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Advection Dominated Accretion Flows around a Rotating Compact Object

A. R. Prasanna[1,2] and Banibrata Mukhopadhyay[1]

[1] Theory Group, Physical Research Laboratory, Navarangpura, Ahmedabad-380009, India
e-mail: prasanna@prl.ernet.in; bm@prl.ernet.in

[2] Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 85748 Garching, Germany.

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Abstract. Using the generalised set of fluid equations that include the 'Coriolis force' along with the centrifugal and pressure gradient forces, we have reanalysed the class of self similar solutions, with the pseudo-Newtonian potential. We find that the class of solutions is well behaved for almost the entire parameter space except for a few selected combinations of γ and α for the co-rotating flow. The analysis of the Bernoulli number shows that whereas it remains positive for co-rotating flow for $f > 1/3$, for the counter-rotating flow it does admit both positive and negative values, indicating the possibility of energy transfer in either direction.

Key words. accretion flows – rotating compact object – perturbation

1. Introduction

The concept of reintroducing the language of inertial forces in general relativity, though not very popular has been found to be useful in explaining some already known results more satisfactorily (Abramowicz 1993, Abramowicz & Prasanna 1990, Miller 1992, Prasanna 1997) wherein particularly the fact that the "Centrifugal force" reverses sign very close to the compact object, could be a plausible reason. As was shown by Abramowicz et al. (1993) it is indeed possible to introduce the concept of inertial forces in a covariant formalism that is valid in general spacetimes, without any particular symmetry requirement. Of these forces, the "Coriolis" force is generally assumed to be of no physical relevance in most of the Newtonian physics. However, in general relativity

Send offprint requests to: A. R. Prasanna the term which gives rise to the Coriolis type field is the well known "Lense-Thirring" effect of dragging of inertial frames (Abramowicz et al. 1988), which couples the angular momentum of the gravitating source with the particle dynamics, through the spacetime curvature. It is indeed interesting to note that a particle in circular orbit around a Kerr black hole at a radius wherein the centrifugal force is zero, has to change its angular velocity depending upon the Kerr parameter 'a' to be in equilibrium. In fact while a co-rotating particle has to decrease its angular velocity slightly with increase in 'a', the counter-rotating ones have to increase it substantially to keep in equilibrium (Prasanna 1997, 2001). Though this is purely a general relativistic effect, which happens very close to black hole, one wonders whether in principle there could be effects of the rotation of the central gravitating source on the particles or fluids at a distance, even in a purely Newtonian approach. In fact it is quite well known that while discussing disks in binary systems and planetary rings one does consider the equations of motion in a rotating co-ordinate system, that includes the Coriolis and centrifugal terms (Pringle 1985, Fridman & Gorkavyi 1999). Even without going to the Abramowicz approach of 3+1 splitting (1988), it may be seen that if one starts with a general set of fluid equations outside a rotating body as given by a stationary axisymmetric spacetime, and takes the Newtonian limit of these equations one finds the terms are very similar to the ones found by writing the Newtonian equation in a rotating coordinate system. An example has been shown by considering the equation in the external field of Hartle-Thorne metric (1968), but restricting the components only up to monopole corrections, in Appendix A.

With this background we now look for possible influence of Coriolis type terms on the fluid flow in accretion disks around compact rotating objects, particularly for the type wherein the fluid disk is in contact with the central body. In fact the topic of accretion disks around compact objects is one of the most actively pursued topics in building models for a wide variety of cosmic objects ranging from cataclysmic variables to active galactic nuclei and Quasars. Of the various scenarios of accretion dynamics, a popular one has been the Advection Dominated Accretion Flow (ADAF) a recent review of which may be found in Narayan et al. (1998). In most of the discussions concerning accretion disks, one considers the set of self similar solutions for the fluid flow (Narayan & Yi 1994). Recently Bhatt & Prasanna (2000) have analysed the self similar solutions of advection dominated flows introducing the effect of the pseudo Newtonian potential (Paczyński & Wiita 1980) perturbatively and have compared them with the exact global numerical solutions, that exist in the literature, demonstrating that the perturbative analysis can give better representation and insight into the parameter space of the solution. As was shown, though self-similarity could be an excellent approximation over a large range of viscosity parameters, for certain combinations of viscosity and gas parameters the perturbation develops a singularity. The present aim of this discussion is to find out whether the introduction of the rotational effects through 'Coriolis' type terms in the equation of motion would alter the situation regarding the singularities in the self similar solution. In fact as will be shown, the introduction of rotation and the coupling of the angular velocity of the central body with that of the fluid flow outside would make a clear distinction between the co-rotating and the counter-rotating flow.

2. Formalism

As is usual in accretion disk theory we consider the height integrated set of equations describing the steady state, axisymmetric fluid distribution that allows one to describe all physical quantities as functions of the cylindrical radius R only.

Mass conservation equation yields

$$-4\pi RH\rho V = \dot{M} = \text{constant}\,, \tag{1}$$

wherein $H = \sqrt{5/2}(c_s/\Omega_K)$, is the vertical half thickness with c_s and Ω_K being the isothermal sound speed and the Keplerian angular velocity respectively.

