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Stability of cold fronts in clusters: is magnetic field necessary?

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ABSTRACT
Cold fronts – sharp discontinuities recently discovered by Chandra in many clusters of galaxies – are believed to be due to a hot gas flow over a colder gravitationally bound gas cloud. We analyze the stability of the fronts with respect to Kelvin-Helmholtz instability and show that an intrinsic width of the interface of the order of a few per cent of the curvature radius strongly limits the growth of perturbation. For the best studied case of a front in the Cluster Abell 3667 we conclude that current observational data on the width and extent of the front can be explained even in the absence of dynamically important magnetic fields.

Key words: galaxies: clusters: individual: A3667, X-rays: galaxies: clusters

1 INTRODUCTION

Cold fronts were discovered as sharp features in the X-ray surface brightness distribution in Chandra observations of the clusters A2142 and A3667 (Markevitch et al., 2000, Vikhlinin, Markevitch & Murray, 2001a), see also Markevitch et al. (2002). Similar features have now been found in several other clusters (e.g. Sun et al., 2002, Kempner, Sarazin & Ricker, 2002). Unlike shocks, these features have lower gas temperature on the X-ray brighter side of the discontinuity. For that reason they are called "cold fronts". It is believed that some cold fronts are formed when a subcluster merges with another cluster and the ram pressure of gas flowing outside the subcluster gives the contact discontinuity the characteristic curved shape. Indeed, features resembling cold fronts are found in numerical simulations of cluster formation (Bialek, Evrard & Mohr, 2002, Nagai & Kravtsov, 2003). Ablation of the gaseous cloud by the hot gas causes characteristic differential motion of the gas inside the subcluster, which transports the low entropy gas from the subcluster core towards the contact discontinuity, thus enhancing the jump in temperature and surface brightness across the discontinuity (Heinz et al. 2003).

Here we address the question of the front stability. As was pointed out by Vikhlinin et al. (2001a,b), Vikhlinin and Markevitch (2002) the observed fronts are narrow and could be unstable to the Kelvin-Helmholtz (KH) instability. For A3667 however the front appears to be narrow (less than ∼5 kpc in width) up to ∼30°from the stagnation point. Magnetic fields can act as a stabilizing agent thus allowing indirect estimates of the field strength near the front (Vikhlinin, Markevitch & Murray,2001b). On the other hand numerical simulations without magnetic field (and in particular relatively high resolution simulations by Heinz et al. 2003) do not show instability of the front within 20-30°. While this discrepancy could be due to numerical effects, we revisit the question of the front stability below and show that the characteristic convex geometry and finite (small) intrinsic width of the interface may help to stabilize the discontinuity with respect to KH instability.

The structure of the paper is as follows. In Section 2 we derive a simple expression for the growth of the KH instability along the curved interface. In Section 3 we argue that diffusion processes are likely to set an approximately constant width for the interface. In Section 4 we discuss astrophysical applications. The last section summarizes our findings.

2 KH INSTABILITY ALONG THE SPHERICAL INTERFACE

For an infinitely thin plane parallel interface separating semi-infinite layers of two incompressible fluids with densities ρ_1 and ρ_2 the dispersion relation for Kelvin-Helmholtz instability is (Landau & Lifshitz, 1987):

$$\omega = kv\frac{\mu \pm i\sqrt{\mu}}{1+\mu}, \quad (1)$$

where $k = 2\pi/\lambda$ is a wave number, λ is the wavelength of the perturbation, v is the velocity of fluid with density ρ_1, while fluid with density ρ_2 is at rest and $\mu = \frac{\rho_1}{\rho_2}$. The frequency ω

