to be small; however, such non-linearities can be measured using non-Gaussian correlations of cosmological perturbations (such as temperature and polarization anisotropy of the cosmic microwave background [1] and density fluctuations in the large-scale structure of the universe [2]). For this reason, precise understanding of the non-linear evolution of cosmological perturbations is of great interest in cosmology.

The so-called “ δN formalism” [3–7] is a popular technique for computing non-linear evolution of cosmological perturbations on large scales. Here, by “large scales,” we mean the scales greater than the Hubble horizon, in a sense that the comoving wavenumber of perturbations, k , is much less than the reciprocal of the comoving Hub-

ble length, i.e., $k \ll aH$. In particular, it enables us to compute the curvature perturbation, ζ , without actually solving the perturbed field equations. In this paper, we show why this is the case by re-deriving the δN formalism using the gradient expansion method as applied to Einstein’s field equations and scalar-field equations in the flat gauge. The usual derivation of the δN formalism is based on the so-called “separate universe” approach [8], which assumes the existence of a locally homogeneous (but not necessarily isotropic [9]) region smoothed over some large length scale. We provide a support for this assumption by considering a global region including many such smoothed local regions, and show that they behave as if they were locally homogeneous regions which evolve independently from each other. In so doing, we point out a subtlety regarding the momentum constraint, which is not *a priori* respected by the δN formalism.

The organization of this paper is as follows. In Sec. II, we describe our basic setup including the metric, gauge, and scalar-field Lagrangian. In Sec. III, we review the gradient expansion method, which constitutes the basis for the δN formalism. In Sec. IV, we re-derive the δN formalism. In Sec. V, we give the sufficient conditions for the validity of the δN formalism and conclude.

* sugiyama@astr.tohoku.ac.jp

II. BASIC SETUP

A. Metric

We write the spacetime metric in the Arnowitt-Deser-Misner (ADM) form, which is the standard (3+1)-decomposition of the metric [10],

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt), \quad (1)$$

where γ_{ij} is decomposed as follows,

$$\gamma_{ij} \equiv a^2 e^{2\psi} (e^h)_{ij}. \quad (2)$$

Here, a is the scale factor which depends only on time, and ψ is the scalar perturbation to the spatial curvature. The traceless tensor, h_{ij} , is further decomposed as

$$h_{ij} = \partial_i C_j + \partial_j C_i - \frac{2}{3} \delta_{ij} \partial_k C^k + h_{ij}^{(T)}, \quad (3)$$

where C_i contains both scalar and vector perturbations, whereas $h_{ij}^{(T)}$ represents tensor perturbations.

We decompose the extrinsic curvature, K_{ij} , into a trace part, K , and a traceless part, \tilde{A}_{ij} , as¹

$$K_{ij} = \frac{\gamma_{ij}}{3} K + a^2 e^{2\psi} \tilde{A}_{ij}. \quad (4)$$

Einstein's field equations written in terms of these variables are summarized in Appendix A.

B. Flat gauge

In this paper, we shall fix the gauge completely (i.e., leaving no gauge degree of freedom) by imposing the following gauge-fixing condition²:

$$\psi = C_i = 0. \quad (5)$$

Therefore, the spatial metric is described only by the scale factor and tensor perturbations as $\gamma_{ij} = a^2 [e^{h^{(T)}}]_{ij}$. This gauge was also used by [11] (see his Eq. (3.2)).

We shall call this gauge the ‘‘flat gauge’’ throughout this paper. Note that a flat-gauge condition in the literature sometimes does not include $C_i = 0$. In such a case a residual gauge degree of freedom would remain. In the flat gauge, the metric is given by

$$ds^2 = -\alpha^2 dt^2 + a^2(t) [e^{h^{(T)}}]_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt). \quad (6)$$

¹ In Salopek and Bond [4], a trace-free part of the extrinsic curvature is denoted as \bar{K}_{ij} . In our notation, we have $\bar{K}_{ij} = a^2 e^{2\psi} \tilde{A}_{ij}$.

² Sometimes another gauge condition, $\dot{\psi} = 0$ and $\beta_i = 0$, is imposed; however, the degree of freedom of this gauge is not completely fixed, as this gauge condition only gives the relation for the time derivatives of the shifts in temporal and spatial coordinates.

The variables in this metric such as α , β_i , and $h_{ij}^{(T)}$ contain *non-linear* perturbations. However, we shall assume that the unperturbed metric is still given by a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \quad (7)$$

Therefore, our argument below does not hold if the unperturbed metric is not given by Eq. (7).

C. Scalar-field Lagrangian

We shall consider a universe filled with scalar fields:

$$\mathcal{L} = -\frac{1}{2} G_{IJ} \partial^\mu \varphi^I \partial_\mu \varphi^J - V. \quad (8)$$

The capital Latin indices (I, J , etc.) denote scalar-field components running from 1 to n where n is the number of scalar fields. Here, G_{IJ} is the metric tensor for scalar-field space. For simplicity we shall take the canonical kinetic term, $G_{IJ} = \delta_{IJ}$, for the moment. We then argue later (in Sec. III C) that the results are also valid for non-canonical kinetic terms in the first order of the gradient expansion.

With this Lagrangian, the stress-energy tensor and the field equation of scalar fields are given by

$$T_{\mu\nu} = G_{IJ} \partial_\mu \varphi^I \partial_\nu \varphi^J + g_{\mu\nu} \left(-G_{IJ} \frac{1}{2} \partial^\alpha \varphi^I \partial_\alpha \varphi^J - V \right), \quad (9)$$

$$\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \varphi^I) - V_I = 0, \quad (10)$$

where $V_I \equiv \partial V / \partial \varphi^I$.

III. GRADIENT EXPANSION METHOD

A. Ordering in the gradient expansion

Since we are interested in non-linear perturbations on super-horizon scales, we shall expand field equations in the number of spatial derivatives: this is called the gradient expansion method [4, 6]. In this method, the ratio of the comoving wavenumber and the comoving Hubble scale,

$$\epsilon \equiv \frac{k}{aH}, \quad (11)$$

is taken to be a small parameter.

Before we proceed, let us emphasize that we assume the validity of perturbative expansion: namely, while we shall deal with non-linear perturbations, we assume that the $(i+1)$ th-order perturbations are smaller than the i th-order perturbations. This means that we have *two*

smallness parameters: one is the number of derivatives, ϵ , and the other is a smallness parameter of perturbation theory, δ , which corresponds to ψ , β^i , $\varphi^I - \bar{\varphi}^I$, etc.

These two parameters should satisfy the following condition:

$$\delta < \epsilon \ll 1. \quad (12)$$

This is because, if we take the smoothing length to be infinitely large, i.e., $\epsilon \rightarrow 0$, then the perturbation must vanish, i.e., $\delta \rightarrow 0$. Then, the metric must approach the unperturbed metric given by Eq. (7) as we take $\epsilon \rightarrow 0$. In other words, the amplitude of the perturbations should be limited by the smoothing length we take.

