On using the Sun to constrain the value of Newton's constant

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Abstract. The goal of the present paper is to investigate whether helioseismology can provide significant constraints on the value of Newton's gravitational constant G. The analysis is based on complete helioseismic inversions using several reference models which differ only in the assumed values of G, of a set of accurate observed acoustic frequencies. We not only consider the effects of small variations of G on the structure of computed solar models, but also take into account other uncertainties in the helioseismic analysis as applied to testing a specific property of the models: in particular, systematic errors and other uncertainties in the assumed physics of the solar interior may cause effects which are larger than, or comparable to, the small effects that are being investigated here and hence cannot in any obvious way be separated from them. Consequently we confirm, as found by previous investigations, that the actual level of precision of the helioseismic inferences does not allow us to constrain G with a precision better than what can be reached with direct experimental measurements. This conclusion is mainly dependent on the accuracy with which solar oscillations are measured, but also reflects the effect of uncertainties in, e.g., the nuclear energy generation rate or opacity used to construct the models.

Key words. Sun: interior – fundamental parameters – helioseismology

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1. Introduction

Extremely precise measurements of the solar oscillation frequencies and the advances both in the inversion techniques resulting in the *seismic structure* of the Sun and in the Standard Solar Models have demonstrated that the models are in very good overall agreement with the seismic inferences (Christensen-Dalsgaard et al. 1996; Bahcall et al. 2001); as a result, the Sun has become a very effective laboratory for stellar and fundamental physics. Some examples are investigations of details of the equation of state (e.g. Basu et al. 1999), the emission of axions (Schlattl et al. 1999), the accumulation of WIMPs (Lopes et al. 2002), the screening of nuclear reaction rates (Fiorentini et al. 2001; Weiss et al. 2001), or the overshooting from the convective envelope (Schlattl & Weiss 1999). In many of these cases the postulated or badly known physical effect is included into the solar model and the thus modified model compared to the seismic structure; this leads to upper limits to the magnitude of the effect investigated, above which the modified model becomes inconsistent with helioseismology.

The Sun is – from the point of stellar evolution theory – a very particular object because so many well-defined global quantities have been measured, such as radius and luminosity, with errors usually well below the per cent level. Among those, the solar mass is comparably ill-determined. From Kepler's third law, $GM_\odot = 1.32712438 \times 10^{26}\,\mathrm{cm}^3\,\mathrm{s}^{-2}$ can be determined and is known to a relative accuracy of $\simeq 10^{-7}$ (IAU 1977). Actually, the quantity $k^2GM_\odot = 0.01720209895^2 \pm 0\,[\mathrm{AU}^3/\mathrm{day}^2]$ is the primary defined constant, and the error arises from the determination of the astronomical unit in physical length ($1\,\mathrm{AU} = 149597871475 \pm 30\,\mathrm{m}$; M. Standish 2003, private communication). Finally, Newton's constant G – as measured in terrestrial laboratories – is known to only $\simeq 10^{-3}$, and thus $M_\odot$ is as uncertain.

Recently, the attention of astrophysicists has been caught by a few new experimental results which lie at the extreme edges of the error range of the standard CODATA (Mhor & Taylor 1999) value ($G = (6.6723 \pm 0.0015) \times 10^{-8}\,\mathrm{cm}^3\mathrm{g}^{-1}\mathrm{s}^{-2}$), used so far. Given the success of using the Sun as a laboratory the idea of looking into "solar" constraints on the value of G is obvious, and Ricci & Villante (2002) and Lopes & Silk (2003) have indeed done so, with the conclusion that the difference between the various experimental values for G is too small to be detectable at present by the standard "solar laboratory" approach. We note, however, that Demarque et al. (1994) and Guenther et al. (1998) used helioseismology to constrain the possible variation with time of G.

As discussed in more detail in Sect. 2 the effects of changing G can be roughly estimated from homology arguments (e.g. Kippenhahn & Weigert 1990). This shows that P and ρ scale as GM^2/R^4 and M/R^3, respectively. Thus the squared sound speed $c^2 \propto P/\rho$ scales as GM/R whereas, assuming the ideal-gas approximation to the equation of state, temperature T scales as T/μ where μ is the mean molecular weight. This already indicates that, according to strict homology, a change in G that leaves GM unchanged will have no effect on the sound speed. Further homology analysis, assuming power-law approximations to the opacity and energy-generation rate according to Teller (1948), leads to a scaling for luminosity as ($L \sim G^7M^5\mu^8$) involving a strong dependency of luminosity on G. The dependence on the molecular weight has been added by us following Kippenhahn & Weigert (1990). For constant μ this is indeed confirmed by numerical results ($L \sim G^{5.6}M^{4.7}$; Degl'Innocenti et al. 1996); however, the fact that GM has to be constant already implies a much weaker sensitivity to changes in G in the case of the Sun. $L_\odot$, the solar surface luminosity, has to be independent of the numerical value for G. This is achieved by modifying the initial helium abundance, and hence the molecular weight; obviously a relative change in G of order 10^{-3} will result in a relative change in μ of the same order.

From this analysis, it is evident that the effects on the seismic properties of solar models due to a change in G at fixed GM are subtle. The change in density would cause a change in the rate of nuclear reactions and hence in the present composition profile of the Sun, essentially reflecting the fact that simple homology provides an incomplete description of the changes to the model. Also, the recalibration to obtain the proper luminosity changes the initial composition. To leading order, there is no effect on the sound speed on which the frequencies principally depend. However, homology is not the whole story, since the change in the composition and the composition profile in the core also affects the sound speed; in particular, the change in the envelope composition changes the sound speed near the surface, due to the change in Γ_1. However, these are second order effects only. Since sound speed is the property to which the oscillation frequencies are most sensitive, this evidently makes a helioseismic test of G far less sensitive; however, we note that helioseismic inversion can also provide inferences of the density which, according to the homology argument, scales as M and hence is directly sensitive to a change in G.

The goal of the present paper is to investigate in some detail the potential for constraining G based on helioseismic inferences. To do so we will not only consider the effect of assuming different values of G in computing the solar model (Sect. 2), but also consider how the helioseismic analysis is applied to test the properties of these models. In particular, we note that the use of a 'seismic solar model', while a good indication, at least in principle might involve systematic errors of some significance to the very subtle effects that are being investigated here. Instead, we show results of helioseismic inversions of a set of accurate observed acoustic frequencies, using as reference several models which differ only for a different assumed value of G, with the purpose of trying to discriminate among these models (Sect. 3).

