Nodal and Periastron Precession of Inclined Orbits in the Field of a Rotating Black Hole and a Rapidly Rotating Neutron Star.

Nail Sibgatullin[1,2]

[1]*Moscow State University, Vorob'evy gory, Moscow, 119899 Russia*
[2] *Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 85740 Garching bei Munchen, Germany*

e-mail:sibgat@mech.math.msu.su

Abstract

The inclination of low-eccentricity orbits is shown to significantly affect the orbital parameters, in particular, the Keplerian, nodal precession, and periastron rotation frequencies, which are interpreted in terms of observable quantities. For the nodal precession and periastron rotation frequencies of low-eccentricity orbits in a Kerr field, we derive a Taylor expansion in terms of the Kerr parameter at arbitrary orbital inclinations to the black-hole spin axis and at arbitrary radial coordinates. The particle radius, energy, and angular momentum in the marginally stable circular orbits are calculated as functions of the Kerr parameter j and parameter s (a minimum latitudinal angle reached at the orbit) in the form of Taylor expansions in terms of j to within $O[j^6]$. By analyzing our numerical results, we give compact approximation formulas for the nodal precession frequency of the marginally stable circular orbits at various s in the entire range of variation of Kerr parameter. We derive a formula for the nodal precession frequency and the Keplerian period of a particle at an arbitrarily inclined orbit in the post-Newtonian approximation in the external field of an oblate rotating neutron star (NS). We also derive formulas for the nodal precession and periastron rotation frequencies of slightly inclined low-eccentricity orbits in the field of a rapidly rotating NS in the form of asymptotic expansions whose first terms are given by the Okazaki–Kato formulas. The NS gravitational field is described by the exact solution of the Einstein equation that includes the NS quadrupole moment induced by rapid rotation. Convenient asymptotic formulas are given for the metric coefficients of the corresponding space-time in the form of Kerr metric perturbations in Boyer–Lindquist coordinates.

Key words: neutron stars, luminosity, disk accretion, X-ray radiation.

Introduction

The X-ray quasi-periodic oscillations (QPOs) discovered in low-mass X-ray binaries (LMXBs) commonly show a variety of modes. In particular, there are horizontal-branch oscillations at low frequencies $\nu_{\rm HBO}$ 1–100 Hz and two peaks at frequencies ν_1 and ν_2 ($\sim$1 kHz) in the power spectrum (van der Klis 2000). In many Z-type sources, as well as in several atoll sources, the frequencies $\nu_{\rm HBO}$ exhibit an almost quadratic dependence on ν_2 (Stella and Vietri 1998; Psaltis *et al.* 1999). For several sources (e.g., for Sco X-1, 4U 1608–52, 4U 1702–43, 4U 1735–44, XTE J2123–058), the frequency difference $\nu_2 - \nu_1 \equiv \Delta\nu$ decreases with increasing ν_2. At the same time, for many Z-type (GX 5-1,GX 17+2, Cyg X-2, GX 340+0, GX 349+2) and atoll (4Γ 0614+09, 4Γ 1636-52, 4Γ 1705-44, Aq1 X-1) sources, the difference $\Delta\nu$ is constant, which allows it to be associated with the stellar rotation frequency (van der Klis 2000). The frequency ν_2 is currently identified with the Keplerian rotation frequency of clumps of matter near a compact object ($\nu_2 \equiv \nu_{\rm K}$) in almost all interpretations.

Cui *et al.* (1998) pointed out that the nodal precession of circular orbits[1] slightly inclined to the equatorial plane of a black hole could be of importance in interpreting the stable QPOs: for several black-hole candidates and microquasars, they found the predicted frequencies of nodal precession to agree (if the Kerr parameter is chosen in the range $0.37 - 0.9$) with the observed QPO frequencies.

Stella and Vietri (1998) justified the formula for the nodal precession frequency $\nu_{\rm nod}$ of orbits slightly inclined to the equatorial plane and identified its even harmonics with the frequency $\nu_{\rm HBO}$. The frequency ν_1 was associated with the periastron rotation frequency $\nu_{\rm per} = \nu_{\rm K} - \nu_r$ of a low-eccentricity orbit (a general-relativity effect). They showed that the theoretical dependence $\nu_r(\nu_\phi)$ for an appropriately chosen mass of the object in the range $1.8 - 2.2 M_\odot$ was in good agreement with the experimental points in the $(\nu_2, \Delta\nu)$ plane for various sources with evolving frequencies. In the Kerr solution, the frequency ν_r becomes zero at the marginally stable orbit. Therefore, when a radiating clump of matter passes from one Keplerian orbit to another[2] by increasing its rotation frequency, the frequency difference between the peaks $\Delta\nu = \nu_r$ decreases and even approaches zero at the marginally stable orbit. This tendency was observed for some of the sources (see above).

Morsink and Stella (1999), Stella and Vietri (1998), and Psaltis *et al.* (1999) showed the theoretical dependences $\nu_{\rm nod}(\nu_{\rm K})$ to agree with the measured source frequencies in the $\nu_2, \nu_{\rm HBO}$ plane if $\nu_{\rm HBO}$ was identified with an appropriate even harmonic of the nodal frequency.

The proper rotation of the source (a neutron star or a black hole) remains a free parameter, and it can be chosen by using the stable frequency observed in LMXBs in the X-ray band during outbursts.

Another free parameter is the orbital inclination s (a minimum latitudinal angle reached at the orbit) to the equatorial plane of a neutron star or a black hole, which significantly affects the observed nodal precession, periastron rotation, and Keplerian frequencies near the marginally stable orbit. Inclined bound orbits in Kerr field were first considered by Wilkins (1972) (the case j=1) and Johnston and Ruffini (1974) (general case covering a charged rotating black hole).

Van Kerkwijk *et al.* (1998) explained the puzzling spindown and spinup of some X-ray pulsars by the fact that the accretion-disk tilt to the equatorial plane in inner regions can become larger than 90°!

Below, we give an example based on our results. If the mass of a black hole or a neutron star is M, then for the Keplerian frequency in the marginally stable orbit $\nu_\phi = 1.2$ $(2.2M_\odot/M)$ kHz, the frequency of nodal precession $\nu_{\rm nod}$ is, respectively, 123 $(2.2M_\odot/M)$ Hz at $s = 0$; 93.6 $(2.2M_\odot/M)$ Hz at $s = 10°$; 65.3 $(2.2M_\odot/M)$ Hz at $s = 30°$; 52.6 $(2.2M_\odot/M)$ Hz at $s = 45°$; 41.18 $(2.2M_\odot/M)$ Hz at $s = 80°$; and 41.11 $(2.2M_\odot/M)$ Hz at $s = 90°$. The nodal precession frequency changes by almost a factor of 3 as the inclination changes from 90° to 0! Thus, by simultaneously measuring three quantities, M, $\nu_{\rm HBO}$, and ν_2, we can determine the

[1]Lense–Thirring (1918) effect.
[2]Through turbulent viscosity and radiative deceleration.

inclination of the marginally stable orbit to the spin axis!

In general, the tilt of an accretion disk to the equatorial plane of a compact object can be finite. The Bardeen–Petterson (1975) hypothesis of the accretion-disk transition into the equatorial plane breaks down when the radiative forces that twist the inner edge of the disk are taken into account. Pringle (1996) showed that a flat disk was unstable to disturbances in the presence of a central radiation source.

The paper consists of two parts. The inclined (tilted) nearly circular orbits are studied in the field of rotating black hole (part I) and in the field of fast rotating neutron star (part II). In the part I we derive formulas for the Keplerian, nodal precession, and periastron rotation frequencies of low-eccentricity orbits at an arbitrary (finite) orbital inclination to the equatorial plane in the form of Taylor expansions in terms of the Kerr parameter. In contrast to previous studies, in which calculations were performed for selected values of the constant Q [with an unclear physical meaning, as was noted by de Felice (1980), and coinciding with the square of the total angular momentum only in the weak-field approximation Q), we consistently use the smallest latitudinal angle $\theta_{-} = s$ reached on a bounded trajectory[3] as a trajectory parameter. At $s = \pi/2$, the derived formulas (51) transform into Taylor expansions of the formulas by Okazaki *et al.* (1987). An analysis of our numerical results has yielded compact approximation formulas for the nodal precession frequency of the marginally stable circular orbits in the entire range of Kerr parameters for several finite orbital inclinations.

However, the results for black holes need to be significantly corrected for neutron stars, because a rapidly rotating NS becomes oblate and a quadrupole moment appears. For orbits with low inclinations to the equatorial plane, Markovic (2000) derived formulas for the nodal and periastron precession frequencies in the post-Newtonian approximation by taking into account the NS quadrupole moment. Morsink and Stella (1999) and Vietri and Morsink (1999) *numerically* calculated ν_r and $\nu_{\rm nod}$ as functions of ν_ϕ for low inclinations and low eccentricities for various NS equations of state.

In the part II *analytic* formulas are derived for ν_r and $\nu_{\rm nod}$ for *low* inclinations and low eccentricities in the form of asymptotic expansions. They transform to the formulas of Okazaki *et al.* (1987) and Kato (1990) at a zero quadrupole moment. The exact quadrupole solution extracted from the more general solution found by Man'ko *et al.* (1994) formed a basis for our study. This solution was reduced to the simplest form in Weyl coordinates. For clarity, the exact quadrupole solution is also given asymptotically as a Kerr metric perturbation in Boyer–Lindquist coordinates.

Sibgatullin and Sunyaev (1998, 2000a, 2000b) (below referred to as SS 98, SS 00a, and SS 00b) provided formulas for the luminosity and spindown rate for various NS equations of state. These authors proposed a method of analytically constructing the quadrupole moment b as a function of the Kerr parameter j and NS rest mass m, $b = b(j, m)$, based on the main NS thermodynamic function — its gravitational mass M as a function of the Kerr parameter and rest mass $M = M(j, m)$. In this case, the space-time geometry for the exact quadrupole solution plays a crucial role in finding the marginally stable orbit. The dependence $b(j, m)$ was constructed using numerical data from Cook *et al.* (1994) and the numerical code by Stergioulas (1998) for the marginally stable orbit. Laarakkers and Poisson (1998) found a parabolic dependence of the quadrupole moment on the Kerr parameter at $j << 1$ by using direct calculations from the formula of Ryan (1995b, 1997) for several equations of state. The

[3] Below, s is called the inclination angle between the angular velocity vector of a compact object and the orbital surface for short.

physical parameters of the marginally stable orbit were studied by Shibata and M. Sasaki (1998) using expansions at large radii.