We now estimate the ordering of perturbation variables in terms of the gradient expansion. First, we demand that all physical quantities do not vanish in the lowest order of the gradient expansion:

$$\begin{aligned} \alpha - 1 &= \tilde{\gamma}_{ij} - \delta_{ij} = \varphi^I - \bar{\varphi}^I = \mathcal{O}(\epsilon^0, \delta), \\ \beta^i &= \mathcal{O}(\epsilon^{-1}, \delta). \end{aligned} \quad (13)$$

We do not include ψ here because we work in the flat gauge. We have defined $\tilde{\gamma}_{ij} \equiv \gamma_{ij}/[a^2(t)e^{2\psi}]$, which is equal to $[e^{h_i^{(T)}}]_{ij}$ in the flat gauge.

One may wonder why we chose to start with $\beta^i = \mathcal{O}(\epsilon^{-1})$, which seems to diverge in the limit of $\epsilon \rightarrow 0$. However, this is not true. As noted earlier, the existence of the perturbation ($\delta > 0$) guarantees $\epsilon > 0$ and we always have $\beta^i = \mathcal{O}(\epsilon^{-1}, \delta) < 1$ from Eq. (12); thus, there is no divergence in the metric. In fact, we recover the standard Friedmann equation in the lowest order approximation. Furthermore, at the end of Sec. IV D, we show that consistency between the Hamiltonian and momentum constraint equations demands $\beta^i = \mathcal{O}(\epsilon^{-1})$.

Note that the shift vector comes with a spatial derivative, ∂_i , in Einstein's field equations and scalar-field equations. As $\partial_i \beta^i = \mathcal{O}(\epsilon^0)$, the spatial derivatives are kept in Einstein's field equations and scalar-field equations for $\alpha - 1 = \tilde{\gamma}_{ij} - \delta_{ij} = \varphi^I - \bar{\varphi}^I = \mathcal{O}(\epsilon^0)$. In other words, as we keep spatial derivatives in our approach, we are considering some global region in which there are many smoothed local regions. Therefore, we do not *a priori* demand that these local regions evolve independently of each other, contrary to what is always demanded by a separate universe approach. Specifically, for a separate universe approach, $\beta^i = \mathcal{O}(\epsilon)$ is always assumed *a priori*.

Similarly, when we decompose the quantities β_i and C_i into scalar and vector components as $\beta_i = \partial_i \beta^{(S)} + \beta_i^{(V)}$ and $C_i = \partial_i C^{(S)} + C_i^{(V)}$, respectively, the scalar components are of order ϵ^{-2} : $\beta^{(S)} = C^{(S)} = \mathcal{O}(\epsilon^{-2})$.

In order to see how Eq. (13) can be relaxed, we now investigate the nature of solutions for β_i and $h_{ij}^{(T)}$.

At the first order in perturbation variables and the lowest order in the gradient expansion, the evolution equation for \tilde{A}_{ij} is given by (see Eq. (A10)):

$$\dot{\tilde{A}}_{ij} + 3H\tilde{A}_{ij} = \mathcal{O}(\epsilon, \delta^2), \quad (14)$$

where H is the Hubble expansion rate, $H \equiv \dot{a}/a$. Here, we have ignored the anisotropic stress term on the right hand side of Eq. (A10), as it is of the second order in the gradient expansion for scalar fields. This is a stronger-than-necessary condition: Eq. (14) is still valid if the anisotropic stress of matter fields is of the first order in the gradient expansion.

It follows from Eq. (14) that the traceless part of the extrinsic curvature, \tilde{A}_{ij} , has a decaying solution, $\tilde{A}_{ij} \propto 1/a^3$ [4].³ On the other hand, the evolution equation for γ_{ij} with $C_i = 0$ yields (see Eq. (A8)):

$$\dot{h}_{ij}^{(T)} = -2\tilde{A}_{ij} + \frac{1}{a^2} \left(\partial_i \beta_j + \partial_j \beta_i - \frac{2}{3} \delta_{ij} \delta^{kl} \partial_k \beta_l \right). \quad (15)$$

As the scalar, vector, and tensor modes are independent in linear theory, the equations for the shift vector and tensor perturbations are given by

$$\begin{aligned} \ddot{h}_{ij}^{(T)} + 3H\dot{h}_{ij}^{(T)} &= \mathcal{O}(\epsilon, \delta^2), \\ \dot{\beta}^i + 3H\beta^i &= \mathcal{O}(\epsilon^0, \delta^2). \end{aligned} \quad (16)$$

Therefore, $\dot{h}_{ij}^{(T)}$ and $\beta^i = \beta_i/a^2$ also have decaying solutions scaling as a^{-3} . This is a consequence of the fact that the unperturbed metric (Eq. (7)) is given by a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker metric. In other words, this result may not hold for anisotropic models such as Bianchi-type metrics.

At the second order in perturbation variables, as the source terms in Eq. (A8) and Eq. (A10) are decaying, the second-order equations for $h_{ij}^{(T)}$ and β_i have approximately the same forms as the first-order equations (Eq. (16) and Eq. (14)), and thus their solutions must also be decaying as a^{-3} . Similarly, n -th order solutions for $n \geq 3$ are also decaying.

These properties allow us to safely ignore, in the lowest order of the gradient expansion and the n -th order of perturbation theory, the traceless part of the extrinsic curvature \tilde{A}_{ij} , the shift vector β^i , as well as a time derivative of tensor perturbations, $\dot{h}_{ij}^{(T)}$, after the decaying solutions become sufficiently small. This means that these quantities must be higher order in the gradient expansion than naively assumed in Eq. (13): $\tilde{A}_{ij} = \mathcal{O}(\epsilon)$, $\beta^i = \mathcal{O}(\epsilon^0)$, and $\dot{h}_{ij}^{(T)} = \mathcal{O}(\epsilon)$.

Now, it turns out that the above argument also applies to the next order of the gradient expansion. At the next order in the gradient expansion:

$$\begin{aligned} \alpha - 1 &= \tilde{\gamma}_{ij} - \delta_{ij} = \varphi^I - \bar{\varphi}^I = \mathcal{O}(\epsilon), \\ \beta^i &= \mathcal{O}(\epsilon^0), \end{aligned} \quad (17)$$

³ As we start with $\beta^i = \mathcal{O}(\epsilon^{-1})$, we need to linearize Eq. (A10) to obtain Eq. (14), showing $\tilde{A}_{ij} \propto 1/a^3$. On the other hand, assuming $\beta^i = \mathcal{O}(\epsilon)$, Hamazaki derives $\tilde{A}_{ij} \propto 1/a^6$ without using perturbation theory (see Eq. (2.54) of [12]).

