New Kinds of Acoustic Solitons

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Abstract. We find that the modified sine–Gordon equation belonging to the class of the soliton equations describes the propagation of extremely short transverse acoustic pulses through the low-temperature crystal containing paramagnetic impurities with effective spin $S = \frac{1}{2}$ in the Voigt geometry case. The features of nonlinear dynamics of strain field and effective spins, which correspond to the different kinds of acoustic solitons, are studied.

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The development of physical acoustics has led to the appearance of technical tools of producing and measuring acoustic pulses about 10–10^2 ps in duration [1, 2]. The characteristics of such pulses are very perspective for diagnostics of fast processes and spectroscopy of solids. This attracts large attention to theoretical study of the interaction of picosecond acoustic pulses with paramagnetic crystals and other nonlinear media [3–8]. Usually, the semiclassical approach is employed to derive the equations governing the evolution of acoustic pulses. Some of these equations occur to be integrable with the help of the inverse scattering transformation (IST) method [9, 10]. In particular, the systems of integrable equations that generalize well-known integrable models of nonlinear coherent optics [11] describe the propagation of transverse-longitudinal picosecond pulses [6, 7].

The duration of picosecond acoustic pulses may be comparable with the oscillation period of the quantum transitions involved into the interaction. Following well-known parallels between the nonlinear phenomena in coherent optics and physical acoustics [12, 13], one has to treat acoustic pulses in this case as extremely short pulses [11, 14]. However, it is necessary in so doing to take into account essential difference between acoustic and optical waves. The linear velocities of the components of the former can differ significantly [15]. Thus, the longitudinal component velocity is normally much higher than the transverse ones. The nonlinear interaction of these components is weak in that case, and, consequently, longitudinal and transverse picosecond acoustic pulses propagate independently. At the same time, transverse components can interact efficiently since their linear velocities are equal under propagation along the acoustic symmetry axis of the crystal.

In this paper we investigate the nonlinear dynamics of the acoustic extremely short pulses in the low-temperature paramagnetic crystal in the external magnetic field presence. In accordance with above mentioned parallels between coherent optics and physical acoustics, we apply here the spectral overlap approximation [16]. This approximation is based on condition

$$\varepsilon \equiv (\omega_0 \tau_p)^2 \ll 1, \tag{1}$$

where ω_0 is the characteristic frequency of quantum transitions created by the external field; τ_p is the pulse duration. The main aim of the present article is to clarify the role of nonlinear interaction of the acoustic pulse components. We suppose for this reason that the pulses are especially transverse.

Let a tetragonal (or cubic) crystal contain paramagnetic impurities with effective spin $S = \frac{1}{2}$. Assume that the Cartesian axes x, y and z are aligned with symmetry axes of the crystal. Let the transverse acoustic pulse propagate along the x axis and the external magnetic field $\boldsymbol{B}$ be parallel to the z axis (Voigt geometry). Consider the one-dimensional case with dynamical variables depending on coordinate x and time t only. Then, the Hamiltonian $\hat{H}$ of the spin-elastic interaction has the form [13]

$$\hat{H} = -\frac{\hbar\omega_0}{2}\left[\hat{\sigma}_z + F_{44}\mathcal{E}_{yx}\hat{\sigma}_y + F_{55}\mathcal{E}_{zx}\hat{\sigma}_z\right]. \tag{2}$$

Here $\omega_0 = g\mu_{\mathrm{B}}B/\hbar$ is the frequency of the Zeeman splitting of the Kramers doublets; g is the Lande factor; μ_{B} is the Bohr magneton; $B = |\boldsymbol{B}|$; $\mathcal{E}_{yx} = \partial u_y/\partial x$ and $\mathcal{E}_{zx} = \partial u_z/\partial x$ are the components of the strain tensor; u_y and u_z are the Cartesian components of the local displacement vector $\boldsymbol{u}$; $F_{44} = g^{-1}(\partial g_{yx}/\partial\mathcal{E}_{yx})_0$ and $F_{55} = g^{-1}(\partial g_{zx}/\partial\mathcal{E}_{zx})_0$ are the components of the tensor of the spin-elastic interaction (in Voigt notation; subscript "0" means differentiation at the absence of acoustic pulse); g_{jk} are the components of the Lande tensor; $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are the Pauli matrices; $\hbar$ is the Planck constant. From the microscopic point of view, the spin-elastic coupling appears in the case $S = \frac{1}{2}$ due to the modulation of the Lande tensor components by the strain field [13].