2. Method

2.1. Solar models with varying G

A comprehensive list of recent estimations of the value of the Newton's constant such as obtained by the laboratory experiments was provided by Lopes & Silk (2003). In particular, the three experiments by Luo et al. (1999), Gundlach & Merkowitz (2000), and Quinn et al. (2001) revived the interest in G, since they yield values differing (relatively) by -0.0004, 0.0004 and 0.0005 from the CODATA98 recommendation (Mhor & Taylor 1999), and in particular do not lie within the mutual error bars. They are within the large CODATA98 error range of $\pm 1.5 \times 10^{-3}$, which is, however, only the consequence of including the Michaelis et al. (1996) result of $G = 6.715 \times 10^{-8}\,\mathrm{cm^3 g^{-1} s^{-2}}$, which deviates from the CODATA value by as much as 0.65%, with a quoted uncertainty of only 8.3×10^{-5}.

For the purpose of this investigation we have assumed the reference values $(GM)_\odot = 1.32712 \times 10^{26}\,\mathrm{cm^3 s^{-2}}$ and the gravitational constant $G_0 = 6.6723 \times 10^{-8}\,\mathrm{cm^3 g^{-1} s^{-2}}$. Consequently, the mass of the Sun has been assumed to be $M_\odot = 1.989 \times 10^{33}$ gm.

Table 1. Typical quantities of the solar models in this work with the indices 'i' and 's' denoting initial and surface values, respectively. In the last row the location of the bottom of the convective envelope r_{cz}/R is provided. The solar models were calibrated to reproduce the solar values $L_\odot = 3.84600 \times 10^{33}$ erg/s, $R_\odot = 6.95508 \times 10^{10}$ cm and $(Z/X)_s = 0.0245$ to an accuracy better than 2×10^{-6}.

G/G_0	0.999	1.000	1.001	1.0065
Y_i	0.275742	0.275432	0.275122	0.273423
Z_i	0.0198765	0.0198874	0.0198981	0.0199576
α_{CM}	0.929370	0.929180	0.928943	0.927793
L [10^{33}erg/s]	3.845997	3.845999	3.8460000	3.846000
R [10^{10}cm]	6.955081	6.955093	6.955080	6.955080
$(Z/X)_s$	0.02449995	0.02449999	0.02449997	0.02450000
Y_s	0.2454467	0.2451290	0.2448116	0.2430690
r_{cz}/R	0.7131326	0.7132300	0.7132772	0.7135028

Different, fully calibrated solar models were calculated using several values for the Newton's constant G – denoted as G' –, while $G'M' = G_0 M_\odot$ has been kept constant. As a consequence, the mass M' differs for each model considered. While Lopes & Silk (2003) considered only the cases with the different measured values for G', Ricci & Villante (2002) used values of $G' = G_0(1. \pm 0.01)$, which are extreme variations in G_0. Given the various determinations of Newton's constant, we followed Ricci & Villante (2002) and applied typical modifications; in particular we used $G' = G_0(1 \pm 0.001)$, i.e. an order of magnitude smaller variation, consistent with the differences found in the most recent experiments, and one extreme case of $G' = 1.0065 G_0$ representing the Michaelis et al. (1996) result as the most extreme experimental case. All models were calibrated, to within a relative error of 2×10^{-6}, to a photospheric radius of $R_\odot = 6.95508 \times 10^{10}$ cm, and a surface luminosity of $L_\odot = 3.84600 \times 10^{33}$ erg/s at an age of 4.57 Gyr.

As our standard solar model we used the latest GARching SOlar MOdel (GARSOM5), which is the follow-up model of GARSOM4. This model has been described by Schlattl (1999). It agrees well with those by Christensen-Dalsgaard et al. (1996) and Bahcall et al. (1998) and also employs the standard physical input, which is that of most standard solar models at this epoch. In particular, particle diffusion for hydrogen, helium and metals is fully taken into account. Inversion procedures have shown an overall excellent agreement of order 0.002 between the sound speed profiles of the Sun and that of this model, with a major discrepancy in the tachocline region between $0.6 \lesssim r/R_\odot \lesssim 0.7$, where the relative difference in squared sound speed c^2 reaches 5×10^{-3}, outside the conservative error range of the seismic model.

GARSOM5 differs from the previous one only in that the solar models were started from the zero age instead of the pre-main sequence and the latest version of the OPAL equation of state (Rogers et al. 1996) of 2001 has been implemented. The differences as compared to the 1996 version of the EOS are the improvement of the activity expansion method for repulsive interactions (Rogers 2001), a relativistic treatment of electrons, and the inclusion of HeH^+ and He^{++} in addition to H_2, H_2^+, and H^-. GARSOM5 in fact shows a slightly degraded agreement with the seismic model (see also Bonanno et al. 2002). Table 1 contains initial and final solar quantities for this and all other models used in this paper. Note that $\alpha_{\rm CM}$ is the mixing length parameter used in the Canuto-Mazzitelli convection theory (Canuto & Mazzitelli 1992).

There are various statements in the literature concerning the dependence of the stellar structure equations on G. As discussed in Sect. 1, and noted also by Ricci & Villante (2002), these effects are described most clearly in terms of homology-scaled variables. We introduce dimensionless variables $(q, x, \hat{P}, \hat{\rho}, \hat{c}, \hat{T})$ by

$$\begin{aligned} m &= Mq\,, & r &= Rx\,, \\ P &= \frac{GM^2}{R^4}\hat{P}\,, & \rho &= \frac{M}{R^3}\hat{\rho}\,, \\ c^2 &= \frac{GM}{R}\hat{c}^2\,, & T &= \frac{GMm_{\rm u}}{k_{\rm B}R}\hat{T}\,. \end{aligned} \tag{1}$$

Here r is distance to the centre, m is mass within a sphere of radius r, $m_{\rm u}$ is the atomic mass unit, $k_{\rm B}$ is Boltzmann's constant. $\hat{T}$ is introduced such that the equation of state can be approximated by the ideal gas law, $\hat{T} \simeq \hat{P}\mu/\hat{\rho}$, where μ is the mean molecular weight. Then the two mechanical equations of stellar structure become

$$\frac{\partial x}{\partial q} = \frac{1}{4\pi\hat{\rho}x^2} \tag{2}$$

$$\frac{\partial \hat{P}}{\partial q} = -\frac{q}{4\pi x^4} \tag{3}$$

where obviously G and M do not appear explicitly. The energy-transport equation can be written

$$\frac{\partial \hat{T}}{\partial q} = -\frac{q\hat{T}}{4\pi x^4 \hat{P}}\nabla\,; \tag{4}$$