one can show that, for scalar fields whose anisotropic stress is of the second order in the gradient expansion, the equations take on the same form as Eq. (14) and Eq. (16):

$$\dot{\tilde{A}}_{ij} + 3H\tilde{A}_{ij} = \mathcal{O}(\epsilon^2, \delta^2), \quad (18)$$

$$\ddot{h}_{ij}^{(T)} + 3H\dot{h}_{ij}^{(T)} = \mathcal{O}(\epsilon^2, \delta^2), \quad (19)$$

$$\dot{\beta}^i + 3H\beta^i = \mathcal{O}(\epsilon, \delta^2). \quad (20)$$

Applying the same argument as above, one finds that \tilde{A}_{ij} , $\dot{h}_{ij}^{(T)}$, as well as β^i decay for the n -th order in perturbation theory. We thus find: $\tilde{A}_{ij} = \mathcal{O}(\epsilon^2)$, $\beta^i = \mathcal{O}(\epsilon)$, and $\dot{h}_{ij}^{(T)} = \mathcal{O}(\epsilon^2)$. However, this argument cannot be extended to the second order of the gradient expansion, as Eq. (18) is valid only when the anisotropic stress term is unimportant. As one can no longer ignore the anisotropic stress of scalar fields at the second order in the gradient expansion, Eq. (18) is no longer valid in that order.

Therefore, Eq. (13) should be revised as

$$\begin{aligned} \alpha - 1 &= \varphi^I - \bar{\varphi}^I = \mathcal{O}(\epsilon^0), \\ \beta^i &= \mathcal{O}(\epsilon), \quad \dot{h}_{ij}^{(T)} = \mathcal{O}(\epsilon^2), \end{aligned} \quad (21)$$

where we have dropped δ in $\mathcal{O}(\dots)$, as the above estimation is valid for all orders of perturbation theory. Note that this result is valid only in the flat gauge given by Eq. (5). In particular, the condition $C_i = 0$ was needed to estimate the gradient-expansion-order of β^i and $\dot{h}_{ij}^{(T)}$.

These results might depend on the details of theory of gravitation, as we have used Einstein's field equations to obtain solutions of β^i and $\dot{h}_{ij}^{(T)}$. Furthermore, as a perturbative expansion is used in estimating β^i and $\dot{h}_{ij}^{(T)}$, the validity of a perturbative description of the metric with the unperturbed metric given by Eq. (7) has been assumed in the above argument.

B. Comparison with previous work

How does Eq. (21) compare with the previous work? Our starting point, Eq. (13), is different from the assumption made in Lyth, Malik and Sasaki [6] (also see [13]). They assume that there exist an approximate set of coordinates with which the metric of any local region can be written as a Friedmann-Lemaître-Robertson-Walker metric. This implies that the shift vector, β_i , vanishes and the quantity, $\tilde{\gamma}_{ij}$, is time-independent in the limit of $\epsilon \rightarrow 0$: $\beta_i = \mathcal{O}(\epsilon)$ and $\dot{\tilde{\gamma}}_{ij} = \mathcal{O}(\epsilon)$. We do not make this assumption *a priori*, and thus our argument is more general than that given in [6]. They then show that, using Einstein's field equations and ignoring the anisotropic stress term, $\dot{\tilde{\gamma}}_{ij}$ decays in the first order of the gradient expansion, concluding $\dot{\tilde{\gamma}}_{ij} = \mathcal{O}(\epsilon^2)$.

In [14], Weinberg uses a broken symmetry argument to show $\beta_i = \mathcal{O}(a^{-2})$ and $\dot{\tilde{\gamma}}_{ij} = \mathcal{O}(a^{-2})$ for generally

covariant theories and with a suitable choice of coordinates, assuming that the unperturbed metric is given by Eq. (7) and the anisotropic stress term is negligible. He then shows that, for Einstein's field equations in a coordinate system in which $\beta_i = 0$ and a certain combination of matter perturbations vanishes, this solution is an attractor. By identifying $\mathcal{O}(a^{-2})$ with $\mathcal{O}(\epsilon^2)$ (because each spatial derivative must come with $1/a$), his argument yields $\beta_i = \mathcal{O}(\epsilon^2)$ and $\dot{\tilde{\gamma}}_{ij} = \mathcal{O}(\epsilon^2)$.

Therefore, our finding agrees with the previous work: if the unperturbed metric is given by Eq. (7), the anisotropic stress term is negligible on large scales, and field equations are given by Einstein's field equations, then $\beta_i = \mathcal{O}(\epsilon)$ and $\dot{\tilde{\gamma}}_{ij} = \mathcal{O}(\epsilon^2)$. Note that Weinberg's estimate for the order of β_i is higher by ϵ ; however, he does not show $\beta_i = \mathcal{O}(\epsilon^2)$ explicitly because he chooses coordinates in which $\beta_i = 0$.

C. Gradient expansion of Einstein's field equations and scalar-field equations in the flat gauge

Now, we apply the gradient expansion to Einstein's field equations and scalar-field equations. We shall work with the flat gauge given by Eq. (5), which gives the gradient-expansion order of perturbation variables given in Eq. (21).

We shall choose the number of e -folds, $N \equiv \int_{t_*}^t H dt'$, as our time coordinates in the flat gauge. The Hamiltonian constraint (Eq. (A5)) and the scalar-field equation (Eq. (10)) in *both* the lowest order and the next order of the gradient expansion are given by

$$3\tilde{H}^2 M_p^2 = \rho, \quad (22)$$

$$\tilde{H}\partial_N (\tilde{H}\varphi_N^I) + 3\tilde{H}^2\varphi_N^I + V_I = 0, \quad (23)$$

where $\tilde{H} \equiv H/\alpha$ is related to a trace of the extrinsic curvature as $\tilde{H} = -K/3$, and the subscript N denotes a partial derivative with respect to N . The energy density, ρ , is given by $\rho = \frac{\tilde{H}^2}{2}G_{IJ}\varphi_N^I\varphi_N^J + V$. Then, \tilde{H} is given by [15],

$$\tilde{H}^2 = \frac{2V}{6M_p^2 - G_{IJ}\varphi_N^I\varphi_N^J}. \quad (24)$$

All we need to do is to solve Eq. (23) coupled with Eq. (24)

On the other hand, the unperturbed equations are

$$3H^2 M_p^2 = \bar{\rho}, \quad (25)$$

$$H\partial_N (H\bar{\varphi}_N^I) + 3H^2\bar{\varphi}_N^I + V_I(\bar{\varphi}) = 0, \quad (26)$$

where $\bar{\rho}$ and $\bar{\varphi}^I$ are the unperturbed energy density and scalar fields, respectively. Apparently, the perturbative

equations (Eqs. (22) and (23)) coincide exactly with the unperturbed equations (Eqs. (25) and (26)). This result shows that each region smoothed by a super-horizon scale, $\epsilon \ll 1$, in the universe evolves independently and behaves like an unperturbed universe, providing a support for the assumption made by a separate universe approach.