An important outstanding question (which is not considered here) is the passage from a viscous accretion disk (Shakura and Sunyaev 1973, 1976) to free particles near the marginally stable orbit. Another complicated question (which is not considered here either) is the behavior of *the marginally stable orbit* for their finite inclination to the equatorial plane of fast rotating neutron star.

1 Nodal and Periastron Precession of Inclined Orbits in the Field of a Rotating Black Hole.

Geodesics in the Kerr solution

It is well known that the equations of geodesics in Riemannian spaces = can be written in Hamiltonian form. The corresponding Hamilton–Jacobi equation is

$$Q \equiv (g^{i,j} S_{,i}\, S_{,j} + 1)/2 = 0, \quad S_{,i} \equiv \frac{\partial S}{\partial x_i}, \quad i, j, ... = 0, 1, 2, 3. \tag{1}$$

The generalized momenta $p_i = S_{,i}$ are related to the 4-velocity components by the Hamilton equations

$$\frac{dx^j}{d\tau} = \frac{\partial Q}{\partial p_j} = g^{jk} p_k, \quad \frac{dp_j}{d\tau} = -\frac{\partial Q}{\partial x^j} = -\frac{\partial g^{kl}}{\partial x^j} p_k p_l. \tag{2}$$

For the Kerr solution in Boyer–Lindquist coordinates, the contravariant metric components are

$$g^{00} = -\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2 c^2 \Delta}, \quad g^{\phi\phi} = \frac{(1 - 2rGM/c^2\rho^2)}{\Delta \sin\theta}, \quad g^{0\phi} = \frac{-2ar}{\rho^2 \Delta},$$
$$g^{rr} = \frac{\Delta}{\rho^2}, \quad g^{\theta\theta} = \frac{1}{\rho^2}. \quad \Delta \equiv r^2 + a^2 - 2rGM/c^2; \quad \rho^2 \equiv r^2 + a^2 \cos^2\theta. \tag{3}$$

The constant a is related to the angular momentum of a rotating black hole described by the Kerr solution by the equality $J = Mac = GM^2 j/c$, where j is the dimensionless Kerr parameter. Because of axial symmetry and stationarity, the Hamilton–Jacobi equation (1) for the Kerr metric has two cyclic coordinates, t and ϕ; therefore, Eq. (1) with the first integrals $p_t = -E = \text{const}$ and $p_\phi = L = \text{const}$ for the Kerr metric can be written as

$$(1-E^2)((r^2+a^2)^2 - \Delta a^2 \sin^2\theta) + 4\,a\,r\,L\,E\,G\,M/c^3 +$$
$$+(\rho^2 - 2GMr/c^2)L^2/(c\sin\theta)^2 \Delta^2 S_{,r}^2 + \Delta S_{,\theta}^2 = 2r(r^2+a^2)\,G\,M/c^2. \tag{4}$$

The existence of the complete integral of the Hamilton–Jacobi equation established by Carter (1968) is less obvious:

$$S = -Etc^2 + L\phi + \int \sqrt{\Theta(\theta)} d\theta + \int \sqrt{R(r)} \frac{dr}{\Delta},$$

$$\Theta(\theta) \equiv Q - \cos^2\theta (a^2 c^2 (1-E^2) + L^2/(\sin\theta)^2), \tag{5}$$
$$R(r) \equiv (cE(r^2+a^2) - La)^2 - \Delta(r^2 + (L - aEc)^2 + Q. \tag{6}$$

Instead of the constant Q, we introduce a constant $s < \pi/2$, which has the meaning of the minimum angle θ (turning point) reached on the bounded trajectory in question [in the notation of Wilkins (1972), the constant $s \equiv \theta_-$; see also Shakura (1987)]. According to (5), the constant Q can be expressed in terms of s:

$$Q = \cos^2 s(a^2c^2(1 - E^2) + L^2/(\sin s)^2). \tag{7}$$

In that case, the angle θ on a bounded trajectory varies over the range $s < \theta < \pi - s$. For stable trajectories on the $r =$ const surfaces, the following conditions must be satisfied:

$$R(r) = 0, \quad dR(r)/dr = 0 \tag{8}$$

(Bardeen *et al.* 1972; Wilkins 1972).

The energy and angular momentum of particles moving along circular orbits in the Kerr solution

In this section, for simplicity, we choose a system of units in which $c = G = M = 1$. To find stable orbits on the $r =$ const surfaces, we must solve the algebraic equations (8). The corresponding solutions in the $s = \pi/2$ equatorial plane were foundin pioneering work of Ruffini and Wheeler (1970) in the case $j = 1$. Later Bardeen *et al.* (1972) found the expressions for energy and angular momentum of particles at circular orbits for arbitrary j. For $s = 0$ and arbitrary j, the particle angular momentum L is zero. The corresponding expression for the energy was derived by Lightman *et al.* (1975). Inclined orbits at $j = 1$ were investigated by Wilkins (1972). In the case of charged rotating black holes the selected tilted bound geodesics were studied by Johnston and Ruffini, who gave also the general expressions for bound trajectories. Finally, Shakura (1987) derived formulas for the energy and angular momentum at arbitrary j and s by a complex, indirect method. Below, we show that derivation expressions for E and L from Eqs. (8) can be reduced to solving a quadratic equation and factorizing the numerator and denominator in the resulting fraction (to be subsequently canceled by a common factor).

Let us introduce new unknowns instead of E and L:

$$x_c \equiv \frac{E}{L}, \quad y_c = \frac{E^2 - 1}{L^2}. \tag{9}$$

Eqs. (8) can then be written as

$$y_c a_1 + x_c a_2 + (x_c^2 - y_c)(a_1 + a_3) + a_0 = 0, \tag{10}$$

$$y_c b_1 + x_c b_2 + (x_c^2 - y_c)(b_1 + b_3) + b_0 = 0, \quad b_i \equiv \frac{\partial a_i}{\partial r}|_{j,s}, \quad i = 0, 1, 2, 3. \tag{11}$$

Here,

$$a_0 = -(r^2 - 2r + \Delta \cos^2 s), \quad a_1 = r^4 + 2j^2 + j^2 r + j^2 \Delta \cos^2 s,$$

$$a_2 = -4jr, \quad a_3 = -p\Delta, \ p \equiv r^2 + j^2 \cos^2 s. \tag{12}$$

Eliminating y_c from Eqs. (10) and (11) yields the quadratic equation

$$x_c^2(a_1b_3 - a_3b_1) + x_c(a_2b_3 - a_3b_2) + \; + a_0b_3 - a_3b_0 = 0,$$

whose solution is given by

$$x_c = \frac{E}{L} = \frac{A + qB}{D\sin s}, \quad q \equiv \sqrt{r - j^2\cos^2 s/r}. \tag{13}$$

In formula (13), we use the notation

$$A \equiv j\sin s(3r^4 - 4r^3 + j^2r^2\cos^2 s(j^2r^2 - r^4)), \quad B \equiv rp\Delta, \tag{14}$$

$$D = -a_1b_3 + a_3b_1 = (r^2 - j^2\cos^2 s)(r^2 + j^2)^2 - 4j^2r^3\sin^2 s. \tag{15}$$

The numerator and denominator in (13) can be factorized as

$$A + qB = r((r^2 + j^2)q + 2jr\sin s)(j\sin sq + r^2 + j^2\cos^2 s - 2r), \tag{16}$$

$$D = r((r^2 + j^2)q + 2jr\sin s)((r^2 + j^2)q - 2jr\sin s). \tag{17}$$

Therefore, cancelling a common factor in the numerator and denominator in (13), we obtain

$$x_c = \frac{E}{L} = \frac{jq\sin s + r^2 + j^2\cos^2 s - 2r}{(r^2 + j^2)q - 2jr\sin s)\sin s}. \tag{18}$$

Multiplying Eq. (10) by b_1 and Eq. (11) by a_1 and subtracting (11) from (10) yields

$$x_c^2 - y_c = \frac{1}{L^2} = \frac{x(a_1b_2 - a_2b_1) + a_1b_0 - a_0b_1}{D} = \frac{A_1 + qB_1}{D\sin s}, \tag{19}$$

$$A_1 \equiv 2jrp(2r^4 - 3r^3 + j^2r^2 + \cos^2 sj^2(r - j^2)), \quad B_1 \equiv p^2(r^3 - 3r^2 + j^2 + j^2r).$$

The expression $A_1 + qB_1$ can also be factorized as

$$A_1 + qB_1 = p((r^2 + j^2)q + 2jr\sin s)(2rjq\sin s + r^3 - 3r^2 + j^2\cos^2 s(r + 1)). \tag{20}$$

Hence,

$$L = \sqrt{\frac{D\sin s}{A_1 + qB_1}} = \frac{((r^2 + j^2)q - 2jr\sin s)\sin s}{\sqrt{p(p - 3r + j^2\cos^2 s/r + 2jqr\sin s)}}. \tag{21}$$

Using formula (18) for E/L, we obtain with (21)

$$E = \frac{p - 2r + jq\sin s}{\sqrt{p(p - 3r + j^2\cos^2 s/r + 2jqr\sin s)}}. \tag{22}$$

Formulas (21) and (22) are the sought-for expressions for the particle energy and angular momentum in inclined circular orbits.