Here, except in a very thin near-surface boundary layer, $\nabla = {\rm d}\ln T/{\rm d}\ln P$ is approximately adiabatic in the convection zone, $\nabla \simeq \nabla_{\rm ad} = (\partial \ln T/\partial \ln P)_{\rm ad}$. In radiative regions,

$$\nabla = \nabla_{\rm rad} = \frac{3\kappa L_r P}{16\pi a\tilde{c}T^4}\left(\frac{1}{Gm}\right)\,. \tag{5}$$

Here κ is opacity, a is the radiation-density constant, $\tilde{c}$ is the speed of light and L_r is the local luminosity which obviously is determined by the energy-generation rate ϵ. Analysis of these thermal equations is complicated by the complex dependence of κ and ϵ on the physical conditions. However, a rough idea about the behaviour can be obtained by approximating κ by a power law, $\kappa \simeq \kappa_0 \rho^\delta T^{-\nu}\mu^\zeta$. Expressing $\nabla_{\rm rad}$ in terms of homology-scaled variables and using the ideal-gas approximation for $\hat{T}$ we obtain

$$\nabla_{\rm rad} = \frac{3\kappa_0 R^{\nu-3\delta}}{16\pi a\tilde{c}(GM)^{3+\nu-\delta}}\left(\frac{k_{\rm B}}{m_{\rm u}}\right)^{4+\nu}\frac{L_r}{\mu^{4+\nu-\zeta}G^{1+\delta}}\frac{\hat{\rho}^{4+\nu+\delta}}{\hat{P}^{3+\nu}q}\,. \tag{6}$$

The surface boundary conditions are often defined by determining the photospheric pressure, at the point where $T = T_{\rm eff}$, the effective temperature, from a more or less sophisticated atmospheric model. In the present case where GM, L and R are fixed, the atmospheric solution is also essentially fixed, apart from the effect of the change in composition required to calibrate the model. It follows that the dimensional photospheric pressure $P_{\rm ph}$ will change little with changing G; consequently, the homology-scaled photospheric pressure is expected to change approximately proportional to G.

Given this analysis, we can estimate the likely effect on the model of a change in G at fixed GM and R, to the extent that this can be estimated from homology. As discussed in the introduction the composition is calibrated such that the surface luminosity is unchanged. Since the temperature is unchanged, to leading order, we expect that the internal distribution L_r of luminosity is also approximately unchanged; according to Eq. (6) this is satisfied if the composition is changed such that $\mu^{4+\nu-\zeta}G^{1+\delta}$ is unchanged. (A more complete description would take into account also the changes in the energy-generation rate; however, because of the strong dependence of ϵ on T this has little effect on the result.) Assuming this calibration the homology-scaled equations are independent of the value of G, and so, therefore, is the expected scaled structure; thus the sound speed would be unaffected, whereas according to Eq. (1) the density would change proportionally to $M \propto G^{-1}$.

To make these statements more precise, we note that helioseismic analyses are typically carried out in terms of differences, denoted by δ_r, between the Sun and the model at fixed r (which, given that R is kept fixed, corresponds to differences at fixed x). Keeping GM fixed, it obviously follows from Eqs. (1) that

$$\frac{\delta_r c^2}{c^2} = \frac{\delta_r \hat{c}^2}{\hat{c}^2} , \quad \frac{\delta_r P}{P} = -\frac{\delta G}{G} + \frac{\delta_r \hat{P}}{\hat{P}} , \\ \frac{\delta_r \rho}{\rho} = -\frac{\delta G}{G} + \frac{\delta_r \hat{\rho}}{\hat{\rho}} . \tag{7}$$

Under strict homology the scaled variables are unaffected by the change; hence $\delta_r c^2/c^2 = 0$ and $\delta_r P/P = \delta_r\rho/\rho = -\delta G/G$.

In fact, for changes to the solar structure to be completely described in terms of homology one must assume that the physical conditions are of the same form throughout the star, with radiative transport controlling the temperature gradient. This is manifestly not satisfied in the convection zone. Here, on the other hand, the properties of the model are simplified by the fact that the stratification is essentially adiabatic, with an adiabatic gradient that may be assumed to be roughly constant in most of the convection zone. As discussed in Appendix A, this leads to a relatively simple description of the model changes in the convection zone. It is found that, as also predicted by simple homology, the sound speed is unaffected by the change of G. The change in ρ can be written as

$$\frac{\delta_r \rho}{\rho} \simeq -\frac{\delta G}{G} - C\frac{\delta r_{\rm cz}}{H_{P,\rm cz}} , \tag{8}$$

where $\delta r_{\rm cz}$ is the change of the radius of the base of the convection zone, $H_{P,\rm cz}$ is the pressure scale height at this point and C is a constant of order unity. Here the first term is as predicted from homology; however, our numerical calculations (see below) show that the second term is of similar magnitude.

The actual computed changes are illustrated in Fig. 1 which shows the relative differences in sound speed and density between the GARSOM5-models with different G'-values and the model which assumes standard G_0 and $M_\odot$, under the condition $G'M' = G_0 M_\odot =$ const. As it can be seen, the differences in the squared sound speed are minimal, reaching 2×10^{-4} only in the innermost core ($r/R_\odot < 0.1$). This results from the combined effects of a change in the distribution of the hydrogen abundance and in temperature, arising from the non-homologous changes in the nuclear reaction rates. Our result compares very well with Ricci & Villante (2002), who found relative changes of order 2×10^{-3} in the same region for a ten times larger ($G' - G$).

The relative change in density is larger than predicted by homology: in particular, the nearly constant value in the convection zone is of the same sign, but almost twice as large, as expected. This is clearly related to the second term on the right-hand side of Eq. (8).