These results might depend on the details of theory of gravitation. While the correspondence between the perturbed and unperturbed equations for other theories of gravitation is an interesting problem, in this paper we shall focus on Einstein's General Relativity. However, these results should not depend on the form of the Lagrangian of scalar fields. This is because the anisotropic stress (i.e., a traceless part of the stress-energy tensor) for scalar fields with arbitrary Lagrangian necessarily comes with two spatial derivatives, and thus it must be $\mathcal{O}(\epsilon^2)$. As a result, \tilde{A}_{ij} has a decaying solution and a Friedmann-Lemaître-Robertson-Walker universe will be restored on large scales.

One can generalize the above results for the canonical case to non-canonical Lagrangians given by $\mathcal{L} = P(X^{IJ}, \varphi^K)$, where $X^{IJ} \equiv -g^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^J$. The Hamiltonian constraint is still given by Eq. (22) while the scalar field equation is given by

$$\tilde{H} \partial_N \left[\tilde{H} \partial_N \varphi_N^J \right] P_{IJ} + \tilde{H}^2 \varphi_N^J \partial_N P_{IJ} + 3\tilde{H}^2 P_{IJ} \varphi_N^J + \frac{P_I}{2} = 0, \quad (27)$$

where $P_{IJ} \equiv \partial P / \partial X^{IJ}$, $X^{IJ} = \tilde{H}^2 \varphi_N^I \varphi_N^J$, and the energy density is defined as $\rho = 2X^{IJ} P_{IJ} - P$. As P and P_{IJ} are functions of X^{IJ} , φ^K , and \tilde{H} , one can write \tilde{H} as a function of X^{IJ} and φ^K if an explicit form of the Lagrangian P is specified.

What are the implications of these results? As the equations take on the same forms, the functional forms of the solutions for the perturbed equations and those for the unperturbed equations must be the same. Therefore, the perturbed solutions, φ^I , are given by the unperturbed solutions, $\bar{\varphi}^I$, with *perturbed initial conditions computed in the flat gauge*, $\varphi_*^I(\mathbf{x})$ and $\varphi_{N*}^I(\mathbf{x})$:

$$\varphi^I(N, \mathbf{x}) = \bar{\varphi}^I(N, \varphi_*^J(\mathbf{x}), \varphi_{N*}^K(\mathbf{x})). \quad (28)$$

This is the fundamental result of the gradient expansion as applied to Einstein's field equations and scalar-field equations in the flat gauge. Here, the subscript $*$ indicates that the quantity is evaluated at some initial time, where all the relevant fields are sufficiently outside their sound horizon, i.e., $k \ll a(t_*)H(t_*)/c_s^I$, where c_s^I is the speed of sound of propagation of an I th scalar field perturbation.

In order to simplify our notations, from now on we shall use the lower-case alphabet indices, such as a, b, c, \dots , to denote the numbers of scalar fields and their time-derivatives:

$$\varphi^a \equiv (\varphi^I, \varphi_N^J),$$

with a running from 1 to $2n$. With this notation, the solution (Eq. (28)) is expressed as $\varphi^a(N, \mathbf{x}) = \bar{\varphi}^a(N, \varphi_*^b(\mathbf{x}))$.

Similarly, we can write the perturbed energy density of multi-scalar fields using the unperturbed energy density solution: $\rho(N, \mathbf{x}) = \bar{\rho}(N, \varphi_*^a(\mathbf{x}))$.

IV. THE δN FORMALISM

We now need to relate perturbed initial scalar fields and their derivatives, $\varphi_*^a(\mathbf{x})$, to the observables. In cosmology, it is now customary to express the observables such as temperature and polarization anisotropies and the large-scale distribution of galaxies in terms of a curvature perturbation in the ‘‘uniform-density gauge,’’ denoted as ζ .

The so-called δN formalism [3–7] achieves this by realizing that ζ is equal to a perturbation to the number of e -folds, N , arising from perturbed initial scalar fields, $\varphi_*^a(\mathbf{x})$, computed in the flat gauge.

A. Conservation of ζ outside the horizon

We define the uniform-density gauge as $\delta\rho = 0$ and $C_i = 0$. (Once again, a uniform-density gauge in the literature sometimes does not include $C_i = 0$. In such a case the gauge is not completely fixed.) Let us denote a value of ψ in the uniform-density gauge as $\psi|_{\delta\rho=C_i=0}$ ⁴, and write ζ as

$$\zeta \equiv \psi|_{\delta\rho=C_i=0}. \quad (30)$$

This quantity is useful for extracting information about the physics of inflation, as it is conserved outside the horizon, provided that the adiabatic condition, $p = p[\rho]$, is satisfied [6, 8]. This is easily seen from the energy conservation equation in the lowest order as well as in the next order gradient expansion with the gauge condition $C_i = 0$:

$$\dot{\rho} + 3(H + \dot{\psi}|_{C_i=0})(\rho + p) = 0. \quad (31)$$

The perturbation to this equation in the uniform-density gauge yields

$$\dot{\psi}|_{\delta\rho=C_i=0} = 0, \quad (32)$$

and thus $\zeta = \psi|_{\delta\rho=C_i=0}$ becomes a constant, provided that the adiabatic condition is satisfied.

⁴ Incidentally, as the gauge is completely fixed for $\psi|_{\delta\rho=C_i=0}$, there is no ambiguity with respect to the residual gauge degree of freedom. Moreover, one can always write perturbation variables (such as ψ) after gauge-fixing as a combination of perturbation variables before gauge-fixing (such as C_i and $\delta\rho$) such that ψ is explicitly gauge-invariant. For example, we have, at the linear order,

$$\zeta \equiv \psi|_{\delta\rho=C_i=0} = \psi - \frac{\partial_i C^i}{3} - \frac{\delta\rho}{\rho_N}, \quad (29)$$

where the right-hand-side of Eq. (29) is the well-known form for a gauge-invariant curvature perturbation in the linear order [16].

Alternatively, Eq. (31) may be integrated with respect to t without imposing $\delta\rho = 0$:

$$\tilde{\zeta} \equiv \psi + \int_{\bar{\rho}}^{\rho} \frac{d\rho}{3(\rho + p[\rho])} = \text{const.} \quad (33)$$

One may then identify this quantity, $\tilde{\zeta}$, as a generalization of ζ when $\delta\rho = 0$ is not imposed; however, $\tilde{\zeta}$ is not gauge-invariant and does not coincide with $\zeta = \psi|_{\delta\rho=C_i=0}$, unless the adiabatic condition is satisfied.

What about scalar fields? As scalar fields do not satisfy the adiabatic condition in general, ζ is not conserved in a universe filled with scalar fields. However, as shown by [17], ζ is generally conserved outside the horizon when inflation was driven by a single scalar field. More precisely, ζ is conserved outside the horizon in a universe dominated by a single scalar field, provided that the slow-roll conditions are satisfied, or that we completely neglect a decaying mode solution without imposing the slow-roll conditions. This implies that the slow-roll conditions correspond effectively to the adiabatic condition and the neglect of a decaying mode solution for a single scalar field.