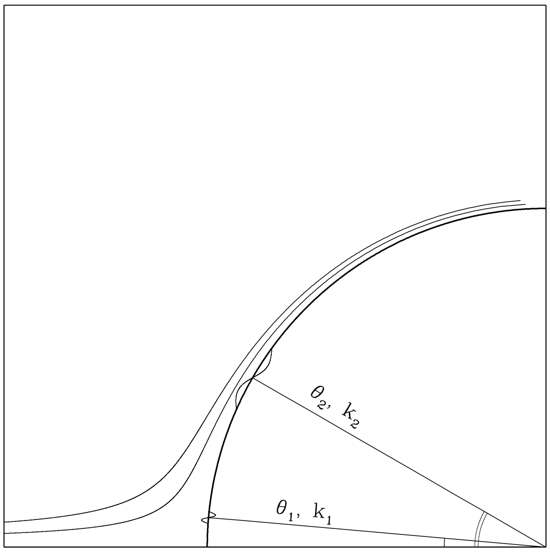

Figure 1. Geometry of the problem. Hot gas is flowing around an approximately spherical cloud of colder gas. Stream lines illustrate the hot gas pattern of motion. The gas inside the cloud is assumed to be still. While the interface is unstable against KHI, observations shows that the front remains thin up to angles of $\sim 30°$. Perturbation with a wave number k_1 at angle θ_1 will have a wave number k_2 when arriving at angle θ_2 due to the increase of the velocity along the interface.

is measured in the frame of the fluid at rest (i.e. with density ρ_2). For cold fronts in clusters we adopt the geometry of the flow following Vikhlinin et al. 2001a as shown in Fig.1. The front has a curvature radius R and the velocity v along the interface is assumed to follow the law $v = v_0 \sin\theta$, where θ is the angle along the interface. For the potential flow of incompressible fluid $v_0 = 1.5\ v_\infty$, where v_∞ is the velocity of the cold gas through the hot gas. For a particular case of the cold front in A3667, Vikhlinin et el. (2001b) argue that due to compressibility v_0 is more close to v_∞. For simplicity we assume that the densities of both fluids remain constant along the interface.

Consider the perturbation of the interface separating cold and hot gas. From eq.(1) it follows that perturbations are advected along the interface with the increasing velocity $v_a = v_{a0} \sin\theta$, where $v_{a0} = v_0 \frac{\mu}{1+\mu}$. This implies that the wave number k of the perturbation decreases as:

$$k(\theta) = k_1 \frac{v_{a1}}{v_a} = k_1 \frac{\sin\theta_1}{\sin\theta}, \qquad (2)$$

where $k_1 = k(\theta_1)$ is the wave number at some initial position θ_1. This result can be formally obtained from the lowest order WKB (Wentzel-Kramer-Brillouin) approximation (see appendix), applicable if $kR\sin\theta \gg 1$. Apart from the exponentially growing factor, the next order WKB approximation allows one to estimate slower changes of the perturbation amplitude δ caused by the changes of the velocity along the interface and stretching of fluid elements in the azimuthal direction (see appendix). For estimates we assume that $\delta \propto \sin\theta^{-\alpha}$, and $\alpha \approx 2$. I.e. $\delta = \delta_1 \left(\frac{\sin\theta_1}{\sin\theta}\right)^2$, where δ_1 is the initial amplitude.

Thus one can evaluate the growth of the perturbation propagating along the interface as:

$$\delta = \delta_1 \left(\frac{\sin\theta_1}{\sin\theta}\right)^2 \exp\left\{\int \gamma(t)dt\right\}, \qquad (3)$$

where $\gamma(t)$ is the increment of KH instability and one has to evaluate it using k from eq.(2). An analogous expression was used by Inogamov & Chekhlov (1991) for the growth of perturbations for Rayleigh-Taylor instability. From eq. (1) and (2) it is clear that the increment $\gamma \propto kv$ does not increase as the perturbation propagates, since the increase of the velocity amplitude is compensated by the decrease of the wave number. Thus $\gamma(t) = k_1 v_{i0} \sin\theta_1 = const$, where $v_{i0} = v_0 \frac{\sqrt{\mu}}{1+\mu}$. Replacing dt with $dt = \frac{dt}{d\theta} d\theta = \frac{R}{v_a} d\theta$ one can write an explicit expression for $\int \gamma(t)dt$:

$$\int \gamma(t)dt = \int k_1 v_{i0} \sin\theta_1 \frac{R}{v_{a0}\sin\theta} d\theta = Rk_1 \sin\theta_1 \frac{1}{\sqrt{\mu}} \ln\left[\tan(\theta/2)\right]. \qquad (4)$$