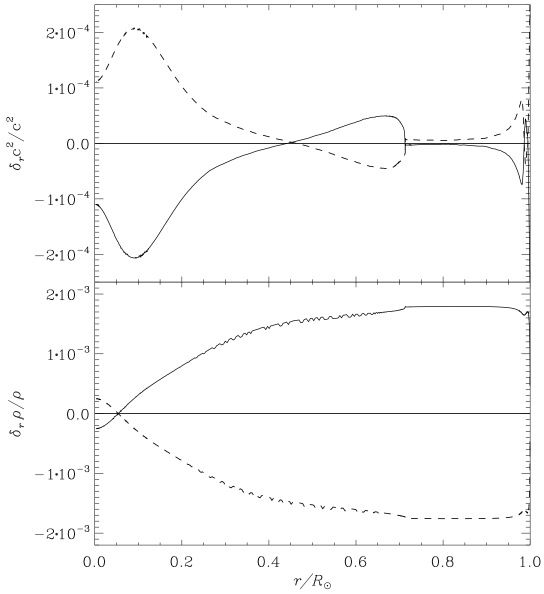

Fig. 1. Influence of modified G on solar model sound speed. In the upper panel there are shown the relative differences in squared sound speed, in the sense (modified model) – (standard model) for two values of G (dashed: $G' = 1.001G_0$; solid: $G' = 0.999G_0$). The lower panel shows the changes in density relative to the standard model.

2.2. Inversion procedure

The ability of helioseismology to constrain G obviously depends on the way in which the oscillation frequencies depend on G. As is common in helioseismic analyses, we consider the oscillations as being adiabatic. In that case, it is easy to show that the oscillation equations can be expressed solely in terms of the homology-scaled structure variables and the adiabatic exponent Γ_1, the oscillation frequency being expressed in terms of the dimensionless frequency σ, defined by

$$\omega^2 = \frac{GM}{R^3}\sigma^2 \,, \tag{9}$$

where ω is the dimensional angular frequency. Thus, to the extent that homology is satisfied and GM is fixed, the frequencies have no direct dependence on G; this underscores the fact that constraining G is a subtle process.

An inversion for solar structure is usually based on the linearization of the equations of stellar oscillations around a known reference model, under the assumption of hydrostatic equilibrium. This results in integral equations that can be analyzed by means of an inverse procedure to determine the corrections which have to be imposed on the reference model in order to obtain the observed oscillation frequencies. Obviously, the results of the inversion depend both on the mode selection $i \equiv (n, l)$ and on the observational errors which characterize the mode set ($i = 1, \ldots, N$) to be inverted.

The above discussion suggests that we express the observed frequencies in dimensionless form, according to Eq. (9) and consider differences $\delta\sigma_i = \sigma_i^{(\mathrm{obs})} - \sigma_i^{(\mathrm{mod})}$ between the observed and the model values. The relative differences $\delta\sigma_i/\sigma_i$ between the frequencies of the Sun and the model are related to the differences at fixed r ($\delta_r\hat{c}^2/\hat{c}^2$, $\delta_r\hat{\rho}/\hat{\rho}$) in scaled sound speed $\hat{c}$ and density $\hat{\rho}$ between the structure of the Sun and the reference model through the following integral equation (e.g. Dziembowski et al. 1990):

$$\begin{aligned}\frac{\delta\sigma_i}{\sigma_i} &= \int_0^1 K^i_{c^2,\rho}(x)\frac{\delta_r\hat{c}^2}{\hat{c}^2}(x)\,\mathrm{d}x \\ &+ \int_0^1 K^i_{\rho,c^2}(x)\frac{\delta_r\hat{\rho}}{\hat{\rho}}(x)\,\mathrm{d}x + F_{\mathrm{surf}} + \varepsilon_i \,,\end{aligned} \tag{10}$$

where ε_i are the observational errors in the data, which are assumed to be independent and Gaussian-distributed with zero mean and variances $\sigma(\sigma_i)^2$. Furthermore, since the mass and radius have been taken out in the homology scaling, the density difference obviously satisfies

$$0 = \int_0^1 x^2 \hat{\rho} \frac{\delta_r \hat{\rho}}{\hat{\rho}} \mathrm{d}x \,, \tag{11}$$

which is formally similar to Eqs (10), with a vanishing 'frequency difference'. The kernels $K^i_{c^2,\rho}$ and K^i_{ρ,c^2} are calculated, for each mode i, from the equilibrium model quantities and oscillation eigenfunctions; as for the dimensionless oscillation equations, the kernels do not depend explicitly on G, M and R. Non-adiabatic effects and other errors in modelling the surface layers, that can give rise to frequency shifts, have been taken into account by including an arbitrary function of frequency F_{surf} in the variational formulation, as suggested by Dziembowski et al. (1990).

Equation (10), which forms the basis for the linearized structure inversion problem, involves three unknown functions: $\delta_r \hat{c}^2/\hat{c}^2$, $\delta_r \hat{\rho}/\hat{\rho}$ and F_{surf}. However, the number of the unknown functions can be reduced to one by adopting the method of optimally localized averages.

This method aims at solving Eq. (10), together with the constraint in Eq. (11), by estimating a localized weighted average of the unknown quantity, $\delta_r \hat{c}^2/\hat{c}^2$ or $\delta_r \hat{\rho}/\hat{\rho}$, at selected target radii x_0, by means of linear combinations of the data $\delta\sigma_i/\sigma_i$ with coefficients $\alpha_i(x_0)$, chosen such as to localize the solution while suppressing the contributions from the other terms. The solutions are characterized by the so-called *averaging kernel* and by the *cross-term kernel*, which in the case of the sound-speed inversion are expressed, respectively, by

$$\mathcal{K}_{c^2,\rho}(x_0, x) = \sum_{i=1}^N \alpha_i(x_0) K^i_{c^2,\rho}(x) \tag{12}$$

and

$$C_{c^2,\rho}(x_0, x) = \sum_{i=1}^N \alpha_i(x_0) K^i_{\rho,c^2}(x) \,. \tag{13}$$

The cross-term kernel measures the influence of the contribution from $\delta_r \hat{\rho}/\hat{\rho}$ to the inferred $\delta_r \hat{c}^2/\hat{c}^2$. Analogous relations can be obtained for the density inversion, so that the averaging kernel is expressed as a linear combination of the mode kernels K^i_{ρ,c^2}, while the cross term, which represents the contribution of the sound speed to the inferred density, is given as a linear combination of $K^i_{c^2,\rho}$.