We determine the coordinate radius of the marginally stable orbit, r_*, from the condition of E given by (40) being at a minimum. Representing r_* as a Taylor expansion in terms of j, we obtain r_* as a function of j, s:

$$\begin{gathered} r_* \approx 6 - 4\sqrt{\frac{2}{3}}j\sin s + .. \approx 6 - 3.266j\sin s - j^2(0.5 + 0.1111\cos 2s) + \\ +j^3(-0.2532\sin s - 0.0567\sin 3s) + j^4(-0.1196 + 0.0206\cos 2s + 0.0162\cos 4s) + \\ +j^5(-0.11\sin s - 0.02\sin 3s + 0.0025\sin 5s)\ldots \end{gathered} \tag{23}$$

At $s=0$, the first expansion terms for r_* are

$$r_* \approx 6 - 0.6111j^2 - 0.0828j^4 - 0.211j^6 - 0.0066j^8 - 0.0023j^{10} + \dots$$

Substituting the Taylor expansion (23) in (21) and (22) yields Taylor expansions for the binding energy and angular momentum in the marginally stable orbit:

$$\begin{gathered}1 - E_* \approx 0.0572 + 0.0321j\sin s + j^2(0.0131 - 0.0087\cos 2s) + \\ +j^3(0.0143\sin s - 0.002\sin 3s) + \\ j^4(0.0065 - 0.006\cos 2s + 0.0003\cos 4s) + j^5(0.0087\sin s - 0.002\sin 3s)\dots, \\ \frac{L_*}{\sin s} \approx 3.4641 - 0.9428j\sin s - j^2(0.1123 + 0.1443\sin^2 s) - j^3(0.1178\sin s + 0.0175\sin^3 s) + \\ j^4(-0.0138 - 0.0878\sin^2 s + 0.0138\sin^4 s) + \\ +j^5(-0.0327\sin s + 0.0467\sin^3 s + 0.016\sin^5 s)\dots \end{gathered} \tag{24}$$

We supplement expansions (23) and (24) with the values of the corresponding functions at $j=1$ numerically constructed by least squares for an arbitrary fixed orbital inclination to the black-hole spin axis:

$$\begin{gathered}1 - E_* \approx 0.4222 - 0.8314\cos^2 s + 0.264\cos^4 s + 0.8605\cos^6 s - 0.656\cos^8 s, \\ r_* \approx 1 + 6.2005\cos^4 s - 7.1816\cos^8 s + 5.1365\cos^{12} s, \\ \\ L_* \approx \sin s\left(1.1876 + 2.5913\cos^2 s - 0.9948\cos^4 s + 0.494\cos^6 s\right). \end{gathered} \tag{25}$$

Note that for $s=0$ and $j=1$, the radius of the marginally stable orbit is $1+\sqrt{3}+\sqrt{3+2\sqrt{3}} \approx 5.2745$ (in units of GM/c^2).

Periastron and nodal precession of circular orbits

Choose the angle θ as a parameter on a bounded trajectory. It then follows from (5)–(7) that

$$\frac{d\phi}{d\theta} = \frac{(2jEr + L(\Delta/\sin^2\theta - j^2))}{\Delta\sqrt{\Theta}}, \tag{26}$$

$$\Theta \equiv (1 - \frac{\sin^2 s}{\sin^2\theta})\left(\sin^2\theta j^2(1-E^2) + \frac{L^2}{\sin^2 s}\right),$$

$$\frac{dt}{d\theta} = \frac{-2jrL + E(r^2 + j^2\cos^2\theta)\Delta + 2rE(r^2+j^2))}{\Delta\sqrt{\Theta}}. \tag{27}$$

Expressions (21) and (22) for the energy and angular momentum in circular orbits must be substituted in these formulas for E and L.

Integrating the right-hand parts of expressions (26-27) over θ from s to $\pi/2$ yields an expression for a quarter of the variation in azimuthal angle $\Delta\phi$ and for a quarter of the period T in which the particle runs from a minimum latitudinal angle s to its maximum $\pi - s$ and back. The integration result can be expressed in terms of the elliptic functions that contain E, L, and s as parameters [instead of the constant Q, we inserted the constant s

into formula (7), which has the meaning of orbital inclination to the black-hole spin axis). Using the notation for the complete elliptic integrals of the first, $K(x)$, second, $E(x)$, and third, $\Pi(n,x)$, kinds, we have

$$T = 4A\left(K(k)(jL - j^2E + (Er^2 + Ej^2 - jL)\frac{(r^2+j^2)}{\Delta}) + (K(k) - E(k))\frac{E}{A^2(1-E^2)}\right);$$

$$\Delta\phi = 4A(L\Pi(-\cos^2 s, k) + (2rEj - j^2L)K(k)), \tag{28}$$

$$A \equiv (L^2/\sin^2 s + j^2(1-E^2))^{-1/2}; k^2 \equiv cos^2 s j^2(1-E^2)A^2.$$

In contrast to Johnston and Ruffini (1974), who first wrote corresponding formulas in form of elliptic integrales for*any* j, in formulas (28), we inserted explicit expressions for all quantities in terms of the turning-point angle $\theta_- \equiv s$ in the orbit. Expressions (21) and (22) must be substituted for E and L, respectively.

For low-eccentricity orbits [4] ($\delta r = \epsilon \sin\xi, \quad \epsilon << r$), it follows from the Hamilton–Jacobi equation (4) that

$$\frac{d\xi}{d\theta} = \frac{\sqrt{-R''/2}}{\sqrt{\Theta}}, \tag{29}$$

$$R'' \equiv \frac{d^2R}{dr^2} = -2\left(6(1-E^2)r^2 - 6r + (1-E^2)j^2\sin s^2 + \frac{L^2}{\sin^2 s}\right).$$

From Eq. (29) for the periastron rotation frequency ν_r, we have

$$\omega_r = 2\pi\nu_r = \frac{\Delta\xi}{T} = 4A\frac{\sqrt{R''/2}}{T}K(k). \tag{30}$$

Note that the frequencies $\Delta\phi/2\pi T$ and $1/T$ are commonly denoted by ν_ϕ and ν_θ, respectively, so the sought-for nodal precession frequency is $\nu_{\rm nod} = \nu_\phi - \nu_\theta$. The periastron precession frequency of an orbit is $\nu_{\rm per} = \nu_\phi - \nu_r$ (Merloni *et al.* 1999).

To clearly present the result, let us derive asymptotic formulas from (28) and (30) in the form of Taylor expansions in powers of j by taking into account the dependence of the first integrals E, L, and Q on j and s given by formulas (21), (22), and (7).

The constants E and $\tilde{L}$ enter into the formulas for $\Delta\phi$ and T via the ratios $E/\tilde{L}$ and $(1-E^2)/L^2, Q/L^2$. Let us write out the series expansions of these ratios with the accuracy that will be required to calculate the nodal precession frequency $(\Delta\phi/2\pi - 1)/T$ up to terms of the order of j^3 inclusive:

$$\frac{E}{L} = \frac{r-2}{r^{3/2}\sin s} + j\frac{3r-4}{r^3} + j^2\frac{(r + 0.5r^2 + \sin^2 s(8 + 7r - 1.5r^2))}{r^{9/2}\sin s} + \ldots,$$

$$\frac{1-E^2}{L^2} = \frac{r-4}{r^3\sin^2 s} + j\frac{8(r-2)}{r^{9/2}\sin s} + \ldots$$

[4]Syer and Clarke (1992) considered the possible existence of stationary disks in the equatorial plane in which the particle orbits were constant-eccentricity ellipses. Clearly, allowance for general-relativity effects will result in the intersection of orbital trajectories in such disks, because the orbital periastron processes. However, in the Newtonian theory, unstable disturbance modes with a radially variable eccentricity disturbance exist in such disks (Lyubarskij *et al.* 1994). Arbitrary accretion-disk disturbances apparently produce spiral waves in the disk structure (Spruit 1987).

$$\frac{Q}{L^2} = \cot^2 s(1 + j^2\frac{(r-4)\sin s}{2r^3} - j^3\frac{4(r-2)\sin s}{r^{9/2}} + \ldots). \tag{31}$$

Using (31), we obtain from expressions (28) and (30) for $\Delta\phi$, T, and $\Delta\xi$

$$\frac{\Delta\phi}{2\pi} - 1 = \frac{2j}{r^{3/2}} - \frac{j^2}{2r^3}3(r-4)\sin s + \frac{j^3}{r^{9/2}}(2 - 1.5r + (18 - 7.5r)\sin^2 s) + \ldots$$

$$\left(\frac{\Delta\xi}{2\pi}\right)^2 = \frac{r-6}{r} + \frac{12j}{r^{5/2}}(r-2)\sin s +$$

$$+\frac{j^2}{r^4}\left(\frac{3}{2}r^2 + 15r - 12 - \sin^2 s(r-4)(\frac{15}{2}r - 21)\right) + \ldots$$

$$\frac{T}{2\pi} = r^{3/2} + 3j\sin s + \frac{j^2}{4r^{3/2}}(8 + 3r + (24 - 9r)\sin^2 s) + \ldots \tag{32}$$

In the limit $s \to 0$, when the spin axis becomes parallel to the tangential plane to the accretion-disk surface, the following formulas hold:

$$\frac{\Delta\phi}{2\pi} - 1 = \frac{2j}{r^{3/2}} + \frac{j^3}{r^{9/2}}(2 - 1.5r) + \frac{3j^5}{r^{15/2}}\left(\frac{3}{2} - \frac{7}{4}r + \frac{19}{32}r^2\right) + \ldots$$

$$\frac{T}{2\pi} = r^{3/2} + \frac{j^2}{4r^{3/2}}(8 + 3r) + \frac{3j^4}{64r^{9/2}}(7r^2 - 48r + 80) + \ldots$$

$$\left(\frac{\Delta\xi}{2\pi}\right)^2 = \frac{r-6}{r} + \frac{3j^2}{2r^4}\left(r^2 + 10r - 8) + \frac{3j^4}{32r^7}(-352 + 576r - 286r^2 + 9r^3)\right) + \ldots \tag{33}$$

From formulas (32) for the Keplerian frequency ν_ϕ, the nodal precession frequency $\nu_{\rm nod}$, and the periastron rotation frequency ν_r of a low-eccentricity orbit around the black-hole spin direction, we derive the compact formulas

$$\omega_\phi = 2\pi\nu_\phi = \frac{\Delta\phi}{T} = \frac{1}{r^{3/2}}\left(1 + \frac{(2 - 3\sin s)j}{r^{3/2}} + \frac{((12 + 9r)\sin^2 s - 8 - 9r)j^2}{4r^6} + \ldots\right);$$

$$(\omega_r)^2 = \frac{r-6}{r^4} + \frac{6j}{r^{11/2}}\sin s(r + 2 + \frac{j^2}{r^7}(12 + 20r - 3\sin^2 s(1+r)(10+r)) + \ldots$$

$$2\pi\nu_{nod} = (\Delta\phi - 2\pi)/T \approx \frac{2j}{r^3} - j^2\frac{1.5\sin s}{r^{7/2}} + j^3\frac{-2 - 3r + (6 + 1.5r)\sin^2 s}{r^6} + \ldots \tag{34}$$

The first term in the formula for nodal precession was found by Lense and Thirring (1918) (see also Wilkins 1972). In a coordinate system rotating with the frequency $\nu_{\rm nod}$, the Keplerian orbit is stationary.