The equation of motion for the fluid in the gravitational field of a slowly rotating compact object is given by

$$\frac{d\vec{V}}{dt} + 2\vec{\varpi}\times\vec{V} + \vec{\varpi}\times\vec{\varpi}\times\vec{R} = -\frac{\nabla p}{\rho} + \vec{F}_g + \nu\nabla^2\vec{V} \tag{2}$$

wherein, $\vec{\varpi}$ is the intrinsic angular velocity of the central body, $\vec{V}$ the 3-velocity of the fluid element, $\vec{F}_g$ the gravitational acceleration and ν is the coefficient of kinematic viscosity $= \alpha c_s^2/\Omega_{kn}$. As we are using the height integrated equations we get from (2) the radial and angular momentum conservation equations

$$V\frac{dV}{dR} - R\Omega^2 - 2\omega\Omega R - \omega^2 R = -R\Omega_{kn}^2 - \frac{1}{\rho}\frac{dp}{dR} \tag{3}$$

and

$$V\frac{d}{dR}\left(\Omega R^2\right) = \frac{1}{\rho RH}\frac{d}{dR}\left(\nu\rho R^3 H\frac{d\Omega}{dR}\right) - 2\omega RV \tag{4}$$

with V and Ω being the radial velocity and angular frequency of the fluid element.

We shall use the quasi-Newtonian potential

$$\phi = GM/(R-R_g), \quad R_g = 2GM/c^2 \tag{5}$$

in terms of which the Keplerian angular velocity is

$$\Omega_{kn}^2 = GM/R(R-R_g)^2. \tag{6}$$

As mentioned earlier we shall now couple ω and Ω through a simple relation $\omega = a\Omega$, a being a constant. Using the equation of state $p = \rho c_s^2$ we can rewrite (3) and (4) as

$$V\frac{dV}{dR} - nR\Omega^2 + \Omega_{kn}^2 R + 2c_s\frac{dc_s}{dR} + \frac{c_s^2}{\rho}\frac{d\rho}{dR} = 0 \tag{7}$$

with $n = (1+a)^2$ and

$$\frac{d\Omega}{dR} = \frac{V\Omega_{kn}}{\alpha c_s^2 R^2}\left[\Omega R^2 + a\int 2\Omega R dR - j\right] \tag{8}$$

where the constant j has the usual interpretation of being the angular momentum per unit accreted mass. However for self similar solutions one assumes j negligible in comparison with the other terms.

Apart from these hydrodynamical equations one also considers the thermodynamical balance between the local viscous heating and radiative cooling that gives rise to advection and is obtained through the energy equation (Narayan & Yi 1994)

$$\frac{\rho V}{\gamma-1}\frac{d}{dR}c_s^2 - Vc_s^2\frac{d\rho}{dR} = \frac{f\alpha\rho c_s^2 R^2}{\Omega_{kn}}\left(\frac{d\Omega}{dR}\right)^2. \tag{9}$$

3. Equilibrium Solutions

The original set of self similar solution were obtained by setting $\frac{R_g}{R}=0$ in the potential, thus taking $\Omega_{kn}=\Omega_K=(GM/R^3)^{1/2}$ and assuming the radial dependence of the other parameters to be

$V = V_0\left(\frac{R_g}{R}\right)^{1/2}$, $c_s = c_{s0}\left(\frac{R_g}{R}\right)^{1/2}$,

$\Omega = \Omega_0\left(\frac{R_g}{R}\right)^{3/2}$, $\rho = \rho_0\left(\frac{R_g}{R}\right)^{3/2}$.

Assuming the same dependences and substituting these in Eqns. (1), (7), (8) & (9), one can get the set of algebraic equations

$$\begin{aligned}
V_0^2 - 1 + 2n\Omega_0^2R_g^2 + 5c_{s0}^2 &= 0\\
\frac{3}{2}c_{s0}^2 &= -V_0(1+4a)/\sqrt{2}\alpha\\
(3\gamma-5)V_0 &= \frac{9}{\sqrt{2}}(\gamma-1)f\alpha\Omega_0^2R_g^2\\
\rho_0 &= -\dot{M}/(4\pi\sqrt{5}c_{s0}V_0R_g^2).
\end{aligned} \tag{10}$$

Solving these, one can get

$$\begin{aligned}
V_0 &= -Ag/(3\sqrt{2}\alpha),\\
c_{s0}^2 &= Ag(1+4a)/9\alpha^2\\
\Omega_0R_g &= (\epsilon' Ag)^{1/2}/3\alpha,\\
\rho_0 &= \frac{63}{88}\alpha^2\left[\frac{5}{2}(Ag)^3(1+4a)\right]^{-1/2}.
\end{aligned} \tag{11}$$

with

$$\begin{aligned}
\epsilon' &= \frac{(\frac{5}{3}-\gamma)}{(\gamma-1)f}\\
A &= 2n\epsilon' + 5(1+4a)\\
g &= -1 + \sqrt{1+\frac{18\alpha^2}{A^2}}.
\end{aligned} \tag{12}$$