We have solved Eq. (10) using the Subtractive Optimally Localized Averages (SOLA) method (Pijpers & Thompson 1992, 1994), fitting the averaging kernel to a target function $\mathcal{G}(x_0, x)$ of appropriate width and centered at the target radius. Thus the inversion coefficients $\alpha_i(x_0)$ are chosen to minimize

$$\int_0^1 \left[\mathcal{K}_{c^2,\rho}(x_0, x) - \mathcal{G}(x_0, x)\right]^2 \mathrm{d}x \tag{14}$$
$$+\beta_1 \int_0^1 C^2_{c^2,\rho}(x_0, x)\, g(x)\, \mathrm{d}x + \beta_2 \sum_{i=1}^N \alpha_i^2(x_0) \sigma(\sigma_i)^2 \,,$$

where $g(x)$ is a suitably increasing function of radius aimed at suppressing the surface structure in the cross-term kernel; we have used $g(x) = (1 + x)^4$. The target functions were chosen to be Gaussians with widths proportional to the sound speed at $x = x_0$, to reflect the variation with depth in the wavelength of the eigenfunctions. The coefficients $\alpha_i(x_0)$ should be chosen such as to suppress the contribution from the cross term, to localize the averaging kernel near $x = x_0$, and to suppress the surface term, while limiting the error in the solution. This can be achieved by the use of the two parameters in Eq. (14): β_1, which controls the contribution of the cross-term kernel, and β_2, which is the trade-off parameter determining the balance between the demands of well-localized averaging kernels and a small error in the solution. Further details on the inversion method were provided by Rabello-Soares et al. (1999).

As a result of the inversion procedure we obtain estimates of the differences in $\hat{c}^2$ and $\hat{\rho}$ between the Sun and the model:

$$\left\langle \frac{\delta_r \hat{c}^2}{\hat{c}^2} \right\rangle (x_0) = \int_0^1 \mathcal{K}^i_{c^2,\rho}(x_0, x) \frac{\delta_r \hat{c}^2}{\hat{c}^2}(x)\, \mathrm{d}x \,, \tag{15}$$

and

$$\left\langle \frac{\delta_r \hat{\rho}}{\hat{\rho}} \right\rangle (x_0) = \int_0^1 \mathcal{K}^i_{\rho,c^2}(x_0, x) \frac{\delta_r \hat{\rho}}{\hat{\rho}}(x)\, \mathrm{d}x \,. \tag{16}$$

From these the solar internal density and sound speed can be estimated as

$$c^2_{\mathrm{inf}}(r) = c^2_{\mathrm{ref}}(r) \left[1 + \left\langle \frac{\delta_r \hat{c}^2}{\hat{c}^2} \right\rangle (r/R) \right] , \tag{17}$$

and

$$\rho_{\mathrm{inf}}(r) = \rho_{\mathrm{ref}}(r) \left[1 - \frac{\delta G}{G} + \left\langle \frac{\delta_r \hat{\rho}}{\hat{\rho}} \right\rangle (r/R) \right] , \tag{18}$$

where $c^2_{\mathrm{ref}}(r)$ and $\rho_{\mathrm{ref}}(r)$ define the structure of the reference model; as usual, we have assumed that GM and R are fixed. It is common practice to regard $c^2_{\mathrm{inf}}(r)$ and $\rho_{\mathrm{inf}}(r)$ as defining a *seismic solar model* and, for example, to test aspects of model modifications by comparing the sound speed of the model with $c_{\mathrm{inf}}(r)$, as was done, for example, by Lopes & Silk (2003) (see also, e.g., Neuforge-Verheecke et al. 2001; Turck-Chièze et al. 2001; Winnick et al. 2002). Also, it was demonstrated by Degl'Innocenti et al. (1997) and Basu et al. (2000) that the resulting inferred structure is relatively insensitive to the assumed reference model. As noted by Antia (1996) the process can be iterated by calculating a new model based on the inferred sound speed and density and using this as reference for a renewed inversion; given the lack of sensitivity of the solution to the original reference, however, such iteration is generally not carried out.

It should be noted, however, that the interpretation in terms of a seismic solar model requires some care. As is evident from Eqs. (15) – (18), what is determined from the inversion unavoidably involves averages over averaging kernels of finite extent. Thus a comparison of a solar model with the resulting c^2_{inf}, neglecting the finite resolution, results in possible systematic errors. This is the reason why, in our analysis, we carry out inversions using each of the modified models as reference. Although this again involves averages, these at least are well-defined averages of the differences between the solar and the model structure. In this way we obtain a consistent estimate of the effects of the model modification under investigation.

The present helioseismic inversions have been carried out by using the oscillation data set published by Basu et al. (1997), which includes modes with $l < 100$ and in particular very accurate frequencies of low harmonic degree, which allow us to resolve the solar core. We used the adiabatic oscillation code of Christensen-Dalsgaard (see Christensen-Dalsgaard & Berthomieu 1991) to calculate the p-mode theoretical eigenfrequencies for each model.

3. Results

3.1. Comparison of frequency differences

A direct way to test a solar model is to consider differences between observed frequencies and those calculated for the theoretical model. Since G' is affecting mostly the sound speed in the central parts of the Sun (see Fig. 1), it is interesting to consider predictions of frequencies of low-degree acoustic modes which are able to penetrate towards the centre. The frequency differences between observed values and those computed for the solar models are shown in Fig. 2 for modes with $l < 6$. The differences are very similar for models with $G' = G_0$ and with $G' = 0.999G_0$. The differences in frequencies appear to be slightly larger in absolute value for the model with $G' = 1.0065G_0$. However, the effects of the changes in G are evidently very small and it should be pointed out that they are unfortunately lower than or comparable with the uncertainty of the observed frequencies; thus this direct comparison provides no significant constraints on the value of G.