B. Relation between ζ and the difference in the number of e -folds

The relation between the curvature perturbation and the number of e -folds is given by the gauge transformation of the spatial metric, γ_{ij} . Under a gauge transformation given by $t \rightarrow T = t + \delta T$ and $x^i \rightarrow X^i = x^i + \xi^i$, the metric transforms as $g_{ij}(t, \mathbf{x}) \rightarrow \hat{g}_{ij}(T, \mathbf{X})$. Let us write the 3-metric in the original coordinates in terms of the 3-metric in the new coordinates:

$$\begin{aligned} \gamma_{ij}(t, \mathbf{x}) = & -\hat{\alpha}^2(T, \mathbf{X}) \frac{\partial \delta T}{\partial x^i} \frac{\partial \delta T}{\partial x^j} \\ & + \hat{\beta}_k(T, \mathbf{X}) \frac{\partial X^k}{\partial x^i} \frac{\partial \delta T}{\partial x^j} + \hat{\beta}_k(T, \mathbf{X}) \frac{\partial X^k}{\partial x^j} \frac{\partial \delta T}{\partial x^i} \\ & + \hat{\gamma}_{kl}(T, \mathbf{X}) \frac{\partial X^k}{\partial x^i} \frac{\partial X^l}{\partial x^j}. \end{aligned} \quad (34)$$

We shall always impose $C_i = 0$, which completely fixes the spatial gauge degree of freedom, and thus we can set $\xi^i = 0$ without loss of generality.

Let us examine each term in terms of the gradient expansion order. The first term is of order $\mathcal{O}(\epsilon^2)$. As shown in Sec. III A, when $C_i = 0$, the shift vector is of $\mathcal{O}(\epsilon)$; thus, the second and third terms are of order $\mathcal{O}(\epsilon^2)$. This means that, up to $\mathcal{O}(\epsilon^2)$, the 3-metric transforms as

$$\gamma_{ij}|_{C_i=0}(t, \mathbf{x}) = \hat{\gamma}_{ij}|_{C_i=0}(T, \mathbf{x}) + \mathcal{O}(\epsilon^2). \quad (35)$$

Recalling $\gamma_{ij} = a^2(t)e^{2\psi}(e^h)_{ij}$ and taking the determinant and logarithm of the both sides of Eq. (35), we find

$$\psi|_{C_i=0}(t, \mathbf{x}) = \hat{\psi}|_{C_i=0}(T, \mathbf{x}) + \ln \left(\frac{a(T)}{a(t)} \right). \quad (36)$$

Thus, $\psi|_{C_i=0}$ approximately transforms as a scalar quantity having $\ln(a)$ as the unperturbed value. It follows from Eq. (36) that the gauge transformation of $\psi|_{C_i=0}$ from the flat gauge (in which $\psi = 0$) into the uniform-density gauge is given by

$$\psi|_{\delta\rho=C_i=0}(T, \mathbf{x}) = \ln \left(\frac{a(t)}{a(T)} \right), \quad (37)$$

where T denotes time coordinates in the uniform-density gauge.

On the other hand, when we go from the flat gauge to the uniform-density gauge, the number of e -folds, $N \equiv \int_{t_*}^t H dt'$, transforms as $N \rightarrow \hat{N}$, where

$$\hat{N} \equiv \int_{t_*}^T H(t') dt' = \ln \left(\frac{a(T)}{a(t_*)} \right). \quad (38)$$

Here, t_* is an arbitrary initial time.

Comparing Eq. (37) to Eq. (38), one finds

$$\begin{aligned} \psi|_{\delta\rho=C_i=0}(T, \mathbf{x}) = & \ln \left(\frac{a(t)}{a(t_*)} \right) - \ln \left(\frac{a(T)}{a(t_*)} \right) \\ = & N - \hat{N} \equiv \delta N. \end{aligned} \quad (39)$$

Therefore, $\zeta = \psi|_{\delta\rho=C_i=0}$ is equal to the difference between the number of e -folds computed in the flat gauge and that computed in the uniform-density gauge. The remaining task is to relate δN to perturbed initial scalar fields in the flat gauge.

C. Relation between δN and perturbed initial scalar fields in the flat gauge

The most important result that came from the gradient expansion of Einstein's field equations and scalar-field equations in the flat gauge is that perturbed quantities can be calculated using their unperturbed solutions with perturbed initial scalar-field values and their time derivatives computed in the flat gauge. Therefore, a perturbed energy density in the flat gauge is given by $\rho(N, \mathbf{x}) = \bar{\rho}(N, \varphi_*^a(\mathbf{x}))$. Here, we choose the number of e -folds as time coordinates.

On the other hand, by definition the energy density in the uniform-density gauge (whose time coordinates are denoted as \hat{N}) is equal to the unperturbed density. Namely, when we go from the flat gauge to the uniform density gauge by changing the time coordinates as $N \rightarrow \hat{N} = N + \delta N$, the density transforms as $\rho(N, \mathbf{x}) \rightarrow \hat{\rho}(\hat{N}, \mathbf{x}) = \bar{\rho}(\hat{N})$. Here, $C_i = 0$ is satisfied in both gauges, and thus there is no ambiguity with respect to the spatial gauge degree of freedom. Now, as the energy density is a four scalar,

$$\rho(N, \mathbf{x}) = \hat{\rho}(\hat{N}, \mathbf{x}) = \bar{\rho}(\hat{N}), \quad (40)$$

which gives $\bar{\rho}(N, \varphi_*^a(\mathbf{x})) = \bar{\rho}(\hat{N})$. Inverting this result yields

$$N = \hat{N}(\bar{\rho}, \varphi_*^a(\mathbf{x})), \quad (41)$$

where the functional form of \hat{N} is the same as that of the unperturbed number of e -folds. That the unperturbed density $\bar{\rho}$ (not t or N) is used as the time coordinates here ensures that the final time slice coincides with the uniform density hypersurface.

With these results, we can finally calculate ζ :

$$\begin{aligned}\zeta &= N - \hat{N} \\ &= \hat{N}(\bar{\rho}, \varphi_*^a(\mathbf{x})) - \hat{N}(\bar{\rho}, \bar{\varphi}_*^a) \\ &= \hat{N}_a \delta\varphi_*^a(\mathbf{x}) + \frac{1}{2} \hat{N}_{ab} \delta\varphi_*^a(\mathbf{x}) \delta\varphi_*^b(\mathbf{x}) + \dots, \quad (42)\end{aligned}$$

where $\delta\varphi_*^a(\mathbf{x}) \equiv \varphi_*^a(\mathbf{x}) - \bar{\varphi}_*^a$ denotes perturbations to initial scalar fields computed in the flat gauge, and \hat{N}_a and \hat{N}_{ab} are defined as

$$\hat{N}_a \equiv \frac{\partial \hat{N}[\bar{\rho}, \bar{\varphi}_*^b]}{\partial \bar{\varphi}_*^a}, \quad \hat{N}_{ab} \equiv \frac{\partial^2 \hat{N}[\bar{\rho}, \bar{\varphi}_*^c]}{\partial \bar{\varphi}_*^a \partial \bar{\varphi}_*^b}. \quad (43)$$

This is the δN formalism, which enables us to relate ζ to the initial scalar field perturbations (i.e., scalar field perturbations at the initial time) computed in the flat gauge, once we know derivatives of the number of e -folds with respect to the initial values of the unperturbed scalar fields, φ_*^I , and their derivatives, φ_{N*}^I .