In order to achieve fairly efficient interaction between paramagnetic impurities and strain field, the Zeeman splitting energy must exceed the thermal one. This implies that paramagnetic crystal has to be at helium temperatures, as it was in the experiments on acoustic self-induced transparency [12]. In that case the self-absorption of hypersound with frequency 10^2 GHz (or the picosecond acoustic pulses) due to anharmonicity, defects, etc. is appreciably lower than the acoustic absorption due to the presence of paramagnetic impurities [13]. Hence, the self-absorption effect playing important role under the room temperatures can be ignored in our case. Also, characteristic phase relaxation time for transitions within the Zeeman multiplets is 10^{-5}–10^{-6} s, and the energy relaxation time is much longer under such conditions [12]. We neglect these dissipative effects in what follows because the duration of the pulses considered is much shorter than all the relaxation times.

According to the general scheme of the semiclassical approach, we describe the evolution of effective spins by the equation on density matrix $\hat{\rho}$:

$$
\mathrm{i}\hbar\frac{\partial\hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]. \tag{3}
$$

On the other hand, the elastic pulse field obeys the classical Hamiltonian equation for continuous medium:

$$
\frac{\partial \boldsymbol{p}}{\partial t} = -\frac{\delta}{\delta \boldsymbol{u}}\left(H_{\mathrm{a}} + \int n\langle\hat{H}\rangle\,\mathrm{d}\boldsymbol{r}\right), \tag{4}
$$

$$
\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\delta}{\delta \boldsymbol{p}}\left(H_{\mathrm{a}} + \int n\langle\hat{H}\rangle\,\mathrm{d}\boldsymbol{r}\right), \tag{5}
$$

where $\boldsymbol{p}$ is the momentum density of the local displacement of the crystal;

$$
H_{\mathrm{a}} = \frac{1}{2}\int\left[\frac{p_y^2 + p_z^2}{\rho} + \rho a^2(\mathcal{E}_{yx}^2 + \mathcal{E}_{zx}^2)\right]\mathrm{d}\boldsymbol{r} \tag{6}
$$

is the Hamiltonian of the free strain field; ρ is the average density of the crystal; n is the concentration of paramagnetic ions; $\langle\hat{H}\rangle = \mathrm{Tr}(\hat{\rho}\hat{H})$ is the quantum average value of $\hat{H}$; a is the linear velocity of transverse acoustic waves. The integration is carried out over the crystal volume.

Let us introduce the Bloch variables

$$
U = \frac{\rho_{21} + \rho_{12}}{2}, \qquad V = \frac{\rho_{21} - \rho_{12}}{2\mathrm{i}}, \qquad W = \frac{\rho_{22} - \rho_{11}}{2},
$$

where ρ_{jk} $(j, k = 1, 2)$ are the elements of the density matrix. Then (3) gives

$$
\frac{\partial U}{\partial t} = (\omega_0 + \Omega_z)V + \Omega_y W, \tag{7}
$$

$$
\frac{\partial V}{\partial t} = -(\omega_0 + \Omega_z)U, \tag{8}
$$

$$
\frac{\partial W}{\partial t} = -\Omega_y U, \tag{9}
$$

where

$$
\Omega_y = \omega_0 F_{44}\mathcal{E}_{yx}, \qquad \Omega_z = \omega_0 F_{55}\mathcal{E}_{zx}.
$$

With (2), (4)–(6) we obtain

$$
\frac{\partial^2\Omega_y}{\partial t^2} - a^2\frac{\partial^2\Omega_y}{\partial x^2} = -\frac{n\hbar\omega_0^2F_{44}^2}{4\rho}\frac{\partial^2 V}{\partial x^2}, \tag{10}
$$

$$
\frac{\partial^2\Omega_z}{\partial t^2} - a^2\frac{\partial^2\Omega_z}{\partial x^2} = \frac{n\hbar\omega_0^2F_{55}^2}{4\rho}\frac{\partial^2 W}{\partial x^2}. \tag{11}
$$

Equations (7)–(11) describe the interaction of the transverse strain field with the paramagnetic crystal in the Voigt geometry case. As it is seen from (7)–(9), y-component Ω_y of the acoustic pulse causes quantum transitions between the Zeeman sublevels, whereas z-component Ω_z shifts dynamically their frequency. For transverse acoustic pulse propagating along the z axis (Faraday geometry), both components of the pulse excite quantum transitions only. The spin-elastic interaction between the components leads in this case to the rotation of the polarization plane of the pulse [4] (acoustic Faraday effect).