An interesting way to conduct such seismic analysis is to consider the asymptotic properties of the oscillation modes. Theory predicts that oscillation frequencies $\nu(n, l)$ of acoustic modes, characterized by radial order n, at low harmonic degree l should satisfy a simple asymptotic approximation (Vandakurov 1967; Tassoul 1980). According to this the oscillation spectrum is characterized by two parameters, known as large and small frequency separations, respectively:

$$\Delta_l(\nu) := \nu_{n,l} - \nu_{n-1,l} \tag{19}$$

$$\delta_l(\nu) := \nu_{n,l} - \nu_{n-1,l+2} \,. \tag{20}$$

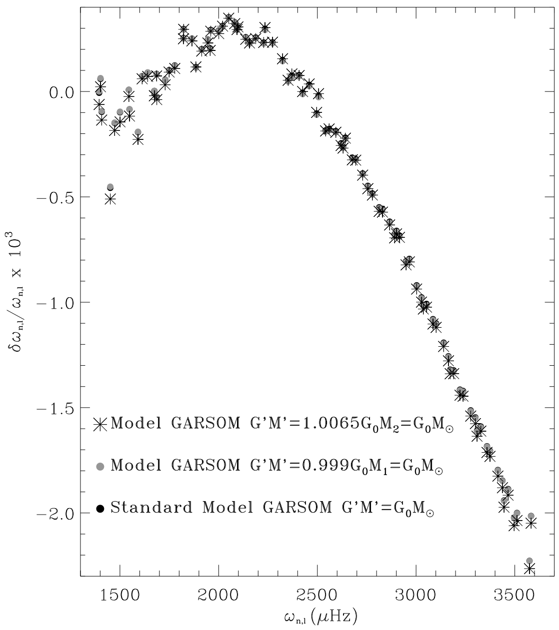

Fig. 2. Relative differences between the observed frequencies of Basu et al. (1997), for $l < 6$, and theoretical frequencies for several models considered, plotted as a function of frequency; the differences are shown in the sense (Sun) – (model).

In particular, the small frequency separation is sensitive to the chemical composition gradient in central regions of the star and hence to its evolutionary state (e.g. Gough 1983). Figure 3 shows the differences between the small frequency separations obtained for the observed set of frequencies and the theoretical frequencies calculated for the models which use different values of G'. The right panel, based on the most recent observations obtained by the BiSON network, provides more details at lower frequencies. The differences are very similar for the models with $G' = G_0$ and with $G' = 0.999G_0$, while slightly smaller differences, in absolute value, are obtained for the model with $G' = 1.0065G_0$, particularly at low frequencies. However, it is evident that no clear conclusion can be obtained from these results, concerning the most appropriate value of G.

3.2. The sound speed and the density

To obtain a more detailed test of the models we have carried out inversion for sound speed and density, as discussed in Sect. 2.2. Figures 4 and 5 show the inferred relative squared sound-speed and density differences, as functions of the fractional radius, between the Sun and three solar models used as reference models in the inversion; these were computed as the GARSOM5 model, but characterized by different values of G'. The vertical error bars correspond to the standard deviations based on the errors in the mode sets, whereas the horizontal bars give a measure of the localization of the solution.

The inversion results for the sound speed indicate that the differences between the solar models and the Sun are extremely small, except below the base of the convection zone ($0.71\,R_\odot$) where mixing beyond the convection zone might reduce the composition gradients established by settling (e.g. Brun et al. 1999; Elliott & Gough 1999). The results obtained with the use of the models with the standard $G' = G_0$ and with $G' = 0.999G_0$ are similar at all radii (we do not show the symmetric case $G' = 1.001G_0$). This indicates that with the present uncertainty with which oscillation frequencies are measured, helioseismic inversions cannot allow us to distinguish between similar models which differ only by 0.1% in the value of G'. Moreover, the sound-speed inversion results for the extreme case of a model with $G' = 1.0065G_0$ visibly deviates in the region with $0.1 \lesssim r/R_\odot \lesssim 0.3$ from the standard case, but the difference between the model and the Sun increases in magnitude by about 30%.

The interpretation of the inversion results for density involves some subtlety. It follows from the discussion after Eq. (9) that the adiabatic oscillation frequencies are determined by the scaled

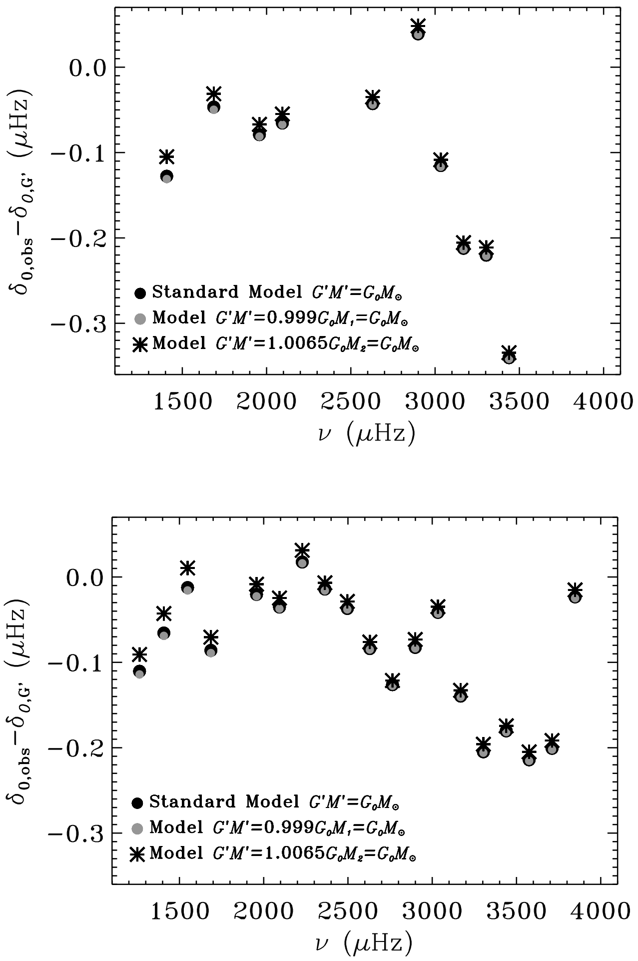

Fig. 3. Differences in small frequency separations δ_l (Eq. 20) obtained for $l = 0$, between the observed set of frequencies and the theoretical frequencies calculated for several models with different modified G' values (indicated by different symbols), plotted as function of the frequency. Left panel: for the data set by Basu et al. (1997); right panel: for the most recent data from the BiSON network (Chaplin et al. 2002).

structure variables, on the assumption that GM and R are fixed. This is also implied by the fact that the inversion imposes the constraint of fixed mass in Eq. (11). Thus, the inversion is only sensitive to the non-homologous part $\delta_r\hat{\rho}$ of the density difference, e.g. as reflected in the second term in Eq. (8), even though the full density difference contains a term in δG. The results are plotted in Fig. 5. The differences between the Sun and the two models with standard $G' = G_0$ and with $G' = 0.999G_0$ are consistent within the errors at all radii; thus the helioseismic inversion for the density hardly allows us to judge the relative correctness of the two models at this level of change in G. On the other hand, for $G' = 1.0065G_0$ the relative density differences between the model and the Sun are increased by about 30% relative to the standard case.