D. Momentum constraint

Perhaps a striking thing about the δN formalism is that we only had to use the Hamiltonian constraint (Eq. (22)) and the scalar-field equation (Eq. (23)) in the gradient expansion. But, should not we also impose the momentum constraint for consistent calculations?

As the momentum constraint comes with a spatial derivative, ∂_i , we need to consider the momentum constraint in $\mathcal{O}(\epsilon^2)$ in order to derive the correct relationship between physical quantities up to $\mathcal{O}(\epsilon)$:

$$\partial_i \tilde{H} = -\frac{\tilde{H}}{2M_p^2} G_{IJ} \varphi_N^I \partial_i \varphi^J + \mathcal{O}(\epsilon^3), \quad (44)$$

where we have used the fact that $\tilde{A}_{ij} = \mathcal{O}(\epsilon^2)$.

On the other hand, the Hamiltonian constraint (Eq. (22)) may be differentiated by ∂_i to give

$$\partial_i \tilde{H} = -\frac{\tilde{H}}{2M_p^2} G_{IJ} \varphi_N^I \partial_i \varphi^J + B_i, \quad (45)$$

where

$$B_i \equiv \frac{\tilde{H}^3}{2V} G_{IJ} (\varphi_N^I \partial_i \varphi_N^J - \varphi_{NN}^I \partial_i \varphi^J). \quad (46)$$

Here, we have used the equation of motion for scalar fields given by Eq. (23), as well as the evolution equation for K given by Eq. (A9), which yields $\tilde{H}_N = -\frac{\tilde{H}}{2M_p^2} G_{IJ} \varphi_N^I \varphi_N^J$ in the gradient expansion in the flat gauge.

Comparing Eq. (45) and Eq. (44), we find that the δN formalism, which does not use the momentum constraint but uses only the Hamiltonian constraint, can violate the momentum constraint by an amount B_i .

How important is B_i ? In order to investigate the behavior of B_i , let us take a spatial derivative of the equation of motion for scalar fields:

$$\begin{aligned}\tilde{H} \partial_N (\tilde{H} \partial_i \varphi_N^I) + 3\tilde{H}^2 \partial_i \varphi_N^I \\ + \left[V_{IJ} - \frac{\tilde{H}}{a^3 M_p^2} \frac{d}{dN} \left(G_{JK} a^3 \tilde{H} \varphi_N^J \varphi_N^K \right) \right] \partial_i \varphi^J \\ + H \varphi^I (B_{iN} + 3B_i) + 2\varphi_{NN}^I H B_i = 0.\end{aligned} \quad (47)$$

By contracting this equation with φ_N^I , one finds

$$\begin{aligned}\partial_N B_i + 3B_i &= 0, \\ \rightarrow B_i &= \frac{a_*^3 B_{i*}}{a^3}.\end{aligned} \quad (48)$$

Therefore, B_i has only a decaying solution.⁵ This is a good news for the δN formalism: while it does not *a priori* respect the momentum constraint, a violation of the momentum constraint rapidly decays away by inflation.

That B_i is a decaying mode may be traced back to the fact that the traceless part of the extrinsic curvature, \tilde{A}_{ij} , is a decaying mode in the gradient expansion in the flat gauge. In other words, it is a consequence of the universe behaving like a Friedmann-Lemaître-Robertson-Walker universe on super-horizon scales, which is guaranteed by Eq. (21).

One may wonder why consistency between the momentum constraint and the Hamiltonian constraint gives a relation only among scalar fields: $G_{IJ} (\varphi_N^I \partial_i \varphi_N^J - \varphi_{NN}^I \partial_i \varphi^J) = 0$, rather than a relation between the metric variables and scalar fields. This is because we have ignored the decaying solutions of β^i and $\tilde{h}_{ij}^{(T)}$. To see this, let us work at the first order in perturbations, and bring β^i back into Einstein's field equations. We find that consistency between the momentum constraint and the Hamiltonian constraint gives

$$B_i + \frac{M_p^2 H^2}{V} \partial_i (\partial_j \beta^j) = 0. \quad (49)$$

Indeed, consistency gives a relation between scalar fields (contained in B_i) and a metric variable (β^i)⁶. This equation also indicates that, when we keep the decaying quantities of order a^{-3} , the gradient-expansion order of β^i is indeed $\mathcal{O}(\epsilon^{-1})$ (see Eq. (13)), as Eq. (49) gives $\partial_j \beta^j = \mathcal{O}(1)$.

⁵ In linear theory, this quantity is equal to $-a^{-3} \partial_i W$ where W is given by Eq. (2.26) of [15], as well as to $-\partial_i \tilde{f}$ where \tilde{f} is given by Eq. (5.23) of [18]. They show $W \propto a^{-3}$ and $\tilde{f} \propto a^{-3}$ in linear theory.

⁶ We thank M. Sasaki for clarifying this point.

Now is the time to answer the following question: what if we demand $\beta^i = \mathcal{O}(\epsilon^0)$? In this case Eq. (49) gives $B_i = \mathcal{O}(\epsilon^2)$, or

$$G_{IJ} (\bar{\varphi}_N^I \delta\varphi_N^J - \bar{\varphi}_{NN}^I \delta\varphi^J) = \mathcal{O}(\epsilon), \quad (50)$$

where $\delta\varphi^I \equiv \varphi^I - \bar{\varphi}^I$ is the perturbation of scalar fields in linear theory. This result indicates that we are not allowed to have a configuration of scalar fields which yields $G_{IJ} (\bar{\varphi}_N^I \delta\varphi_N^J - \bar{\varphi}_{NN}^I \delta\varphi^J) = \mathcal{O}(\epsilon^0)$. On the other hand, if we start with $\beta^i = \mathcal{O}(\epsilon^{-1})$, then we can show that $G_{IJ} (\bar{\varphi}_N^I \delta\varphi_N^J - \bar{\varphi}_{NN}^I \delta\varphi^J)$ becomes negligible as it is a decaying mode. Another way of saying this is that, if we start with $\beta^i = \mathcal{O}(\epsilon^0)$, then $G_{IJ} (\bar{\varphi}_N^I \delta\varphi_N^J - \bar{\varphi}_{NN}^I \delta\varphi^J) = 0$ gives only one solution for $\delta\varphi^I$, and another solution, which corresponds to a decaying mode, does not exist.⁷ Therefore, demanding that the number of independent solutions (which is two) not reduce, one should start with $\beta^i = \mathcal{O}(\epsilon^{-1})$.