Thus the final expression for the growth factor for a perturbation starting at θ_1 with the wave number k_1 measured at position θ_2 is:

$$\text{Growth factor} = \left(\frac{\sin\theta_1}{\sin\theta_2}\right)^2 \exp\left\{Rk_1 \sin\theta_1 \frac{1}{\sqrt{\mu}} \ln\left[\tan(\theta/2)\right]\Big|_{\theta_1}^{\theta_2}\right\}. \qquad (5)$$

One can compare this expression with eq. (10) from Vikhlinin & Markevitch (2002) - i) an additional factor appears in front of the exponential and more importantly ii) the argument of the exponential is changed due to the variation of the velocity along the interface, thus **reducing the growth factor** for a fixed k_1 and θ_1. We note here that from the point of view of the degree of nonlinearity of a perturbation one can use the ratio of the amplitude to the wavelength: $\delta/\lambda = \delta \frac{k}{2\pi}$. Therefore the expression for the degree of nonlinearity of a perturbation will contain an additional factor proportional to $\frac{k_2}{k_1} = \frac{\sin\theta_1}{\sin\theta_2}$ in front of the exponential.

Alternatively, since $k_1 \sin\theta_1 = k_2 \sin\theta_2$, we may regard the growth factor as a function of k_2 specified at the angle θ_2. The formal condition for the growth factor (5) to diverge for small θ_1 is $Rk_2 \sin\theta_2/\sqrt{\mu} > 2$. Thus in spite of the **suppression** of the growth factor for a fixed **initial** wave number, the growth factor can be infinitely large for a fixed **final** wave number. There are however several factors which can limit the growth of instability. We argue below that a small, but finite, thickness of the interface is the most important factor in the conditions relevant for cold fronts in clusters.

2.1 Finite thickness of the interface

Any real interface can not be infinitely thin. If the thickness of the interface is h, then only modes with $k \le k_{max} \sim 1/h$ are unstable. Thus for a fixed θ_2 and $k_2 < k_{max}$ the minimum angle at which KH instability starts is such that $\sin\theta_{min} = \frac{k_2}{k_{max}} \sin\theta_2$. In Section 3 we argue that diffusive

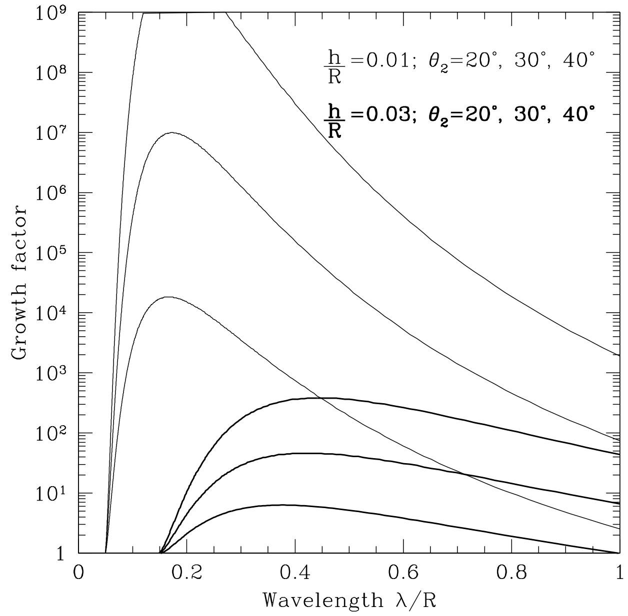

Figure 2. Growth factor as a function of the wavelength λ_2 for several values of angle θ_2 and intrinsic with h. λ and h are expressed in units of the front curvature radius. The factor $\frac{v_{i0}}{v_{a0}}$ in eq.(7) is set to 1.43.