In the limit $s \to 0$, when the black-hole spin axis becomes tangential to the disk surface, we can derive the following expansions for the nodal precession frequency, the periastron rotation frequency, and the Keplerian latitudinal frequency from formulas (28) and (30) using (33)[5]

$$2\pi\nu_{\rm nod} = (\Delta\phi - 2\pi)/T \approx \frac{2j}{r^3}\left(1 - j^2\frac{3r+2}{2r^3} + j^4\frac{8 + 54r + 27r^2}{16r^6} + \ldots\right)$$

$$(2\pi\nu_r)^2 = \frac{r-6}{r^4} + \frac{4j^2}{r^7} - \frac{3j^4}{r^{10}}(r^3 + 128r^2 + 172r + 32) + \ldots$$

[5]Caution must be exercised when passing to the limit $s \to 0$; for example, when $s \to 0$, $L/(\sqrt{\Theta}\sin^2\theta) \to \delta(\sin\theta)$, where $\delta(x)$ denotes the Dirac δ function.

$$2\pi\nu_\theta = \frac{1}{r^{3/2}} - \frac{j^2}{4r^{9/2}}(3r+8) + \frac{j^4}{64r^{15/2}}(16+336r+15r^2)+\dots \qquad (35)$$

To compare formulas (34) and (35) with observations, we should eliminate the radius from them and express the frequencies $\nu_{\rm nod}$ and ν_r as functions of the Keplerian frequency ν_ϕ. The functions $\nu_{\rm nod}(\nu_\phi)$ and $,\nu_r(\nu_\phi)$ may be said to be given by (34) and (35) in parametric form. An increase in ν_2 and a decrease in the frequency difference between the two peaks $\nu_2-\nu_1$ during observations implies the transition of a radiating clump to an orbit closer to the black hole and the approach of its orbit to the marginally stable orbit. When observations are accumulated, formulas (34) and (35) make it possible to determine the orbital inclination to the spin axis of a slowly rotating black hole or neutron star. When analyzing the energy release in the boundary layer and in an extended disk for a rapidly rotating neutron star, we should took into account the appearance of an intrinsic quadrupole moment that exceeds the Kerr one (see Sibgatullin and Sunyaev 1998, 2000a, 2000b). We emphasize that formulas (34) and (35) are not related to the weak-field approximation and are valid up to the marginally stable orbit.

If the disk almost lies in the equatorial plane of a black hole ($s=\pi/2$), then formulas (34) give Taylor expansions of the formulas by Okazaki *et al.* (1987) and Kato (1990) (see also Merloni *et al.* 1999):

$$\omega_\phi = \frac{1}{r^{3/2}+j},$$

$$\omega_{nod} = \frac{1-\sqrt{1-4j/r^{3/2}+3j^2/r^2}}{r^{3/2}+j},$$

$$\omega_r^2 = \frac{1-6/r+8j/r^{3/2}-3j^2/r^2}{(r^{3/2}+j)^2}. \qquad (36)$$

The nodal frequency of the marginally stable orbit for an arbitrary orbital inclination angle s

The following relations (Sibgatullin and Sunyaev 1998, 2000a) hold for the marginally stable orbit in the equatorial plane in a Kerr field, in which the reciprocal radius of the marginally stable orbit acts as a parameter:

$$j = \frac{4\sqrt{x}-\sqrt{3-2x}}{3x}, \quad E=\sqrt{\frac{3-2x}{3}}, \quad L=\frac{2}{3\sqrt{3x}}(2\sqrt{x}\sqrt{3-2x}+x), \quad x\equiv 1/r.$$

Let us substitute these expressions in (36):

$$\omega_\phi = \frac{3x^{3/2}}{3-\sqrt{x}\sqrt{3-2x}+4x},$$

$$\omega_{nod} = \omega_\phi(1+\sqrt{\frac{2}{3}}(\sqrt{x}-\sqrt{3-2x})), \quad \nu_r=0. \qquad (36a)$$

If the parameter x is eliminated from (53a), then we derive the following dependence by least squares:

$$\omega_{nod} \approx 0.1872(\omega_\phi-\sqrt{6}/36)+1.7246(\omega_\phi-\sqrt{6}/36)^2+1.2064(\omega_\phi-\sqrt{6}/36)^3. \qquad (36b)$$

We emphasize that formula (36b) accurately describes the limiting dependence of the nodal precession frequency on Keplerian frequency at $s = \pi/2$ in the marginally stable orbit over the entire ω_ϕ range from 1/26 to 0.5, which corresponds to the range of Kerr parameter from -1 to 1. To derive dimensional dependences, we must make the following substitutions in all formulas: $23.1\omega_{\rm nod} = \nu_{\rm nod}(M/1.4M_\odot)$, $23.1\,\omega_\phi = \nu_\phi(M/1.4M_\odot), \ldots$, where frequencies $\nu_\phi, \nu_{\rm nod}$ will be given hereafter in kHz.

To determine the nodal precession frequency of the marginally stable orbit for an arbitrary orbital inclination s, we use formulas (28) in which for E, L, r, we substitute their expressions in the marginally stable orbit as functions of j and s, (23) and (24). Below, we give the highly accurate approximation dependences for the nodal precession frequency and the Keplerian frequency of the marginally stable orbit, in kHz, on Kerr parameter in the entire j range $(-1 \le j \le 1)$ that we derived by analyzing our numerical calculations for various inclinations of this orbit to the black-hole spin axis:

$$
\begin{aligned}
s = 0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + 0.0448j^3 + 0.0308j^5 + 0.007j^7),\\
\nu_\phi \approx & \frac{1.4M_\odot}{M}(1.5716 + 0.1933j + 0.1539j^2 + 0.0712j^3 + 0.0852j^4 + 0.0533j^5);\\
s = 10^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + 0.0431j^2 + 0.0674j^3 + 0.054j^4 +\\
& 0.0004j^5 - 0.0414j^6 + 0.038j^7 + 0.0432j^8),\\
\nu_\phi \approx & \frac{1.4M_\odot}{M}(1.5716 + 0.3875j + 0.2206j^2 + 0.0675j^3 + 0.1681j^4 + 0.1289j^5);\\
s = 30^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.1394j^2}{1 - 0.82j}),\\
s = 45^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.1923j^2}{1 - 0.87j}),\\
s = 60^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.2499j^2}{1 - 0.92j}),\\
s = 80^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.2535j^2}{1 - 0.976j}).
\end{aligned} \tag{37}
$$

Note that, in contrast to the preceding formulas, the last two formulas in (37) approximate the numerical data in the range $-1 \le j \le 0.99$.

The corresponding limiting dependence of the nodal precession frequency on Keplerian frequency for $s = \pi/6, \pi/4, \pi/3, 4\pi/9$ can be derived from numerical calculations by least squares (here, $\tilde{\omega_\phi} \equiv \omega_\phi - \sqrt{6}/36$):

$$
\begin{aligned}
s = 30^0: \quad & \omega_{nod} \approx 0.3189\tilde{\omega_\phi} + 0.5205(\tilde{\omega_\phi})^2 + 4.37(\tilde{\omega_\phi})^3, \quad 0.049 \le \omega_\phi \le 0.171;\\
s = 45^0: \quad & \omega_{nod} \approx 0.2416\tilde{\omega_\phi} + 1.5953(\tilde{\omega_\phi})^2 - 0.2723(\tilde{\omega_\phi})^3, \quad 0.044 \le \omega_\phi \le 0.257;\\
s = 60^0: \quad & \omega_{nod} \approx 0.2031\tilde{\omega_\phi} + 1.7744(\tilde{\omega_\phi})^2 + 0.5149(\tilde{\omega_\phi})^3, \quad 0.041 \le \omega_\phi \le 0.414;\\
s = 80^0: \quad & \omega_{nod} \approx 0.1846\tilde{\omega_\phi} + 1.7929(\tilde{\omega_\phi})^2 + 1.0862(\tilde{\omega_\phi})^3, \quad 0.039 \le \omega_\phi \le 0.485.
\end{aligned} \tag{37a}
$$

In formulas (37a), we give the ranges of Keplerian frequencies that correspond to the range of Kerr parameter from -1 to $+1$. Consider an example. Let the black-hole mass be $2.2M_\odot$ and the Keplerian rotation frequency be 1.2 kHz. Since the rotation frequency in the marginally stable Keplerian orbit of a nonrotating black hole with $M = 2.2M_\odot$ is $23.1 \times 1.4/2.2 \times \sqrt{6}/36 \approx 1$ kHz, $\tilde{\omega_\phi} = 0.2 \times 2.2/(1.4 \times 23.1) \approx 0.0136$. According to (37a), the nodal precession frequency of the marginally stable orbit is: $\nu_{\rm nod} \approx 42$ Hz for $s = \pi/2$; $\nu_{\rm nod} \approx 52.6$ Hz for $s = \pi/4$; $\nu_{\rm nod} \approx 65.3$ Hz for $s = \pi/6$; $\nu_{\rm nod} \approx 93.6$ Hz for $s = \pi/18$; and $\nu_{\rm nod} \approx 123$ Hz for $s \to 0$. We see that the nodal precession frequency significantly depends on the orbital inclination and changes by a factor of 3 as the inclination changes! If the nodal precession frequency is identified with the horizontal-branch oscillation frequency $\nu_{\rm HBO}$, then it becomes possible to determine the inclination of the marginally stable orbit from the known black-hole mass, the Keplerian frequency, and the nodal precession frequency. At $j << 1$, the nodal precession frequency of the marginally stable orbit is related to the Keplerian frequency by

$$\omega_{nod} \approx \frac{2}{2 + 9\sin s}\tilde{\omega_\phi}. \tag{37b}$$

At a fixed inclination s, the precession frequency ν at the marginally stable orbit monotonically increases with j and reaches a maximum at $j = 1$. For the mass $M = 1.4M_\odot$ and $s = 0$, it is 296 Hz.