E. Slow-roll conditions and momentum constraint

Interestingly, we can show that B_i vanishes when the slow-roll conditions are satisfied. The slow-roll equations of motion for the canonical scalar fields are

$$3M_p^2 \tilde{H}^2 \approx V, \quad 3\tilde{H}^2 \varphi_N^I + V_I \approx 0. \quad (51)$$

This implies that

$$\varphi_N^I \approx -M_p^2 \frac{V_I}{V}, \quad \varphi_{NN}^I \approx (2\varepsilon_{IJ} - \eta_{IJ}) \varphi_N^J, \quad (52)$$

where the slow-roll parameters, ε_{ab} and η_{ab} , are defined as

$$\varepsilon_{IJ} \equiv \frac{M_p^2}{2} \frac{V_I V_J}{V^2}, \quad \eta_{IJ} \equiv M_p^2 \frac{V_{IJ}}{V}. \quad (53)$$

Then we find that the following relation is satisfied under the slow-roll condition,

$$\varphi_N^I \partial_i \varphi_N^I \approx \varphi_N^I (2\varepsilon_{IJ} - \eta_{IJ}) \partial_i \varphi^J \approx \varphi_{NN}^I \partial_i \varphi^I, \quad (54)$$

which yields $B_i \approx 0$. In this sense, the slow-roll conditions are equivalent to the momentum constraint.

What does this imply? This implies that the δN formalism happens to respect the momentum constraint, if the slow-roll conditions are satisfied at the initial time, t_* . This may provide a partial explanation as to why the δN formalism has been successful in computing ζ for a wide variety of slow-roll inflation models.

V. CONCLUSION

The necessary and sufficient condition for the validity of the δN formalism is that, with a suitable choice of coordinates, the perturbed Hamiltonian constraint and matter-field equations on large scales coincide with the corresponding unperturbed equations.

That perturbed solutions in the long-wavelength limit can be obtained from unperturbed solutions was found and investigated by pioneering work in 1998 [15, 18, 19]. While their work was restricted to linear theory (and to quasi-linear theory [15]), we have extended their work to include non-linear (but still perturbative) perturbations. Such extension is also explored by [6], who use the so-called ‘‘separate universe approach’’ [8]. As we have described in Sec. III B, our starting point is more general than theirs.

In this paper, using the flat gauge ($\psi = C_i = 0$) and choosing the number of e -folds, N , as our time coordinates, we have shown that the perturbed Hamiltonian constraint and matter-field equations on large scales coincide with the corresponding unperturbed equations, as long as (at least) the following conditions are satisfied:

1. The unperturbed metric is given by a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker metric.
2. The final results for the curvature perturbation are not affected by decaying-mode terms, such as the shift vector, a time derivative of tensor perturbations to $g_{ij}/a^2(t)$, or the violation of the momentum constraint.
3. Evolution of scalar-field perturbations outside the horizon can be treated using the lowest order or the next order of the gradient expansion.
4. Matter fields are given by scalar fields with an arbitrary form of Lagrangian (whose anisotropic stress is of order $\mathcal{O}(\epsilon^2)$) or, in the n th-order gradient expansion where $n = 0$ or 1 , by some fluids with anisotropic stress of order $\mathcal{O}(\epsilon^{n+1})$.
5. Theory of gravitation determining the physics of inflation is Einstein’s theory, or modified gravitational theories which can be transformed into Einstein’s theory.

The fourth and fifth conditions are sufficient conditions: we had to use Einstein’s field equations to explicitly show that the fourth condition implies that the shift vector and a time derivative of tensor perturbations are decaying modes. It is possible that other theories of gravitation require different conditions for the shift vector and a time derivative of tensor perturbations to be decaying modes.

⁷ Assuming $\beta^i = \mathcal{O}(\epsilon)$, Kodama and Hamazaki [18] also find this property from the momentum constraint. See their Eq. (5.13) and the argument given below it.

The discussion in this paper should also apply to vector-field models (see [20] for a review and references therein), as long as their anisotropic stress is of order $\mathcal{O}(\epsilon^{n+1})$ in the n th-order gradient expansion where $n = 0$ or 1.

The second condition is naturally expected in any inflation scenarios. However, when the slow-roll conditions are violated, it is known that decaying-mode solutions in the second order of the gradient expansion cannot be neglected in the power spectrum [21]. In such cases, since there are no gauges in which a Friedmann-Lemaître-Robertson-Walker universe can be obtained in the second order of the gradient expansion, we can no longer use the δN formalism [22].

When all of the above conditions are satisfied, one can calculate ζ using initial scalar-field perturbations computed in the flat gauge, and derivatives of the number of e -folds with respect to the initial values of the unperturbed scalar fields, φ_*^I , and their derivatives, φ_{N*}^I .

ACKNOWLEDGMENTS

We would like to thank Y. Itoh, D. Lyth, J. Meyers, A. Naruko, T. Tanaka, M. Sasaki, and S. Weinberg for useful discussion and comments. We would also like to thank T. Hamazaki for bringing Ref. [12, 18] to our attention after we have posted our article to arXiv. This work is supported in part by the GCOE Program ‘‘Weaving Science Web beyond Particle-matter Hierarchy’’ at Tohoku University, a Grant-in-Aid for Scientific Research from JSPS (Nos. 18072001, 20540245 for TF), Core-to-Core Program ‘‘International Research Network for Dark Energy,’’ as well as by NSF grant PHY-0758153. TF would like to thank Luc Branchet and Institute of Astronomical Observatory, Paris, and NSS and EK would like to thank M. Sasaki and Yukawa Institute for Theoretical Physics, for their warm hospitality during the last stage of this work.

Appendix A: Einstein’s field equations

In this Appendix, we give Einstein’s field equations in terms of the variables of the ADM formalism (see Eq. (1) for the ADM metric). We use Latin indices for the 3D spatial components running from 1 to 3, and Greek indices for the 4D spacetime components running from 0 to 3.

We decompose the three-space metric tensor as

$$\gamma_{ij} \equiv a^2(t)e^{2\psi}\tilde{\gamma}_{ij}, \quad (\text{A1})$$

where $a(t)$ is the scale factor. We define $\tilde{\gamma}_{ij}$ such that $\det[\tilde{\gamma}_{ij}] = 1$; thus, $\det[\tilde{\gamma}_{ij}]$ can be written as $\det[(e^h)_{ij}] = e^{\text{Tr}[h]}$ with $\text{Tr}[h] = 0$. We further decompose h_{ij} as

$$h_{ij} = \partial_i C_j + \partial_j C_i - \frac{2}{3}\delta_{ij}\partial_k C^k + h_{ij}^{(T)}, \quad (\text{A2})$$

where $h_{ij}^{(T)}$ denotes a tensor mode and C_i has a scalar mode and a vector mode.