processes across the interface may set a finite thickness of the interface, which is almost independent of the position (angle). One can approximately account for the finite width of the interface by using θ_{min} as a lower bound in eq.(5). But in fact the largest contribution to the growth factor comes from small angles and this part of the integral has to be evaluated more accurately than simply introducing a cutoff in θ. One can use for instance the dependence of increment on the wave number and thickness in the form given by Rayleigh (see "The Theory of Sound", e.g. 1945 edition) for two fluids with similar densities:

$$\gamma(k,h) = \frac{v}{2h}\sqrt{e^{-2kh} - (kh-1)^2}, \qquad (6)$$

where $k \leq 1.2785/h$ is the condition for instability. Thus the growth factor is:

$$\mathrm{GF} = \left(\frac{\sin\theta_1}{\sin\theta_2}\right)^2 \exp\left\{\frac{R}{h}\frac{v_{i0}}{v_{a0}}\int_{\theta_1}^{\theta_2}\sqrt{e^{-2kh} - (kh-1)^2 d\theta}\right\}, \quad (7)$$

where $\theta_1 = \theta_{min}$ is such that $k_2\frac{\sin\theta_2}{\sin\theta_1} = 1.2785/h$. In this case the growth factor has to be calculated numerically. In Fig.2 we show the total growth factor calculated for several values of θ_2 and h. For $\theta_2 \sim 30°$the instability grows very strongly if the thickness of the interface is less than ∼1.5-2% of the curvature radius. We note that all the above expressions for the growth factor do not depend explicitly on the absolute value of v_0, indicating that the appearance of the front may be similar for clouds moving with different velocities, if the relation between the width and the curvature of the interface is the same.

An interesting question is if the nonlinear evolution of KH instability itself can be responsible for establishing the width of the interface (along the lines of reasoning given by Nulsen, 1982) of order of few percent of R, smoothing an interface and suppressing growth of longer modes. We leave this question for subsequent studies and in the next section consider the width of the interface set by diffusion.

3 WIDTH OF THE FRONT DUE TO TRANSPORT PROCESSES

For flow past a flat plate (with high Reynolds number) the width of the boundary layer is known to vary as the square root of the distance x from the front edge of the plate (Blasius law):

$$h \sim \sqrt{x\nu/U}, \qquad (8)$$

where ν is the viscosity. For a rounded body, the flow velocity increases linearly with distance from the stagnation point, $U \sim v_0 x/R$, making the width of the boundary layer constant (Landau & Lifshitz, 1987). Thus $h \sim \sqrt{\nu R/v_0}$, where R is the curvature radius. Substituting further $\nu \sim v_i\lambda_i$, where v_i and λ_i are the rms velocity and mean free path for ions and assuming that $v_i \sim v_0$ one gets an order of magnitude estimate of the expected width $h \sim \sqrt{\lambda_i R}$. In the absence of magnetic field the viscosity of the hot gas is larger than for the cold gas and most of the velocity drop may take place on the hotter side in the form of a broad layer. We note here that observationally it is almost impossible to "see" such a layer.

Analogously one can consider the case of a scalar diffusion (e.g. thermal conduction) with a constant diffusion coefficient D. Assuming that conditions inside the cloud (where the fluid is not moving) are frozen and seeking a stationary solution of a diffusion equation in outer gas it is easy to show that $h \sim \sqrt{\frac{DR}{v_0}}$.

Both expressions, when applied to the case of A3667 and compared with the upper limit on the front width (Vikhlinin et al., 2001a), indicate significant suppression of transport processes across the front. More accurate calculations and a detailed comparison with observations are beyond the scope of this paper and will be reported elsewhere. In what follows we assume that the interface has a constant width independent of θ and the expression (7) for the growth factor is applicable.