The larger the inclination s at fixed j, the larger the precession frequency $\nu_{\rm nod}$. As an illustration, let us write out the approximation formula for $\nu_{\rm nod}$ at $j = 0.998$ [Thorne (1974)][6]:

$$\nu_{nod} \approx 0.29 + 0.7783\sin s + \frac{17.1094\sin^6 s}{1 + 1.35\sin^4 s} \text{in kHz}. \tag{38}$$

2 Nodal and Periastron Precession of Inclined Orbits in the Field of a Rapidly Rotating Neutron Star.

The external gravitational fields of rotating neutron stars

The efficiency of the exact quadrupole solution in describing the external fields of rapidly rotating neutron stars with masses larger than 1 $M_\odot$ was demonstrated in SS 98, SS 00a, and SS 00b. In contrast to the Kerr solution, this solution contains an additional constant, which has the meaning of intrinsic (non-Kerr) NS quadrupole moment, and is symmetric relative to the equatorial plane. The solution under discussion is contained as a special case in the five-parameter (mass, angular momentum, quadrupole moment, electric charge, and magnetic dipole) solutions obtained by Man'ko *et al.* (1994) using the method developed in our book (Singatullin 1991)[7]. However, the expressions for the metric coefficients and the 4-potential of the electromagnetic field are cumbersome. To solve problems with disk accretion in the equatorial plane, we used simple expressions for the metric of the quadrupole solution in the equatorial plane (SS 98, SS 00a). Since the importance of the quadrupole solution in describing the fields of rotating NS gravitational fields is beyond question, we

[6]Studying the evolution of a black hole under the effect of disk accretion in the equatorial plane, Thorne adduced arguments that the Kerr parameter cannot be larger than 0.998 in this case when the radiation from inner disk regions is taken into account. Since the black hole mostly captures photons with a negative angular momentum (opposite to the black-hole rotation), the black-hole spinup by disk accretion is counteracted by its spindown via the predominant capture of the photons propagating against the rotation.

[7]For the relationship of this method to the soliton solutions of Kramer and Neugebauer (1980), see Ernst (1994) and Man'ko and Ruiz (1998).

give here the corresponding exact solution of the Einstein equations in the entire space [8]. Thus, the metric coefficients in the square of the interval in vacuum[9],

$$ds^2 = -f(dt - \omega d\phi)^2 + \frac{\rho^2}{f}d\phi^2 + + \frac{\exp 2\gamma}{f}(d\rho^2 + dz^2) \qquad (39)$$

in the exact quadrupole solution are[10]

$$f = \mathrm{Re}\left(\frac{A-B}{A+B}\right), \quad f\omega = 2\mathrm{Re}\left(\frac{j(A-B)+iC}{A+B}\right), \quad e^{e\gamma} = \frac{A\bar{A} - B\bar{B}}{16k_+^4 k_-^4 R_+ R_- r_+ r_-}. \qquad (40)$$

In formulas (40), we use the following notation

$$\begin{aligned} A &= k_-^2(R_- + r_+)(R_+ + r_-) - - k_+^2(R_- - r_-)(R_+ - r_+), \\ B &= k_+k_-((k_+ + k_-)(r_+ + r_-) + + (k_- - k_+)(R_- + R_+), \\ C &= zB + k_+k_-(k_-(R_+r_+ - R_-r_-) + k_+(R_+r_- - R_-r_+) + 2b(R_+ - R_- + r_- - r_+)), \end{aligned} \qquad (41)$$

$$\begin{aligned} R_\pm &= \sqrt{\rho^2 + (z \pm (k_+ + k_-)/2)^2}(\sqrt{1-j^2} \pm ij), \quad k_- \equiv \sqrt{1-j^2-4b}, \quad k_+ \equiv \sqrt{1-j^2}, \\ r_\pm &= \sqrt{\rho^2 + (z \pm (k_+ - k_-)/2)^2}(\sqrt{1-j^2} \pm ij). \end{aligned} \qquad (42)$$

The metric coefficient ω becomes zero on the symmetry axis at $|z| > (k_+ + k_-)/2$. We assume that ω on the symmetry axis becomes zero everywhere outside the rotating NS (the condition for the absence of conical points). Solution (40)–(42) was obtained for the Ernst potential on the symmetry axis:

$$\mathbf{E} = \frac{A-B}{A+B}|_{\rho=0} = \frac{z^2 + (ij-1)z + b^2}{z^2 + (ij+1)z + b^2}.$$

Note that $(j(A-B)+iC)/(A+B) = i$ on the symmetry axis.

In the special case where $b = (1-j^2)/4$, the constant k_- becomes zero. In this case, it is convenient to pass to the coordinates $\rho = \sin\theta\sqrt{(r-1)^2 - (1-j^2)/4}$, $z = (r-1)\cos\theta$. Passing to the limit $k_- \to 0$, the metric coefficients in the square of the interval

$$ds^2 = -f(dt - \omega d\phi)^2 + \frac{\rho^2}{f}d\phi^2 + \frac{g}{f}\left(\frac{dr^2}{(r-1)^2 - (1-j^2)/4} + d\theta^2\right)$$

may be represented as

$$f = \mathrm{Re}\left(\frac{a}{b}\right), \quad f\omega = 2\mathrm{Re}\left(\frac{c}{b}\right), \quad g = \frac{\mathrm{Re}(ab^*)}{4(4(r-1)^2 - (1-j^2)m^2)^3.} \qquad (43)$$

[8] The metric of the exact solutions for the Einstein equations with a finite set of multipole moments is given in general form in SS 00. The method of constructing exact asymptotic flat solutions for the Einstein-Maxwell set of equations with data specified on the symmetry axis was developed in our book (Sibgatullin 1991). The entire problem reduces to solving the only homogeneous singular equation with the Cauchy kernel on a segment with an additional normalization condition.

[9] in Papapetru form.

[10] Here, we use a system of units with $M = c = G = 1$.

Here, the asterisk denotes a complex conjugate and the following notation is used with $m \equiv \cos\theta$:

$$
\begin{aligned}
a = & -27 - 5j^2 + 18ijm - 2ij^3m - 6ijm^3 + 6ij^3m^3 - j^2m^4 + j^4m^4 + 108r + 4j^2r - \\
& - 48ijmr + 4ijm^3r - 4ij^3m^3r - 144r^2 + 48ijmr^2 + 80r^3 - 16ijmr^3 - 16r^4, \\
b = & -3 + 3j^2 + 14ijm + 2ij^3m - 2ijm^3 + \; + 2ijm^3 - j^2m^4 + j^4m^4 + 10r - 4j^2r - \\
& - 48ijmr + 4ijm^3r - 4ij^3m^3r - 48r^2 + 48ijmr^2 + 48r^3 - 16ijmr^3 - 16r^4, \\
c = & \; ij^4m(-1+2m^2) - im(-3+2r)(-1+2r)^3 - 2ij^2m(-2+m^2+2r) + \\
& + j(-13 + m^4(1-2r) + \; + 46r - 48r^2 + 16r^3) + j^3(-3 + 2r + m^4(-1+2r)).
\end{aligned}
$$

Let us again turn back to the general exact quadrupole solution (40)–(42) and pass to Boyer–Lindquist coordinates $z = (r-1)\cos\theta$, $\rho = \sin\theta\sqrt{(r-1)^2 - (1-j^2)}$.

Let us represent the square of the interval as

$$
\begin{aligned}
ds^2 = & -f dt^2 + 2f\omega dt d\phi + \sin^2\theta \left(\frac{r^2 - 2r + j^2}{f} - f\omega^2 \right) d\phi^2 + \\
& + \frac{\Gamma}{f} \left(\frac{dr^2}{r^2 - 2r + j^2} + d\theta^2 \right),
\end{aligned} \tag{44}
$$

where

$$
f = \frac{F}{r^2 + j^2\cos^2\theta}, \quad \omega = \frac{W}{r^2 + j^2\cos^2\theta}. \tag{45}
$$

We will seek the functions F, Γ, and W using the exact solution (40)–(42) in the form of series in inverse powers of the radius. At $b = 0$, the series for F, Γ, and W break off on the second or third term, giving the exact Kerr solution

$$
F_k = \Gamma_k = r^2 - 2r + j^2\cos^2\theta, \quad W_k = -2jr\sin^2\theta.
$$

At $b \neq 0$, the series for F, G, and W contain an infinite number of terms and diverge on the event horizon. This fact is related to the so-called *no hair theorem*[11] (see Misner *et al.* (1973), Novikov and Frolov (1986), Sibgatullin (1984) for a more detailed discussion of the black-hole theory). A collapsing star loses its magnetic and quadrupole moment and its field asymptotically approaches the field of a rotating black hole (Regge and Wheeler 1957; Ginzburg and Ozernoi 1964; Doroshkevich *et al.* 1965).

[11] According to Wheeler's figurative expression, "a black hole has no hair ".

Cumbersome transformations (in these formulas, $m \equiv \cos\theta$) yielded

$$F = r^2 - 2r + j^2m^2 + b\left(\frac{3m^2-1}{r^2}(r+1) + \frac{1}{r^3}\left(\frac{3}{4}(1+j^2+b) - \right.\right.$$
$$\left.\left. -\frac{1}{2}(3+25j^2+15b)m^2 + \frac{1}{4}(-5+63j^2+35b)m^4\right) + \mathrm{o}\left[\frac{1}{r^4}\right]\right);$$
$$\Gamma = r^2 - 2r + j^2m^2 + b(1-m^2)\left(\frac{3}{2r^3}(5m^2-1)(r+2) + \frac{1}{4r^4}(-17-j^2+8b+\right.$$
$$\left. +2(35+10j^2+56b)m^2 + (35-35j^2-180b)m^4) + \mathrm{o}\left[\frac{1}{r^5}\right]\right);$$
$$W = 2j(1-m^2)\left(-r + b\left(\frac{1}{4r^3}(5m^2-1)(4r+7) + \frac{1}{8r^4}(-24-b+\right.\right.$$
$$\left.\left. +2(54+20j^2+63b)m^2 + (12-72j^2-189b)m^4) + \mathrm{o}\left[\frac{1}{r^5}\right]\right)\right). \tag{46}$$

Periastron and nodal precession of orbits with low inclinations to the equatorial plane and with low eccentricities