If we put

$$S = W + \mathrm{i}U,$$

then (7) and (9) yield

$$\frac{\partial S}{\partial t} = \mathrm{i}(\omega_0 + \Omega_z)V + \mathrm{i}\Omega_y S. \tag{12}$$

Let us assume that $\tau_p \sim 10\,\mathrm{ps}$ and the orders of ω_0 and Ω_z are comparable. Taking $\omega_0 \sim 10^{10}\,\mathrm{s}^{-1}$ (that is $B \sim 10^3\,\mathrm{Gs}$) [4, 12, 13], we see that condition (1) is valid. In that case the first term in the rhs of (12) can be neglected in the approximation of zeroth-order with respect to ε [17]. Then we have

$$S = W_0 \mathrm{e}^{\mathrm{i}\theta},$$

or

$$U = W_0 \sin\theta, \qquad W = W_0 \cos\theta, \tag{13}$$

where

$$\theta = \int_{t_0}^{t} \Omega_y \, \mathrm{d}t', \tag{14}$$

W_0 ($|W_0| \le 1/2$) is the inversion of population of the spin sublevels in the acoustic pulse absence. Substitution (13) into (8) gives

$$\frac{\partial V}{\partial t} = -W_0(\omega_0 + \Omega_z)\sin\theta. \tag{15}$$

To simplify further the equations we deal with, let us carry out some numerical estimations. Assuming $W \sim U$, $\partial/\partial t \sim 1/\tau_p$ we find from (9) that $\Omega_y \sim 1/\tau_p$. Therefore, the ratio η_y of the rhs of equation (10) to the terms in its lhs is estimated as $\eta_y \sim \sqrt{\varepsilon} n\hbar\omega_0 F_{44}^2/4\rho a^2$. The value of similar parameter of (11) is estimated as $\eta_z \sim \sqrt{\varepsilon} n\hbar\omega_0 F_{55}^2/4\rho a^2$. For paramagnetic ions Co^{2+} in cubic crystal MgO at helium temperatures we use the following experimental data [4, 13]: $n \sim 10^{19}\,\mathrm{cm}^{-3}$, $\omega_0 \sim 10^{10}\,\mathrm{s}^{-1}$, $\rho \sim 1\,\mathrm{g/cm}^3$, $a \sim 5\cdot 10^5\,\mathrm{cm/s}$, and $F_{44} \sim F_{55} \sim 10^3$. If $\tau_p \sim 10^{-11}\,\mathrm{s}$, then $\eta_y \sim \eta_z \sim 10^{-2}$. Since parameters η_y and η_z are much less than unity, we shall reduce the order of derivatives in (10) and (11) with the help of the unidirectional propagation approximation [18].

Having introduced new independent variables $\tau = t - x/a$ and $\zeta = \eta x$, where $\eta = \max(\eta_y, \eta_z)$, we obtain

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = -\frac{1}{a}\frac{\partial}{\partial \tau} + \eta\frac{\partial}{\partial \zeta}.$$

In the first order in η, we write

$$\frac{\partial^2}{\partial x^2} \approx \frac{1}{a^2}\frac{\partial^2}{\partial \tau^2} - 2\frac{\eta}{a}\frac{\partial^2}{\partial \tau \partial \zeta}, \qquad \frac{\partial^2}{\partial x^2} \approx \frac{1}{a^2}\frac{\partial^2}{\partial \tau^2}$$

for the lhs and rhs of equations (10) and (11), respectively. Integration of the wave equations obtained in this way with respect to τ, substitution of expressions (13) and taking into account (15) give us the following system in the terms of variables τ and x:

$$\frac{\partial \Omega_y}{\partial x} = -\beta_y(\omega_0 + \Omega_z)\sin\theta, \tag{16}$$

$$\frac{\partial \Omega_z}{\partial x} = \beta_z \Omega_y \sin\theta, \tag{17}$$

where $\beta_y = -W_0 n\hbar\omega_0^2 F_{44}^2/(8\rho a^3)$, $\beta_z = \beta_y F_{55}^2/F_{44}^2$.