4. Discussion and conclusions

The extremely accurate and extensive data from observations of solar oscillations provide unique constraints on solar internal structure. From the dependence of the oscillation frequencies on the structure it follows that the immediately accessible properties are the 'mechanical' structure of the Sun, as characterized by, e.g., the sound speed and density as a function of distance to the centre. However, of greater interest is obviously to constrain the underlying physical properties and processes which lead to that structure, including also the physical constants on which these processes depend. In such analyses all possible uncertainties affecting the results must in principle be taken into account. Thus, without further assumptions or constraints, we should not expect

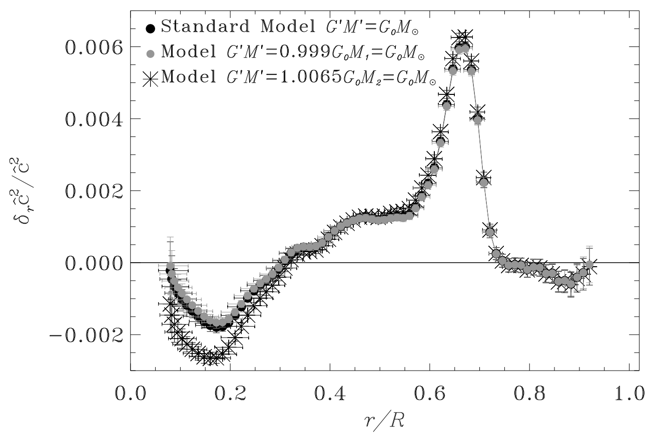

Fig. 4. The relative squared sound-speed difference between the Sun and solar models, in the sense (Sun)–(model), as obtained by inversion of the set of data by Basu et al. (1997) with modes $l < 100$. The black dots are obtained for a standard solar model with $G_0 = 6.6723 \times 10^{-8}$ and $M_\odot = 1.9891 \times 10^{33}$. The grey dots are obtained for a model with $G' = 0.999G_0$ and the asterisk for the model with the value of $G' = 1.0065G_0$, with corresponding values of M' such that $GM = (GM)_\odot = G_0M_\odot$. The vertical error bars correspond to the standard deviations based on the errors in the mode sets, whereas the horizontal bars give a measure of the localization of the solution.

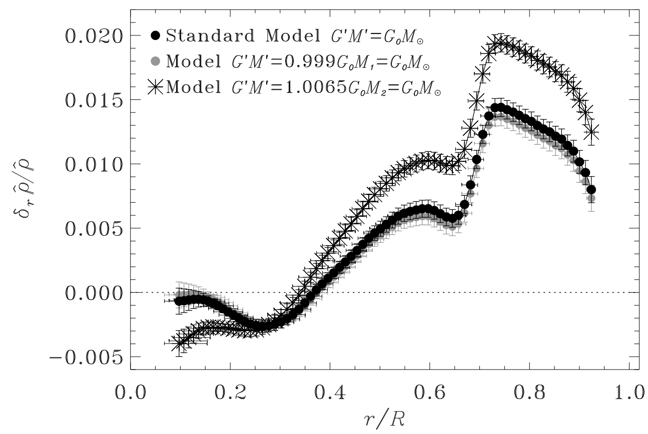

Fig. 5. The relative density differences between the Sun and solar models, in the sense (Sun)–(model), as obtained by inversion of a set of data with modes $l < 100$. Models and symbols are the same as in Fig. 4.

to be able to obtain as strict a constraint on any given quantity as apparently warranted by the precision of the data. Also, it is obviously important to understand the relation between the effect under consideration and the frequencies.

The analysis of any given physical effect is usually carried out through the computation of solar models which are compared with the helioseismic inferences. This is often done by means of 'seismic solar models', obtained by applying helioseismically determined corrections to a reference model; the resulting profile of, for example, sound speed is then compared with sound-speed profiles in the models under investigation. Although this is a simple and efficient technique, it may suffer from systematic errors introduced by the finite resolution of the inversion. In our application of the 'solar physics laboratory' we have used the more cumbersome, but more consistent, method of using each model as a reference in the inversion. In this way we obtain a well-defined average of the difference between the Sun and that particular model, thus providing a measure of the physical effects being investigated.

Newton's gravitational constant G, which enters into the calculation of stellar models, is amongst the most uncertain of the fundamental constants; thus it is of interest to investigate whether significant constraints on its value can be obtained from helioseismic analysis. To contribute to this investigation, motivated also by recently claimed rather discrepant experimental values of G, we have considered the effects of changes to G on solar models and carried out helioseismic tests of these models. As an obvious constraint on these calculations we have kept fixed the product GM, known with extreme accuracy from planetary motion, and furthermore fixed the solar radius and luminosity. Homology arguments, as well as analysis of the response of the convection zone to the changes, show that to a first approximation the sound speed is unaffected by the change to G. Also, to the extent that the changes to the models follow homology scaling the adiabatic oscillation frequencies are not affected.

Thus the potential helioseismic effects are subtle. In particular, the modifications to the sound speed, on which the oscillation frequencies predominantly depend, are at most 20 % of the change in G and concentrated in the core of the model; they result predominantly from indirect effects arising from the change in composition in the core, in part related to the changed initial composition required to obtain the correct solar luminosity at the present age. There are somewhat larger non-homologous changes in the density, of a magnitude similar to the change in G. In the extreme case of Michaelis et al. (1996), who proposed a change in G of 0.65 % relative to the normally accepted value, this has a formally significant effect on the helioseismically inferred density, leading generally to substantially worse agreement between the model and the Sun. Smaller, and perhaps more reasonable, changes in G have noticeable, but barely significant, effects on solar structure, at the present level of precision of helioseismic inferences. This was also concluded by Ricci & Villante (2002) and Lopes & Silk (2003). A fundamental assumption in these studies is that G does not vary with time. It is interesting to recall that Guenther et al. (1998) used analysis of solar oscillation frequencies to constrain the temporal variation in G, at a level of precision highly significant compared with other tests.