4 APPLICATION TO A3667

We now apply the above results to the particular case of the cold front in A3667. We set the following parameters based on the results of Vikhlinin et al., 2001a,b, Vikhlinin & Markevitch 2002: R = 410 kpc, $\theta_2 = 30°$, $\frac{v_{i0}}{v_{a0}} = 1.43$. They give the width of the front (upper limit) in terms of the standard deviation of a gaussian smoothing, as $\sigma \leq 5$ kpc. In our notation, $h = \sqrt{2\pi}\sigma \sim 2.5\sigma$. In Fig.3 we plot the growth factor calculated for several values of σ and angle θ_2. This is largely the same figure as Fig.2 but now in natural units. For comparison we show (upper dotted curve) the growth factor calculated for an infinitely thin interface, assuming fixed wave number k and θ_2 =30°. One can see that accounting for varying wave number and the finite thickness

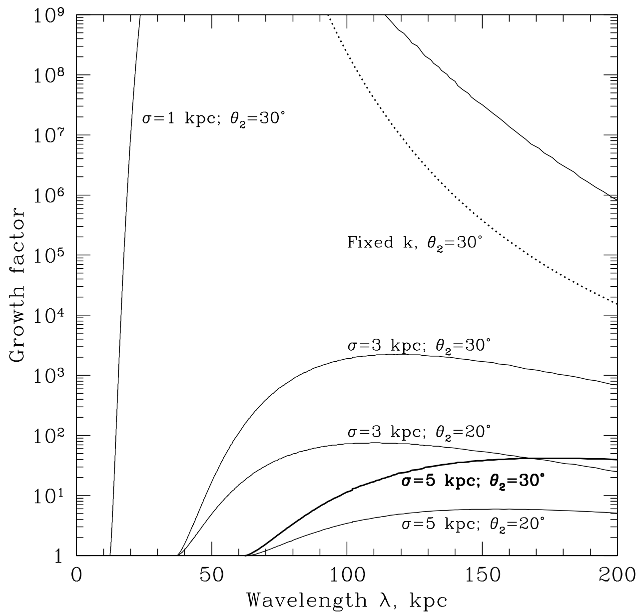

Figure 3. Growth factor for A3667 parameters. Observational limit on the width of the interface is $\sigma \leq$ 5kpc. Upper dotted curve corresponds to the estimate of the growth rate for fixed k.

of the interface (within the observational limits, i.e. $\sigma \leq 5$ kpc) drastically reduces the growth factor. For these reduced values it is not obvious that small initial perturbations can grow strongly nonlinearly and disturb the sharp appearance of the front around $\theta_2 \sim 30^\circ$. For larger angles or for smaller widths of the interface the growth factors are large.

4.1 Role of the cloud internal motions

While we assumed above that the gas inside the cloud is still, internal motions are likely to appear during the formation of the front and especially during later periods of the cloud evolution when continuous stripping at the sides of the cloud is compensated by the cold gas flow along the interface to replenish the losses. E.g. in simulations by Heinz et al. (2003) the velocity of the gas inside the cloud ~0.2 of the velocity of the surrounding gas at the moment when the morphology of the front and the temperature distribution resemble those observed in A3667[1]. The internal motions have a two-fold effect: i) they reduce the shear rate and ii) they increase the group speed of the perturbation in the interface, both of which reduce the total growth of a perturbation arriving at the angle θ_2. Accounting for internal motions can easily reduce the argument of the exponential in equation (7) by 30-50% thus strongly reducing the final growth factor compared to Fig.3.

[1] We note here that in the absence of gravity the velocities of the colder and hotter fluids would be related as $v_{cold} = v_{hot}\sqrt{\mu}$ as follows from the Bernoulli equation. Due to gravity the velocity of the motions in cold gas should be lower.