Equations (16) and (17) possess the integral of motion:

$$\Omega_z^2 + 2\omega_0\Omega_z + \frac{F_{55}^2}{F_{44}^2}\Omega_y^2 = f(\tau), \tag{18}$$

where function $f(\tau)$ is determined by the boundary conditions. The similar integral was revealed in [7]. Defining new variables

$$\tau' = \int_0^\tau \sqrt{1 + f(\tilde{\tau})/\omega_0^2}\,\mathrm{d}\tilde{\tau},$$

$$\Omega_y' = \frac{\Omega_y}{\sqrt{1 + f(\tau)/\omega_0^2}},$$

$$\Omega_z' = \frac{\omega_0 + \Omega_z}{\sqrt{1 + f(\tau)/\omega_0^2}} - \omega_0,$$

one can prove that $f(\tau)$ is supposed equal to zero without loss of the generality [7]. Then, we find from (18):

$$\Omega_z = -\omega_0\left(1 - \sqrt{1 - \tau_c^2\Omega_y^2}\right), \tag{19}$$

where

$$\tau_c = \frac{F_{55}}{\omega_0 F_{44}}.$$

(It is seen that inequality $|\Omega_z| \leq 2\omega_0$ is fulfilled.) Finally, using (14), (16) and (19), we obtain

$$\frac{\partial^2\theta}{\partial x\partial\tau} = -\omega_0\beta_y\sqrt{1 - \tau_c^2\left(\frac{\partial\theta}{\partial\tau}\right)^2}\,\sin\theta. \tag{20}$$

This equation is reduced to the famous sine–Gordon (SG) equation [9, 10] if $\tau_c = 0$. Equation (20) with $\tau_c \neq 0$ is known as the modified SG (mSG) equation [19–22] and belongs to the class of equations integrable by the IST method. Its first physical application was found recently in [23], where (20) was shown to describe the propagation of electromagnetic extremely short pulses through the anisotropic media. In [19–22], this equation was derived in the course of mathematical study of the Bäcklund transformation of the SG equation.

Being integrable with the help of the IST method, (20) admits the zero curvature representation

$$\frac{\partial \hat{L}}{\partial x} - \frac{\partial \hat{A}}{\partial \tau} + [\hat{L}, \hat{A}] = 0, \tag{21}$$

where matrices $\hat{L}$ and $\hat{A}$ are defined as given

$$\hat{L} = \frac{1}{2\lambda}\begin{pmatrix} \mathrm{i}\lambda\Omega_y & \sqrt{1-\tau_c^2\Omega_y^2} - \mathrm{i}\tau_c\Omega_y \\ \sqrt{1-\tau_c^2\Omega_y^2} + \mathrm{i}\tau_c\Omega_y & -\mathrm{i}\lambda\Omega_y \end{pmatrix},$$

$$\hat{A} = -\frac{\omega_0\beta_y}{2}\begin{pmatrix} -\mathrm{i}\tau_c\sin\theta & \lambda \mathrm{e}^{\mathrm{i}\theta} \\ \lambda\mathrm{e}^{-\mathrm{i}\theta} & \mathrm{i}\tau_c\sin\theta \end{pmatrix},$$

and λ is the spectral parameter. Equation (21) is nothing but the compatibility condition of the following Lax pair

$$\begin{cases} \dfrac{\partial\xi}{\partial\tau} = \hat{L}\xi, \\ \dfrac{\partial\xi}{\partial x} = \hat{A}\xi, \end{cases} \tag{22}$$

where $\xi = \xi(\lambda,\tau,x) = (\xi_1,\xi_2)^T$.