The helioseismic effects being subtle, one has to be concerned about the numerical precision of the solar models. While the difference between standard solar and seismic models is of the 10^{-3} level (Fig. 4), that between models with different G'-values is one order of magnitude smaller (Fig. 1). During our calculations it turned out that the precision of the GARSOM models did not suffice to completely separate numerical effects from those arising from variations in G of the magnitude considered. We therefore had to improve our models in that respect, most importantly by ensuring identical spatial and temporal resolution. The models have been compared with those calculated by the solar model code of Christensen-Dalsgaard (Christensen-Dalsgaard et al. 1996), which intrinsically is more accurate that the Garching code, which still is a general stellar evolution code.

As discussed above, physical inferences based on helioseismology must take into account other potential uncertainties in the modelling. In the case of the temporal variation in G this was noted by Guenther et al. (1998) who, however, argued that any such effect is unlikely to have a significant influence on the constraint that they obtained. In the present case of testing the value of G, the effects of changes to G are small compared with, and cannot in any obvious way be separated from, effects of other uncertainties in the assumed physics of the solar interior, such as nuclear reaction parameters or opacities. As noted by, e.g., Lopes & Silk (2003) further constraints on the structure of the solar core are provided by the measurements of neutrino fluxes, but only to the extent that the details of neutrino oscillations are well understood. Needless to say, we may expect improvements in the observational oscillation-frequency and neutrino constraints, as well as likely further independent improvements in our knowledge of aspects of solar internal physics. In this way the parameters controlling solar modelling, including the gravitational constant, will undoubtedly become better known; however, it is not obvious that the dominant improvement in the determination of G will arise from the solar data rather than from improved direct experimental measurements.

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Appendix A: Changes in the convective envelope

The homology scalings applied in Sect. 2.1 assume that the solar interior is described by a uniform set of physics. This is manifestly not the case in the outer convection zone, whose structure may therefore be expected to change in a non-homologous manner. In this appendix we discuss some aspects of these changes; the analysis is based on a similar analysis by Christensen-Dalsgaard (1997).

The bulk of the convective envelope can be characterized by an adiabatic relation between pressure and density which we write

$$P = K\rho^\gamma \ , \tag{A.1}$$

assuming that K and γ are constants; this is approximately satisfied below the ionization zones of hydrogen and helium, for $r \lesssim 0.97R$. From this relation, and the equation of hydrostatic support, it follows approximately that

$$GM\left(\frac{1}{r} - \frac{1}{R}\right) \simeq \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \frac{\gamma}{\gamma-1}K^{1/\gamma}P^{1-1/\gamma} \ , \tag{A.2}$$

where we furthermore neglected the mass in the convective envelope and assumed $m \simeq M$. From the first part of this equation it follows that $c^2 = \gamma P/\rho$ at fixed r is unaffected by changes to the model which leave GM and R unchanged, as is the case for the changes to G considered here. It also follows that

$$\frac{\delta_r P}{P} = \frac{\delta_r \rho}{\rho} \simeq -\frac{1}{\gamma-1}\frac{\delta K}{K} \ ; \tag{A.3}$$

in particular, $\delta_r P/P$ and $\delta_r\rho/\rho$ are approximately constant in the convection zone (see also Fig. 1).

Further relations between the model changes can be derived by noting that at the base of the convection zone, $r = r_{\rm cz}$, $\nabla_{\rm rad} = \nabla_{\rm ad}$ which can be taken to be fixed. It follows from Eq. (5) that, given that L and $Gm \simeq GM$ are fixed, the value of $\kappa P/T^4$ at the base of the convection zone is fixed. Assuming again that $\kappa \simeq \kappa_0\rho^\delta T^{-\nu}\mu^\zeta$, and using the ideal gas law, this yields the following relation satisfied by the change $\delta P_{\rm cz}$ in the value of the pressure at the base of the convection zone:

$$\begin{aligned} 0 = \ & -\frac{\gamma-1}{\gamma}\left(4+\nu-\frac{\gamma+\delta}{\gamma-1}\right)\frac{\delta P_{\rm cz}}{P_{\rm cz}} - \frac{1}{\gamma}(4+\nu+\delta)\frac{\delta K}{K} \\ & -(4+\nu-\zeta)\frac{\delta\mu}{\mu} \ . \end{aligned} \tag{A.4}$$

From Eq. (A.2) it furthermore follows that

$$\frac{1}{\gamma}\frac{\delta K}{K} + \frac{\gamma-1}{\gamma}\frac{\delta P_{\rm cz}}{P_{\rm cz}} \simeq -\frac{R}{d_{\rm cz}}\frac{\delta r_{\rm cz}}{r_{\rm cz}} \simeq -\frac{\gamma}{\gamma-1}\frac{\delta r_{\rm cz}}{H_{P,{\rm cz}}} \ , \tag{A.5}$$

where $H_{P,{\rm cz}}$ is the pressure scale height at the base of the convection zone; in the last equality we used again Eq. (A.2). Using this to eliminate $\delta P_{\rm cz}/P_{\rm cz}$ in Eq. (A.4) we obtain

$$\begin{aligned} 0 = \ & -\frac{\delta+1}{\gamma-1}\frac{\delta K}{K} \\ & -(4+\nu-\zeta)\frac{\delta\mu}{\mu} + \left(4+\nu-\frac{\gamma+\delta}{\gamma-1}\right)\frac{\gamma}{\gamma-1}\frac{\delta r_{\rm cz}}{H_{P,{\rm cz}}} \ . \end{aligned} \tag{A.6}$$

Also, $(4+\nu-\zeta)\delta\mu/\mu \simeq -(1+\delta)\delta G/G$ from the homology argument for constant luminosity based on Eq. (7). Using Eq. (A.3) we finally obtain that in the convection zone

$$\begin{aligned} \frac{\delta_r P}{P} &= \frac{\delta_r\rho}{\rho} \\ &\simeq -\frac{\delta G}{G} - \frac{\gamma}{\delta+1}\left(4+\nu-\frac{\gamma+\delta}{\gamma-1}\right)\frac{\delta r_{\rm cz}}{H_{P,{\rm cz}}} \ . \end{aligned} \tag{A.7}$$

Here the first term is identical to the result obtained from homology; however, in general the change in the model will result in a non-zero change in the radius at the base of the convection zone and hence to a non-homologous contribution to $\delta_r\rho/\rho$.

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