The extrinsic curvature K_{ij} is defined as

$$K_{ij} \equiv -\nabla_i n_j = \frac{1}{2\alpha}(D_i\beta_j + D_j\beta_i - \dot{\gamma}_{ij}), \quad (\text{A3})$$

where $n^\mu = (1/\alpha, -\beta^i/\alpha)$ is the unit vector normal to the t -constant hypersurface, and ∇ and D are the covariant differential operators constructed by using $g_{\mu\nu}$ and γ_{ij} respectively. The dots denote time derivatives with respect to t .

It is useful to decompose the extrinsic curvature K_{ij} into a ‘‘trace’’ part K and a ‘‘trace free’’ part \tilde{A}_{ij} as

$$K_{ij} = \frac{\gamma_{ij}}{3}K + a^2e^{2\psi}\tilde{A}_{ij}, \quad (\text{A4})$$

where the indices of \tilde{A}_{ij} are raised/lowered by $\tilde{\gamma}_{ij}$, and $\tilde{A}^i_i = 0$ is satisfied.

By using the above notations, we write down Einstein’s field equations. The Hamiltonian constraint is

$$R^{(3)} - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = \frac{2}{M_p^2}T_{\mu\nu}n^\mu n^\nu. \quad (\text{A5})$$

The momentum constraint is

$$D_j\tilde{A}^j_i - \frac{2}{3}\partial_i K = -\frac{1}{M_p^2}T_{i\nu}n^\nu. \quad (\text{A6})$$

The dynamical equation for ψ is

$$(\partial_t - \beta^i\partial_i)\psi + H = \frac{1}{3}(-\alpha K + \partial_i\beta^i). \quad (\text{A7})$$

The dynamical equation for $\tilde{\gamma}_{ij}$ is

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k, \quad (\text{A8})$$

which yields the dynamical equation for K :

$$\begin{aligned} (\partial_t - \beta^k\partial_k)K &= \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) \\ &\quad - \gamma^{ij}D_iD_j\alpha + \frac{\alpha}{2M_p^2}(T_{\mu\nu}n^\mu n^\nu + \gamma^{ij}T_{ij}), \end{aligned} \quad (\text{A9})$$

as well as the dynamical equation for \tilde{A}_{ij} :

$$\begin{aligned} &(\partial_t - \beta^k\partial_k)\tilde{A}_{ij} \\ &= \frac{1}{a^2e^{2\psi}}\left[\alpha\left(R_{ij}^{(3)} - \frac{\gamma_{ij}}{3}R^{(3)}\right) - \left(D_iD_j\alpha - \frac{\gamma_{ij}}{3}D_kD^k\alpha\right)\right] \\ &\quad + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k \\ &\quad - \frac{\alpha}{a^2e^{2\psi}M_p^2}\left(T_{ij} - \frac{\gamma_{ij}}{3}\gamma^{kl}T_{kl}\right). \end{aligned} \quad (\text{A10})$$

Here, $T_{ij} - \frac{\gamma_{ij}}{3}\gamma^{kl}T_{kl}$ is the anisotropic stress.

The 3-dimensional Ricci scalar, $R^{(3)}$, is constructed from γ_{ij} , and M_p^2 is the reduced plank mass defined as $1/8\pi G$ where G is the gravitational constant.

-
- [1] E. Komatsu, *Class. Quant. Grav.* **27**, 124010 (2010), arXiv:1003.6097.
- [2] V. Desjacques and U. Seljak, *Classical and Quantum Gravity* **27**, 124011 (2010), arXiv:1003.5020.
- [3] A. A. Starobinsky, *Phys. Lett.* **B117**, 175 (1982); *JETP Lett.* **42**, 152 (1985).
- [4] D. S. Salopek and J. R. Bond, *Phys. Rev.* **D42**, 3936 (1990).
- [5] M. Sasaki and E. D. Stewart, *Prog. Theor. Phys.* **95**, 71 (1996), arXiv:astro-ph/9507001.
- [6] D. H. Lyth, K. A. Malik, and M. Sasaki, *JCAP* **0505**, 004 (2005), arXiv:astro-ph/0411220v3.
- [7] D. H. Lyth and Y. Rodriguez, *Phys.Rev.Lett.* **95**, 121302 (2005), arXiv:astro-ph/0504045 [astro-ph].
- [8] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, *Phys.Rev.* **D62**, 043527 (2000), arXiv:astro-ph/0003278 [astro-ph].
- [9] K. Dimopoulos, M. Karciauskas, D. H. Lyth, and Y. Rodriguez, *JCAP* **0905**, 013 (2009), arXiv:0809.1055 [astro-ph].
- [10] R. L. Arnowitt, S. Deser, and C. W. Misner, (1962), *gravitation: an introduction to current research*, Louis Witten ed. (Wiley 1962), chapter 7, pp 227-265, arXiv:gr-qc/0405109 [gr-qc].
- [11] J. M. Maldacena, *JHEP* **05**, 013 (2003), astro-ph/0210603.
- [12] T. Hamazaki, *Phys.Rev.* **D78**, 103513 (2008), arXiv:0811.2366 [astro-ph].
- [13] D. H. Lyth and A. R. Liddle, *The primordial density perturbation: Cosmology, inflation and the origin of structure* (Cambridge University Press, Oxford, UK, 2009).
- [14] S. Weinberg, *Phys.Rev.* **D78**, 123521 (2008), arXiv:0808.2909 [hep-th]; *Phys.Rev.* **D79**, 043504 (2009), arXiv:0810.2831 [hep-ph].
- [15] M. Sasaki and T. Tanaka, *Prog.Theor.Phys.* **99**, 763 (1998), arXiv:gr-qc/9801017.
- [16] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, *Phys. Rev.* **D28**, 679 (1983).
- [17] S. Weinberg, *Phys.Rev.* **D67**, 123504 (2003), arXiv:astro-ph/0302326 [astro-ph].
- [18] H. Kodama and T. Hamazaki, *Phys.Rev.* **D57**, 7177 (1998), arXiv:gr-qc/9712045 [gr-qc].
- [19] A. Taruya and Y. Nambu, *Phys.Lett.* **B428**, 37 (1998), arXiv:gr-qc/9709035 [gr-qc]; Y. Nambu and A. Taruya, *Class.Quant.Grav.* **15**, 2761 (1998), arXiv:gr-qc/9801021 [gr-qc].
- [20] E. Dimastrogiovanni, N. Bartolo, S. Matarrese, and A. Riotto, *Adv.Astron.* **2010**, 752670 (2010), arXiv:1001.4049 [astro-ph.CO].
- [21] S. M. Leach, M. Sasaki, D. Wands, and A. R. Liddle, *Phys.Rev.* **D64**, 023512 (2001), arXiv:astro-ph/0101406.
- [22] Y. Takamizu, S. Mukohyama, M. Sasaki, and Y. Tanaka, *JCAP* **1006**, 019 (2010), arXiv:1004.1870; Y. Takamizu and J. Yokoyama, *Phys.Rev.* **D83**, 043504 (2011), arXiv:1011.4566.