4.2 Role of gravity and compressibility

In the above discussion we have completely neglected the role of gravity, although we assumed that it plays an important role in preserving the integrity of the cold cloud when it passes through the hot gas. For an infinitely thin interface, gravity modifies the dispersion relation (1) to:

$$\omega = kv\frac{\mu}{1+\mu} \pm k\sqrt{\frac{g}{k}\frac{1-\mu}{1+\mu} - v^2\frac{\mu}{(1+\mu)^2}}, \tag{9}$$

where g is gravitational acceleration. For a fixed k_2 and θ_2 and given that $k = k_2 \sin\theta_2 / \sin\theta_1$ and $v = v_0 \sin\theta$ the role of gravity is more important for small angles θ, effectively introducing a lower limit on θ_1. The growth factor calculated with the above dispersion relation and using the values of g and v_0 from Vikhlinin & Markevitch (2002) is very large, implying that gravity does not strongly suppress KH instability, in agreement with their conclusion.

For an interface of finite thickness, gravity makes the region near the stagnation point stable for perturbations of any k. One can estimate the Richardson number of the flow assuming that density and velocity both vary over a layer of thickness h:

$$Ri \sim \frac{\frac{-g}{\rho}\frac{\partial\rho}{\partial z}}{\left(\frac{\partial v}{\partial z}\right)^2} \approx \frac{\Delta\rho}{\rho}\frac{gh}{v^2}. \tag{10}$$

The sufficient condition for stability is $Ri > 0.25$ everywhere in the flow. Again plugging in the numbers from Vikhlinin & Markevitch (2002) we find that for $\sigma = 1, 3, 5$ kpc the front is stable up to angles of ~6,11,14°respectively. Simple estimates show that account for this stable region would further reduce the growth factor compared to Fig.3, although not dramatically. This is expected given that gravity alone (for the thin interface case - see above) does not strongly suppress KH instability. Accurate calculations of the growth factor taking account of gravity and the finite thickness of the interface require knowledge of the density and velocity profiles of the interface and are beyond the scope of this paper. We note only that stretching of the flow in the stable region near the stagnation point insures that initial perturbations in the unstable regions further downstream are small.

Since we are considering motion near the stagnation point the velocities are small compared to the gas sound velocity (even for colder gas) and compressibility does not play a significant role. Further downstream the compressibility starts to be more important and it reduces the growth factor of the perturbation (see dispersion relation in Miles, 1958, Gerwin, 1968 or Nulsen 1982). E.g. for the density ratio $\mu = 0.5$ the quantity $\frac{\mathrm{Im}[\omega]}{\mathrm{Re}[\omega]}$ is equal to $\frac{1}{\sqrt{\mu}} = 1.41$ for small shear velocity and it drops to ~ 1.13 when the shear velocity approaches the sound speed of the colder gas[2].

5 CONCLUSIONS

We have shown that accounting for wave number changes along the curved interface and finite (but small) width of the interface significantly reduces the KH instability growth

[2] An argument of the exponential in eq.(5) is proportional to this factor

factor. We also argue that if diffusion sets the intrinsic width of the interface then this width does not vary much along the interface (at least for small angles).

For a set of parameters relevant for the front in A3667 the growth factor is not large enough to guarantee nonlinear growth at angles $\sim 30^\circ$, provided that the intrinsic width of the front is of the order of a few percents of the curvature radius (i.e. not much less than the existing observational limit). Therefore dynamically important magnetic fields may not be necessary to stabilize this particular cold front. The growth factor (and therefore possible limits on the magnetic fields) **depends critically on the width of the interface** and much more weakly on the angular extent over which the front remains sharp. A factor of 2-3 improvement in the observational constraints on the interface width is therefore extremely important.

For a given width of the interface the value of the growth factor does not depend strongly on the absolute value of the shear velocity. This probably explains the ubiquity of fronts both in observations and hydrodynamical simulations.