The eikonal equation $g^{ij}S_{,i}S_{,j} = -1$ in stationary spaces with axial symmetry and with the square of the interval (39) have solutions of the form $S = -Et + L\phi + \tilde{S}(\rho, z)$, where $\tilde{S}$ satisfies the equation

$$(\tilde{S}^2_{,\rho} + \tilde{S}^2_{,z})fe^{-2\gamma} = V(\rho, z) \equiv \frac{E^2}{f} - \frac{f}{\rho^2}(L-\omega E)^2 - 1. \tag{47}$$

Consider nearly circular orbits in the equatorial plane. For the latter, $V = V_{,z} = V_{,\rho} = 0$ at $z = 0$. Consequently, expanding the right-hand part of Eq. (47) for perturbations in a Taylor series to within quadratic terms, we have

$$(\tilde{S}^2_{,\rho} + \tilde{S}^2_{,z})fe^{-2\gamma} = \frac{1}{2}(V_{,\rho\rho}(\Delta\rho)^2 + V_{,zz}z^2) + \mathrm{const.} \tag{48}$$

The constant in Eq. (48) is related to the perturbation of V when the constants E and L are perturbed for nonequatorial, noncircular orbits. Equation (48) can be solved by the separation of variables, $\tilde{S} = S_1(\Delta\rho) + S_2(z)$. Consider the Hamiltonian system associated with Eq. (11). We will seek $\Delta\rho$ and z in the form $\Delta\rho = \epsilon_1 \sin\xi$, $z = \epsilon_2 \sin\zeta$. We then obtain

$$\begin{aligned} &\frac{d\zeta}{ds} = \sqrt{-fe^{-2\gamma}V_{,zz}/2}, \quad \frac{d\xi}{ds} = \sqrt{-fe^{-2\gamma}V_{,\rho\rho}/2}, \\ &\frac{d\phi}{ds} = \frac{f}{\rho^2}(L-\omega E), \quad \frac{dt}{ds} = \frac{E}{f} + \frac{f\omega}{\rho^2}(L-\omega E). \end{aligned} \tag{49}$$

Note that the right-hand parts of Eqs. (49) are constant.

Let us now make use of the expression for the energy and angular momentum of particles in equatorial circular orbits (SS 98, SS 00a):

$$E = \frac{\sqrt{f}}{\sqrt{1 - f^2\,p^2/r}}, \quad L - \omega E = pE, \tag{50}$$

$$p \equiv \rho^2(-\lambda + \sqrt{\lambda^2 + \mu - \mu^2 r})/n, \quad \lambda \equiv f\dot{\omega}, \quad \mu \equiv \dot{f}/f, \quad n \equiv f - r\dot{f}.$$

Here, the dot denotes a derivative with respect to ρ^2.

For the angular velocity of a particle in a Keplerian equatorial circular orbit, the following formula can be derived from Eqs. (49):

$$\omega_\phi = \frac{p}{(\rho^2/f^2 + \omega p)}. \tag{51}$$

Below, it is convenient to introduce a new quantity instead of p: $\tau \equiv \rho^2/(fp)$. The formula (51) can then be rewritten as

$$\nu_\phi = \frac{1}{T}, \quad T/(2\pi) = \omega + \tau/f, \quad \tau \equiv (\lambda + \sqrt{\lambda^2 + \mu - \mu^2 r})/\mu. \tag{52}$$

For the rotation frequency from periastron to periastron, $2\pi\nu_r = d\xi/dt$, we obtain from Eqs. (49)

$$\nu_r^2 = \nu_\phi^2 \, M, \tag{53}$$

where we use the notation

$$M \equiv 2e^{-2\gamma}\left(-gf\left(\frac{1}{f}\right)_{,qq}\tau^2 + \frac{q^3}{f}\left(\frac{f}{q}\right)_{,qq} - \right.$$
$$\left. -4q^2\left(\frac{f}{q}\right)_{,q}\tau\omega_{,q} - 2q\tau f\omega_{,qq} + 2\tau^2 f^2(\omega_{,q})^2\right)|_{z=0}, \quad q \equiv \rho^2.$$

In exactly the same way, for the rotation frequency of the maximum rise in z, $2\pi\nu_\theta = d\zeta/dt$, we obtain from Eqs. (49) using the explicit expression (47) for V

$$\nu_\theta^2 = \nu_\phi^2 \, N, \tag{54}$$
$$N = \frac{e^{-2\gamma}}{f}(f_{,zz}(\rho^2 + \tau^2) - 2f\omega_{,zz}\tau)|_{z=0}.$$

For the periastron and nodal precession frequencies, we have

$$\nu_{\rm per} = \nu_\phi(1 - \sqrt{M}), \quad \nu_{\rm nod} = \nu_\phi(1 - \sqrt{N}). \tag{55}$$

We will seek the functions $T(r)$, $N(r)$, and $M(r)$ in coordinates (44) in the form of expansions in terms of inverse powers of r. At $b = 0$, the series break off to give the exact formulas of Okazaki *et al.* (1987) and Kato (1990) for the Kerr metric

$$M = 1 - \frac{6}{r} + \frac{8j}{r^3/2} - \frac{3j^2}{r^2}, \quad N = 1 - \frac{4j}{r^{3/2}} + \frac{3j^2}{r^2}, \quad T/(2\pi) = j + r^{3/2}.$$

Let $b \neq 0$; using coordinates (44), we obtain

$$M=1-\frac{6}{r}+\frac{8j}{r^{3/2}}-\frac{3(j^2+b)}{r^2}+3b\left(-\frac{5}{r^3}+\frac{8j}{r^{7/2}}-\frac{1}{2r^4}(16+7j^2+2b)+\right.$$
$$\left.\frac{12j}{r^{9/2}}-\frac{1}{4r^5}(51+3j^2+31b)+\mathrm{o}\left[\frac{1}{r^{11/2}}\right]\right); \qquad (56)$$
$$N=1-\frac{4j}{r^{3/2}}+\frac{3(j^2+b)}{r^2}+3b\left(\frac{2}{r^3}-\frac{5j}{r^{7/2}}+\frac{1}{2r^4}(8+7j^2+2b)-\frac{9j}{r^{9/2}}+\mathrm{o}\left[\frac{1}{r^5}\right]\right); \qquad (57)$$
$$T/(2\pi)=r^{3/2}+j+b\left(-\frac{3}{4r^{1/2}}-\frac{5}{4r^{3/2}}+\frac{3j}{2r^2}-\frac{3}{32r^{5/2}}(10+10j^2+b)+\right.$$
$$\left.\frac{5j}{r^3}-\frac{3}{16r^{7/2}}(29+13j^2+4b)+\mathrm{o}\left[\frac{1}{r^4}\right]\right). \qquad (58)$$

Using Eqs. (55) and (56) for the nodal velocity of slightly inclined orbits, we derive a Taylor series in inverse powers of the radius:

$$2\pi\nu_{\mathrm{nod}}=\frac{2j}{r^3}-\frac{3(b+j^2)}{2r^{7/2}}-\frac{3b}{r^{9/2}}+\frac{3j}{2r^5}(-j^2+5b)-$$
$$-\frac{3}{8r^{11/2}}(16b-3j^4+11j^2b+4b^2)+\mathrm{o}\left[\frac{1}{r^6}\right]. \qquad (59)$$

At $b=0$, we obtain a Taylor expansion of the formula by Okazaki *et al.* (1987) and Kato (1990) from Eq. (55-58).

For orbits with low inclinations to the equatorial plane, Markovic (2000) derived formulas for the periastron and nodal precession frequencies in the *post-Newtonian approximation*. His formula for the nodal precession frequency is equivalent to Eq. (59) if the first two terms are retained in it. The latter are the sum of the Newtonian precession of a slightly inclined orbit in a gravitational field with a quadrupole moment and the precession of Lense and Thirring (1918).

Thus, for orbits with low inclinations to the equatorial plane and with low eccentricities, allowance for NS oblateness in Eqs. (55-59) for the nodal and periastron precession of orbits with rapid rotation gives a **large contribution for the stiff equations of state**. The method for constructing the dependence of quadrupole moment on the NS rest mass and Kerr parameter is described in SS 00a; for the equations of state A and FPS, specific functions $b(j,m)$ were constructed at rest masses larger than the solar mass and smaller than the critical mass according to the static stability criterion. Laarakkers and Poisson (1998) found an almost quadratic dependence of quadrupole moment on the Kerr parameter at small j. In SS 98, the dependence $b(j)$ was studied for normal sequences with $1.4M_\odot$ in the static limit and for normal sequences unstable in the static limit *over the entire range of Kerr parameters* in which the equatorial rotation velocity on the stellar surface is lower than the equatorial Keplerian velocity.

Nodal precession of orbits with arbitrary inclinations to the equatorial plane in the post-newtonian approximation .

Consider the non-Kerr terms in Eqs. (47) as small Kerr metric perturbations: $F=F_k(1+v)$; $\Gamma=\Gamma_k(1+z)$; $W=W_k(1+w)$. The eikonal equation (47) in Boyer–Lindquist coordinates can be written as

$$(\tilde{S}^2_{,r}\Delta+\tilde{S}^2_{,\theta})(1+v-z)=\frac{(E\,(r^2+j^2)\;-\;L\,j)^2}{\Delta}-(L-jE)^2+\cos^2\theta\left((E^2-1)j^2-\frac{L^2}{\sin^2\theta}\right)-$$
$$-v\left(\left(\frac{(r^2+j^2)^2}{\Delta}-j^2\sin^2\theta\right)E^2+\frac{L^2}{\sin^2\theta}-\frac{L^2j^2}{\Delta}\right)-\frac{4w\,(2j^2\,r^2\,\sin^2\theta+E\,L\,j\,r)}{\Delta}. \quad (60)$$

The asymptotic result for the nodal precession frequency that follows from Eq. (22) can also be obtained as follows.