To investigate the nonlinear dynamic of the transverse strain field components and effective spins, we construct the soliton solutions of (20). It is well known that the multi-soliton solutions of the integrable equation can be found using the algebraic methods. Here we apply the Darboux transformation (DT) technique [24]. Let $\varphi = (\varphi_1,\varphi_2)^T$ be a solution of (22) with $\lambda = \tau_p$. The Lax pair (22) is covariant with respect to DT $\{\xi_1,\xi_2,\theta\} \to \{\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\theta}\}$ of the form

$$\begin{aligned} \tilde{\xi}_1 &= (\lambda\xi_1 - \tau_p\varphi_1\xi_2/\varphi_2)\exp[\mathrm{i}(\tilde{\theta}-\theta)/2], \\ \tilde{\xi}_2 &= (\lambda\xi_2 - \tau_p\varphi_2\xi_1/\varphi_1)\exp[\mathrm{i}(\theta-\tilde{\theta})/2], \\ \tilde{\theta} &= \theta + \mathrm{i}\ln\frac{\tau_p\varphi_1^2 - \tau_c\varphi_1\varphi_2}{\tau_p\varphi_2^2 - \tau_c\varphi_1\varphi_2}. \end{aligned} \tag{23}$$

This implies that relation (23) gives us new solution $\tilde{\theta}$ of the mSG equation (20) if θ is its known solution and φ is a solution of the Lax pair.

In the zero background case (i.e., $\theta = 0$), we obtain from (23) the following expression for the one-soliton solution of the mSG equation:

$$\theta = 2\arccos\frac{q - \tanh\chi}{\sqrt{\Delta}},$$

where $q = \tau_c/\tau_p$, $\chi = (t - x/v)/\tau_p$, $\Delta = 1 - 2q\tanh\chi + q^2$. Velocity v of the soliton and its free parameter τ_p defining the duration are connected by the relation

$$v^{-1} = a^{-1} + \omega_0\beta_y\tau_p^2.$$

The corresponding formula for y-component of the transverse strain field is

$$\Omega_y = \frac{2\,\mathrm{sech}\,\chi}{\tau_p}\,\frac{1 - q\tanh\chi}{\Delta}. \tag{24}$$

For "time area" $A_y \equiv \int_{-\infty}^{\infty}\Omega_y\,\mathrm{d}t$ of this component of the acoustic pulse we find

$$A_y = \begin{cases} \pm 2\pi & \text{for } |\tau_p| > \tau_c \\ 0 & \text{for } |\tau_p| < \tau_c \end{cases}.$$

The last formula indicates that the acoustic extremely short pulses are divided into two families. The family with $A_y = \pm 2\pi$ exists for the SG equation also and corresponds to unipolar 2π-pulses (kinks and antikinks). The pulses of the family with $A_y = 0$ are bipolar 0π-pulses. Unlike the breathers of the SG equation (for them $A_y = 0$ as well), these pulses are steady-state. The solitons of this kind were called as neutral kinks in [23].

In the cubic crystal, one has $|F_{55}| = |F_{44}|$. Then $|\tau_p| < \tau_c$ due to condition (1), and neutral kinks exist only in such a crystal. In the crystals with tetragonal symmetry, both types of the solitons are possible.

Let us discuss in details the properties of the acoustic solitons. In the case $|\tau_p| > \sqrt{2}\tau_c$, component Ω_y (24) of the unipolar one-soliton solution has a single maximum, whose value is smaller than $1/\tau_c$. Accompanying dynamics of effective spins is very similar to that of the SG equation: the leading edge of the pulse of y-component inverts completely the populations of the Zeeman sublevels, and the trailing one returns them to the initial state. The z-component is small as compared to ω_0, and its role is insignificant.

Under $\tau_c < |\tau_p| < \sqrt{2}\tau_c$, component Ω_y has two symmetric peaks (see solid line in figure 1a) with the largest possible amplitude $1/\tau_c$ determined by (18). The peaks are separated by the time interval

$$2|\tau_p| \operatorname{arcsinh}\sqrt{\frac{q^2}{1-q^2} - 1}.$$

The first peak inverts the populations of the spin sublevels while the component Ω_z grows in amplitude and reaches the absolute value $2\omega_0$ in the center of the soliton (figures 1b and 1c). The component Ω_z has an asymmetry on polarity: it decreases the transition frequency ($\Omega_z < 0$) and shifts the Zeeman sublevels so that the ground sublevel becomes excited. On account of this, the paramagnetic ions are in the ground state between the peaks. (The positions of the spin sublevels 1 and 2 of the Kramers doublet under the pulse passage are pictured in figure 1c.) When the second peak has come, the z-component vanishes reverting the mutual position of the sublevels to the initial state. Finally, the second peak of y-component causes the back transitions from excited sublevel to the ground one.