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APPENDIX A: WKB APPROXIMATION FOR PLANE ACCELERATING FLOWS

Consider a plane boundary between two incompressible, inviscid, irrotational fluids with densities ρ_1 and ρ_2 moving in the x direction with velocities $u_1(x,y)$ and $u_2(x,y)$. The fluid '1' is accelerating along the interface: $\frac{\partial u_1}{\partial x} = u_1/x = const$. To match pressures at the boundary we have to assume that the velocity of fluid '2' is $u_2 = u_1\sqrt{\rho_1/\rho_2} = u_1\sqrt{\mu}$ and is also accelerating along the interface. We now consider small perturbations of the boundary $\xi(x,t)$ and velocity potentials of two fluids $\phi_{1,2}(x,y,t)$. The governing equations are:

$$\Delta\phi_{1,2} = 0 \tag{A1}$$

$$\frac{\partial\phi_{1,2}}{\partial y} = \frac{\partial\xi}{\partial t} + u_{x,1,2}\frac{\partial\xi}{\partial x} - \xi\frac{\partial u_{y,1,2}}{\partial y} \tag{A2}$$

$$\rho_1\frac{\partial\phi_1}{\partial t} + \rho_1 u_{x,1}\frac{\partial\phi_1}{\partial x} = \rho_2\frac{\partial\phi_2}{\partial t} + \rho_2 u_{x,2}\frac{\partial\phi_2}{\partial x}, \tag{A3}$$

where $u_{x,1,2}$ and $u_{y,1,2}$ are the velocity components in x and y directions for two flows. The second equation is the kinematic boundary condition and the third equation is the Bernoulli equation. These equations are evaluated at the boundary ξ.

We are seeking a solution of these equations in the form $\xi = b(x)e^{i(S_t(t)+S_x(x))/\varepsilon}$ and $\phi_{1,2} = a_{1,2}(x,y)e^{i(S_t(t)+S_x(x)+S_{y,1,2})/\varepsilon}$, which decays for $y=\pm\infty$, where a,b, and all S are slowly varying functions and ε is a small parameter. Using these expressions and keeping only the terms of order $1/\varepsilon$ we recover a dispersion relation, which we write in a "spatial" form, denoting $\frac{\partial S_t}{\partial t}/\varepsilon = -\omega$, $\frac{\partial S_x}{\partial x}/\varepsilon = k$:

$$k = w\frac{\rho_1 u_{x,1} + \rho_2 u_{x,2} \pm i(u_{x,1}-u_{x,2})\sqrt{\rho_1\rho_2}}{\rho_1 u_{x,1}^2 + \rho_2 u_{x,2}^2} \tag{A4}$$

Now collecting higher order terms and using the relations $\frac{\partial a_{1,2}}{\partial y} = \pm\frac{1}{i}\frac{\partial a_{1,2}}{\partial x}$ from the Laplace equation and $\frac{\partial a_2}{\partial x} = \frac{\rho_1 u_{x,1}}{\rho_2 u_{x,2}}\frac{\partial a_1}{\partial x}$ from the Bernoulli equation we get a transport equation for spatial variations of the amplitude b:

$$(\rho_1 u_{x,1}^2 + \rho_2 u_{x,2}^2)\frac{db}{dx} - (\rho_1 u_{x,1}\frac{\partial u_{y,1}}{\partial y} + \rho_2 u_{x,2}\frac{\partial u_{y,2}}{\partial y})b = 0. \tag{A5}$$

Thus

$$\frac{d\ln b}{dx} = \frac{\rho_1 u_{x,1}\frac{\partial u_{y,1}}{\partial y} + \rho_2 u_{x,2}\frac{\partial u_{y,2}}{\partial y}}{\rho_1 u_{x,1}^2 + \rho_2 u_{x,2}^2} \tag{A6}$$

Incompressibility implies that $\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x} = -u_x/x$. The last equality is due to assumed form of acceleration. Therefore

$$b \propto x^{-1}, \tag{A7}$$

i.e. the amplitude decreases with the distance as $1/x$. If additional uniform stretching in the 3rd dimension is present then one has to set $\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x} - \frac{\partial u_z}{\partial z}$. In the case of interest for us (potential flow past a sphere) one can simply set $\frac{\partial u_y}{\partial y} = -2u_x/x$. Substituting this into eq. (A6) we get

$$b \propto x^{-2}. \tag{A8}$$