Consider the energy integral for a particle moving in the gravitational field of a point mass with a quadrupole moment in the Newtonian approximation:

$$\frac{1}{2}\left(r^2\left(\frac{d\theta}{dt}\right)^2+\frac{L^2}{r^2\sin^2\theta}+\left(\frac{dr}{dt}\right)^2\right)-\frac{1}{r}+\frac{b}{2r^3}(3\cos^2\theta-1)=H, \quad H=\text{const}. \quad (61)$$

In the absence of quadrupole moment for circular orbits, we have

$$H=-1/2; \quad p_r=0; \quad L^2=r\sin^2 s; \quad \cos\theta=\cos s\cos(t/r^{3/2}). \quad (62)$$

We restrict our analysis to the case $b/r^2 << 1$ and denote $\tau\equiv t/r^{3/2}$. The energy integral, to within quantities of the second order of smallness in b/r^2, can be written as

$$\left(\frac{d\theta}{d\tau}\right)^2=\left(1-\frac{L^2}{r\sin^2\theta}\right)\left(1-4\frac{\delta r}{r}\right)+2r\delta H-\frac{b}{r^2}(3\cos^2\theta-1). \quad (63)$$

We write the constant H via its value at the turning point $d\theta/d\tau=0$:

$$1+rH=\frac{L^2}{r\sin^2 s}+\frac{b}{2r^2}(3\cos^2 s-1). \quad (64)$$

With (64), the energy integral (63) takes the form

$$\left(\frac{d\theta}{d\tau}\right)^2=\frac{\sin^2\theta-\sin^2 s}{\sin^2\theta}\left(\frac{L^2}{r\sin^2 s}-4\frac{\delta r}{r}+\frac{3b}{r^2}\sin^2\theta\right). \quad (65)$$

An analog of circular orbits in a quadrupole field are perturbed orbits with a zero mean deviation δr from a sphere of constant radius:

$$\int\delta r\,d\,\phi(t)=0. \quad (66)$$

In Eq. (66), the integral is taken over the meridional period T_θ in which the particle again returns to the minimum angle s. Given (66), the azimuthal angle ϕ changes in time $T_\theta/4$ by

$$\Delta\phi=\int_s^{\pi/2}\sqrt{\frac{L^2}{r\sin^2 s}\frac{d\theta}{\sin^2\theta}}\left((\cot^2 s-\cot^2\theta)\left(\frac{L^2}{r\sin^2 s}+\frac{3b}{r^2}\sin^2\theta\right)\right)^{1/2}\approx$$
$$\approx\frac{\pi}{2}\left(1-\frac{3b}{2r^2}\sin s\right). \quad (67)$$

Consequently, the nodal precession because of NS oblateness (the presence of a quadrupole moment) is

$$2\pi\nu_{\rm nod} = -\frac{3}{2r^{7/2}}\, b\, \sin s.$$

Combining this result with the formula for the precession frequency of inclined circular orbits in a Kerr field [see Eq. (34)], we **derive the sought-for formula for the nodal precession frequency in the post-Newtonian approximation**:

$$2\pi\nu_{\rm nod} = \frac{2j}{r^3} - \frac{3}{2r^{7/2}}(b+j^2)\, \sin s. \tag{68}$$

Formula (68) at $s = \pi/2$ transforms to the formula of Markovic (2000). In exactly the same way, for the azimuthal (Keplerian) period at an inclined orbit in the field of an oblate star in the post-Newtonian approximation, we obtain using (66)

$$1/(2\,\pi\,\nu_\phi) = r^{3/2} + (3\sin s - 2)j - \frac{3}{4}\frac{b}{r^{1/2}}(2-\sin s) + \frac{9}{4\,r^{1/2}}\cos^2 s\,(j^2+b). \tag{69}$$

At $s \to \pi/2$, Eq. (69) matches Eq. (34) in the principal terms.

Conclusions

According to the ideas explicitly formulated by Bardeen and Petterson (1975) and Bardeen (1977), a geometrically tilted accretion disk is most commonly modeled as a set of rings with the center at the coordinate origin that smoothly turn with changing radius under the effect of viscous torques and gravitomagnetic forces. This approach underlies the existing theories of tilted accretion disks [Papaloizou and Pringle (1983) and Markovic and Lamb (1998, 2000) for small disk deviations from the equatorial plane; Pringle (1992, 1997) for finite disk deviations from the equatorial plane), which are based on a vector equation for the conservation of angular momentum. Some ideas developed by Shakura and Sunyaev (1973, 1976) for flat disks can be used to study the physics of tilted accretion disks. Van Kerkwijk *et al.* (1998) explained the puzzling spindown and spinup of some X-ray pulsars by the fact that the accretion-disk tilt in the inner regions could become larger than 90 degrees!

The effects considered above take place for the inner nonstationary part of the disk, where viscous torques may be disregarded. Matter is not accumulated in the marginally stable orbit because of its instability. In contrast to the statement by Stella *et al.* (1999) that the orbital inclination affects weakly the nodal precession frequency, our detailed analysis shows that this effect is significant. Kuznetsov (2001a, 2001b) has estimated effect of circular orbits inclination on variability of X-Ray radiation in Scorpius X-1 and X-2. There are probably preferential inclinations of the marginally stable orbits, and there is no need to introduce even harmonics of the nodal frequency to reconcile observations with theory.

Acknowledgments

I wish to thank Prof. Sunyaev for fruitful discussions.

References

[1] J. M. Bardeen, W. H. Press, and S. Teukolsky, Astrophys. J. **178**, 347 (1972).

[2] J. M. Bardeen and J. A. Petterson, Astrophys. J. **195**, L65 (1975).

[3] B. Carter, Phys. Rev. **174**, 1559 (1968).

[4] G. B. Cook, S. L. Shapiro, and S. A. Teukolsky, Astrophys. J. **424**, 823 (1995).

[5] W. Cui, S. N. Zhang, and W. Chen, Astrophys. J. Lett. **492**, L53 (1998); astro-ph/9811023.

[6] A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Zh. @Eksp. Teor. Fiz. **49**, 170 (1965) [Sov. Phys. JETP **22**, 122 (1966)].

[7] V. L. Ginzburg and L. M. Ozernoi, Zh. Eksp. Teor. Fiz. **47**, 1030 (1964) [Sov. Phys. JETP **20**, 689 (1964)].

[8] F. de Felice, J. Phys. A **13**, 1701 (1980).

[9] W. Israel, Phys. Rev. D **2**, 641 (1970).

[10] M.Johnston, R.Ruffini, Phys. Rev. D **10**, 2324 (1974).

[11] S. Kato, Publ. Astron. Soc. Japan. **42**, 99 (1990).

[12] D. Kramer and G. Neugebauer, Phys. Lett. A **75**, 259 (1980).

[13] S.I.Kuznetsov//Preprints of Cosmic Research Institut (Russia) N 2044, N 2045 (2001).

[14] W. Laarakkers and E. Poisson, Astrophys. J. **512**, 282 (1999); gr-qc/9709033 (1997).

[15] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Nauka, Moscow, 1980; Pergamon, Oxford, 1975).

[16] J. Lense and H. Thirring, Phys. Z. **19**, 156 (1918).

[17] A. F. Lightman, W. N. Press, R. H. Price, and S. A. Teukolsky, *Problem Book on Relativity and Gravitation* (Princeton Iniv. Press, Princeton, 1975).

[18] Y. E. Lyubarskij, K. A. Postnov, and M. E. Prokhorov, Mon. Not. R. Astron. Soc. **266**, 583 (1994).

[19] V. S. Man'ko and E. Ruis, Class. Quantum Grav. **15**, 2007 (1998).

[20] V. C. Man'ko, N. R. Sibgatullin, *et al.*, Phys. Rev. D **49**, 5144 (1994).

[21] D. Markovic, astro-ph/0009450.

[22] D. Markovic and F. K. Lamb, astro-ph/0009169.

[23] D. Markovic and F. K. Lamb, Astrophys. J. **507**, 316 (1998); astro-ph/9801075.

[24] A. Merloni, M. Vietri, L. Stella, and D. Bini, Mon. Not. R. Astron. Soc. **304**, 155 (1999); astro-ph/9811198.

[25] Ch. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).

[26] Sh. M. Morsink and L. Stella, Astrophys. J. **513**, 827 (1999); astro-ph/9808227.

[27] A. T. Okazaki, S. Kato, and J. Fukue, Publ. Astron. Soc. Japan. **39**, 457 (1987).

[28] J. C. B. Papaloizou and J. E. Pringle, Mon. Not. R. Astron. Soc. **202**, 1181 (1983).

[29] J. A. Petterson, Astrophys. J. **214**, 550 (1977).

[30] J. E. Pringle, Mon. Not. R. Astron. Soc. **258**, 811 (1992).

[31] J. E. Pringle, Mon. Not. R. Astron. Soc. **281**, 357 (1996).

[32] J. E. Pringle, Mon. Not. R. Astron. Soc. **292**, 136 (1997).

[33] D. Psaltis, R. Wijnands, J. Homan, *et al.*, Astrophys. J. **520**, 763 (1999); astro-ph/9903105.

[34] Regge T., Wheeler J.A., Phys. Rev. **108**, 1063 (1957).

[35] R. Ruffini and J. A. Wheeler, Bull. Am. Phys. Soc. **15** (11), 76 (1970).

[36] F. D. Ryan, Phys. Rev. D **52**, 5707 (1995).

[37] F. D. Ryan, Phys. Rev. D **55**, 6081 (1997).

[38] N. I. Shakura, Pis'ma Astron. Zh. **13**, 245 (1987) [Sov. Astron. Lett. **13**, 99 (1987)].

[39] N. I. Shakura and R. A. Sunyaev, Astron. Astrophys. **24**, 337 (1973).

[40] N. I. Shakura and R. A. Sunyaev, Mon. Not. R. Astron. Soc. **175**, 613 (1976).

[41] M. Shibata and M. Sasaki, Phys. Rev. D **58**, 10401 (1998).

[42] N. R. Sibgatullin, *Oscillation and Waves in Strong Gravitational and Electromagnetic Fields* (Springer Verlag, Heisenberg, 1991) .

[43] N. R. Sibgatullin, Pis'ma Astron. Zh. **27**, 918 (2001) [Astron. Lett. **27**, 799 (2001)].

[44] N. R. Sibgatullin and R. A. Sunyaev, Pis'ma Astron. Zh. **24**, 894 (1998) [Astron. Lett. **24**, 774 (1998)].

[45] N. R. Sibgatullin and R. A. Sunyaev, Pis'ma Astron. Zh. **26**, 813 (2000a) [Astron. Lett. **26**, 699 (2000a)].

[46] N. R. Sibgatullin and R. A. Sunyaev, Pis'ma Astron. Zh. **26**, 898 (2000b) [Astron. Lett. **26**, 772 (2000b)].

[47] H. C. Spruit, Astron. Astrophys. **184**, 173 (1987).

[48] L. Stella, astro-ph/0011395.