When $|\tau_p| < \tau_c$ and the interval between the peaks of neutral kink surpasses its duration, the dynamics of the strain fields and effective spins is similar to the second case described above. The only difference is that the peaks of Ω_y are opposite in sign. The time interval between the peaks is

$$2|\tau_p| \operatorname{arccosh}\sqrt{1 + \frac{q^2}{q^2 - 1}}.$$

If we take duration of such a soliton to be shorter, then the peaks are brought closer together and the degree of excitation of the paramagnetic ions decreases (see dotted lines in figure 1).

When $|\tau_p| \to \tau_c$, the interval between the peaks grows indefinitely large, and y-component (24) consists of a single peak with amplitude equal to $1/\tau_c$ and with absolute

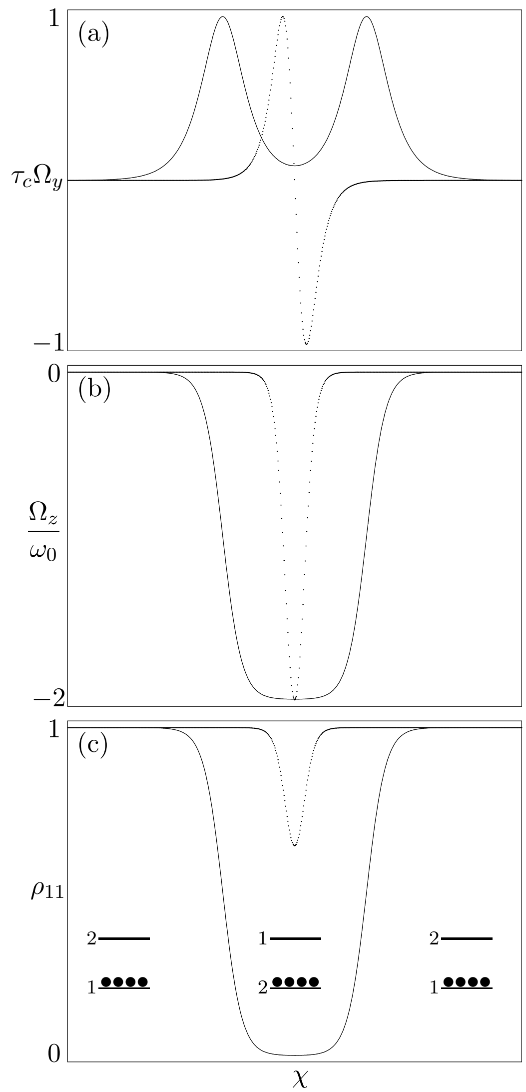

Figure 1. Profiles of the components of the strain field and the population ρ_{11} for the one-soliton pulses with $\tau_c < |\tau_p| < \sqrt{2}\tau_c$ (solid lines) and $|\tau_p| < \tau_c$ (dotted lines)

value of time area A_y equal to π. This case stresses especially the role of the component Ω_z of the acoustic pulses considered. The peak of Ω_y inverts almost completely the population of the spin sublevels. This state of the effective spins is unstable in the absence of the strain field. But, the z-component, whose amplitude tends to $2\omega_0$, shifts the levels of the Kramers doublets in a such manner that the energy of the excited sublevel becomes lesser than the energy of the ground one. Owing to this, the state of effective spins after the passage of the y-component peak becomes stable.

The form of the acoustic solitons in the case $f(\tau) \neq 0$ (see integral (18)) tends to their form in the case $f(\tau) = 0$ at $t \to \infty$ since the pulse velocity v differs from linear velocity a of the transverse waves. As it follows from the previous consideration, this means in particular that the amplitude of the pulses is bounded, and the component of the strain field parallel to the external magnetic field has the asymmetry on polarity.

In this paper we considered the propagation of the transverse acoustic extremely short pulse through paramagnetic crystal in a direction perpendicular to external magnetic field. It was shown that the dynamics of the strain field and effective spins is governed by the modified sine–Gordon equation (20). The soliton solutions of this equation reveal strong nonlinear coupling between the components of the acoustic pulse. As a result of this, the behaviour of paramagnetic impurities and elastic fields during the interaction exhibits new features.

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