[49] L. Stella and M. Vietri, Astrophys. J. Lett. **492**, L59 (1998); asto-ph/9709085.

[50] L. Stella, M. Vietri, and Sh. M. Morsink, Astrophys. J. **524**, L63 (1999); astro-ph/9907346.

[51] D. Syer and C. J. Clarke, Mon. Not. R. Astron. Soc. **255**, 92 (1992).

[52] S. Thorne Kip, Astrophys. J. **191**, 507 (1974).

[53] M. van der Kliss, Astron. Astrophys.(Annual Review) **38**, 716 (2000); astro-ph/0001167.

[54] M. H. van Kerkwijk, Deepto Chakrabarty, J. E. Pringle, and R. A. M. Wijers, Astrophys. J. Lett. **499**, L67 (1998);

astro-ph/9802162.

[55] D. C. Wilkins, Phys. Rev. D **5**, 814 (1972).

[56] M. N. Zaripov, N. R. Sibgatullin, and A. Chamorro, Prikl. Mat. Mekh. **59** (5), 750 (1995).

Application.
The newtonian analog of the Kerr solution and its potentials

In the Newtonian limit, $E^2 \approx 1 + 2H/c^2$ and $\Delta \approx r^2 + a^2$. Substituting these expressions in Eq. (4) yields an approximate Hamilton–Jacobi equation:

$$2H = \frac{L^2}{(r^2+a^2)\sin^2\theta} + \frac{r^2+a^2}{r^2+a^2\cos^2\theta} S_{,r}^2 + \frac{1}{r^2+a^2\cos^2\theta} S_{,\theta}^2 + \\ \frac{4\,a\,r\,L\,G\,M/c}{(r^2+a^2)(r^2+\ a^2\ \cos^2\theta)} - \frac{2GMr}{r^2+a^2\cos^2\theta} + \frac{4(G\,M\,r\,a\,\sin\theta)^2}{(r^2+a^2\cos^2\theta)c^2(r^2+a^2)}. \qquad \text{(A 1)}$$

Note that Eq. (A 1) describes the trajectories of test particles in the field of a flat disk with radius a in special curvilinear coordinates r, θ, ϕ, which are related to Cartesian coordinates x, y, z by

$$r = \frac{1}{2}(\sqrt{x^2+y^2+(z-ia)^2} + + \sqrt{x^2+y^2+(z+ia)^2}), \qquad \text{(A 2)}$$

$$a\sin\theta = \frac{1}{2i}(\sqrt{x^2+y^2+(z+ia)^2} - \sqrt{x^2+y^2+(z-ia)^2}), \quad \phi = \arctan y/x. \qquad \text{(A 3)}$$

When passing from Cartesian coordinates x, y, z to orthogonal curvilinear coordinates r, θ, ϕ using formulas (A 2) and (A 3), the nonzero metric tensor components are expressed in terms of the curvilinear coordinates as

$$g_{rr} = \frac{r^2+a^2\cos^2\theta}{r^2+a^2}, \quad g_{\theta\theta} = r^2+a^2\cos^2\theta, \quad g_{\phi\phi} = (r^2+a^2)\sin^2\theta. \qquad \text{(A 4)}$$

Hamiltonian (A 1) contains the contravariant metric tensor components (A 4), and it can be written in Cartesian coordinates as

$$H = \frac{(\mathbf{p}+\mathbf{A}/c)^2}{2} - \Phi.$$

The corresponding Hamilton equations are

$$\frac{dp_k}{dt} = \Phi_{,k} - (\mathbf{p} + \mathbf{A}/c)\frac{\partial \mathbf{A}/c}{\partial x^k}, \quad \frac{dx^k}{dt} = p_k + \mathbf{A}_k/c. \tag{A 5}$$

Eqs. (A 5) can be rewritten as the equations of motion in a gravitomagnetic field in the quasi-Newtonian approximation:

$$\frac{d\mathbf{v}}{dt} = \nabla\Phi + \mathrm{curl}\,\mathbf{A} \times \mathbf{v}/c, \tag{A 6}$$

where the Newtonian potential Φ and the gravitomagnetic vector potential $\mathbf{A}$ in a vacuum satisfy the Laplace equation. The equations of relative motion in a rotating coordinate system can be derived from Eqs. (A 6) if $\mathbf{\Omega} \times \mathbf{R}$, $\mathbf{R} = (x, y, z)$, is substituted for $\mathbf{A}$. In that case, $\mathrm{curl}\,\mathbf{A}/c = -2\mathbf{\Omega}$ and the second term on the right-hand side of (A 6) represents the Coriolis force.

For a flat Kerr disk, the Newtonian potential Φ is (Israel 1970; Zaripov *et al.* 1995)

$$\Phi = \frac{GM}{2\sqrt{x^2 + y^2 + (z + ia)^2}} + \frac{GM}{2\sqrt{x^2 + y^2 + (z - ia)^2}}. \tag{A 7}$$

The vector $\mathbf{A}$ consists of the components $-g_{0\alpha}/g_{00}$, $\alpha = 1, 2, 3$ (Landau and Lifshitz 1980). In the Newtonian limit in a vacuum, the vector of the gravitomagnetic field Ψ can be introduced instead of the vector potential $\mathbf{A}$:

$$\mathrm{curl}\,\mathbf{A} = 2\nabla\Psi. \tag{A 8}$$

The scalar Ψ in an axisymmetric case matches the imaginary part of the Ernst complex potential. For the special case of a Kerr disk, the scalar Ψ is (Zaripov *et al.* 1995):

$$\Psi = \frac{GM}{2i\sqrt{x^2 + y^2 + (z - ia)^2}} - \frac{GM}{2i\sqrt{x^2 + y^2 + (z + ia)^2}}. \tag{A 9}$$

Clearly, expressions (A 7) and (A 9) for Φ and Ψ satisfy the Laplace equation.

As follows from definition (A 2), the $r = \mathrm{const}$ surface is neither a sphere nor an ellipsoid.

The precession of circular keplerian Orbits in the field of a gravitating rotating mass

For the Newtonian analog of a supercritical Kerr disk, $a > GM/c^2$. However, since $a < GM/c^2$ for black holes, in this subsection, we discard terms of the order of a^2 in expressions (A 7) and (A 9) for Φ and Ψ lest the order of accuracy of the model be exceeded. In that case, $\Phi \approx GM/r$, $\quad \Psi \approx -azGM/r^3$.

Let us introduce a coordinate system with the origin at the gravitating mass with the z axis directed along the spin axis. The equations of motion for free particles in this coordinate system are

$$\begin{aligned}
\frac{du}{dt} &= -\frac{GMx}{r^3} + \left(6a(yw - zv)\frac{z}{r^5} + 2v\frac{a}{r^3}\right)\frac{GM}{c}, \\
\frac{dv}{dt} &= -\frac{GMy}{r^3} + \left(6a(zu - xw)\frac{z}{r^5} - 2u\frac{a}{r^3}\right)\frac{GM}{c}, \\
\frac{dw}{dt} &= -\frac{GMz}{r^3} + \left(6a(xv - yu)\frac{z}{r^5}\right)\frac{GM}{c}.
\end{aligned} \tag{A 10}$$

In our approximation, the complete integral of the Hamilton–Jacobi equation is [cf. formulas (5)–(6)]

$$S = -Ht + L\phi + \int \sqrt{\Theta(\theta)}d\theta + \int \sqrt{2R(r)}dr,$$

$$\Theta(\theta) \equiv L^2(\cot^2 s - \cot^2 \theta), \tag{A 11}$$

$$R(r) \equiv H + \frac{GM}{r} - \frac{L^2}{2r^2 \sin^2 s} - 2\frac{L\,a\,G\,M}{r^3 c}. \tag{A 12}$$

In formulas (A 11) and (A 12), we chose the turning point for the angle θ as a constant: according to (A 11), θ varies over the range $s < \theta < \pi - s$, with $0 < s < \pi/2$.

For stable circular trajectories, the equalities $R(r) = 0$ and $dR(r)/dr = 0$ must be satisfied. Using expressions (A 11, A 12) for a fixed radius r in these equations yields the corresponding energy H and angular momentum L:

$$H \approx -\frac{GM}{2r} - a\sqrt{(GMr)^3}\frac{\sin s}{c\,r^4}, \quad L \approx \sin s\sqrt{G\,M\,R} - 3\,a\,G\,M\frac{\sin^2 s}{c\,r}. \tag{A 13}$$

Consider the precession of circular orbits. From the Hamilton equations $\dot{\theta} = \partial H/\partial p_\theta$ and $\dot{\phi} = \partial H/\partial L$, we have

$$\frac{d\theta}{d\,t} = \frac{L}{r^2}\sqrt{\cot^2 s - \cot^2 \theta}, \tag{A 14}$$

$$\frac{d\phi}{dt} = \frac{L}{r^2 \sin^2 \theta} + \frac{2aGM}{r^3 c}. \tag{A 15}$$

Integrating Eq. (A 14) yields

$$\phi - \phi_0 = \int \frac{Ldt}{r^2 \sin^2 \theta} + \frac{2\,a\,G\,M\,t}{r^3 c} = \int \frac{d\theta}{\sin^2 \theta\sqrt{\cot^2 s - \cot^2 \theta}} + \frac{2\,a\,G\,M\,t}{r^3\,c} =$$
$$\arcsin\left(\frac{\cot\theta}{\cot s}\right) + \frac{2\,a\,G\,M\,t}{r^3 c}. \tag{A 16}$$

When changing the variable in formula (A 16), we used Eq. (A 14).

Consider the case where all particle at $t = 0$ were in the $\sin\phi\sin\theta = \cos\theta\cot s$ plane or, introducing Cartesian coordinates, in the $y = z\cot s$ plane. According to equality (A 16), the particles will lie on the surface $y\cos(2\,a\,G\,M\,t/r^3 c) - x\sin(2\,a\,G\,M\,t/r^3\,c) = z\cot s$ at time t. The closer the particles to the rotating gravitating center, the faster their precession. For a given radius r, the particle motion in an accretion disk can be represented as the rotation of a circumference inclined to the z spin axis at angle s with angular velocity $2\,a\,G\,M/r^3\,c$. In a coordinate system corotating with the circumference, the particle rotates with a Keplerian velocity. Thus, if viscous friction between adjacent orbits is disregarded, a tilted accretion disk cannot be in a steady state.