Figs. 1 and 2 show the behaviour of the various physical parameters for the cases $a \geq 0$ and $a \leq 0$ respectively. The behaviours are shown as a function of R for fixed values of α and γ. The general trend of the solutions are similar for all values of a (to be expected) except for the fact that with $a \neq 0$ the relative values tend to decrease with higher a

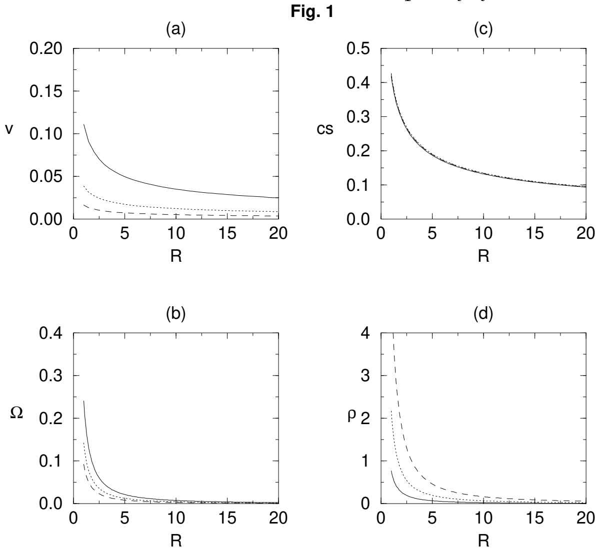

Fig. 1. Plots of radial velocity V, angular velocity Ω, sound speed c_s and density ρ as a function of R for different values of the rotation parameter 'a' for fixed $\alpha = 0.3$ and $\gamma = 1.5$; $a = 0$ (———), $a = 0.5$ (........) and $a = 1.5$ (– – – – –).

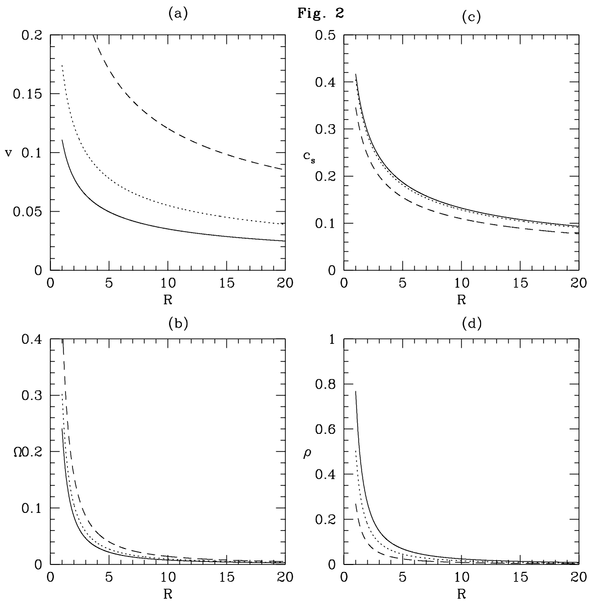

Fig. 2. Plots similar as Fig. 1 but for $a = 0$ (———), $a = -0.1$ (........) and $a = -0.2$ (– – – – –).

values for the fluid velocities (Fig. 1ab) in the co-rotating case, but increase for the counter-rotating flow (Fig. 2ab). Whereas the sound speed shows no variation for the co-rotating case (Fig. 1c) but very small change in the counter-rotating case (Fig. 2c), the density shows an increase with the values of a for the co-rotating flow (Fig. 1d) and decrease in the counter-rotating cases (Fig. 2d), the change being pronounced only close to the compact object.

4. Perturbation

Following the approach of Bhatt & Prasanna (2000) we now study the effects of non-Newtonian potential on the self similar solutions considered above, through perturbations

$p \to p_0 + \delta p$, $V \to V_0 + \delta V$, $\Omega \to \Omega_0 + \delta\Omega$, $c_s \to c_{s0} + \delta c_s$

in the equation of motion and mass and energy conservation equations. We then simplify by retaining the terms linear in the perturbations, after using the background solution (11) and obtain the equations governing the perturbations. Perturbation in the Keplerian velocity to the first order in $\frac{R_g}{R}$ is given by

$\delta\Omega_{kn}^2 \approx 2\frac{R_g}{R}\Omega_K^2$, $\delta\Omega_{kn} = \frac{R_g}{R}\Omega_K$

where Ω_K is taken as $\left(GM/R^3\right)^{1/2}$.

Thus the set of equations governing the perturbations δV, $\delta\Omega$, δc_s and $\delta\rho$ are: [1]

$$\frac{\delta H}{H} + \frac{\delta\rho}{\rho} + \frac{\delta V}{V} = 0; \quad \frac{\delta H}{H} = \frac{\delta c_s}{c_s} - \frac{R_g}{R},$$

$$V\frac{d}{dR}\delta V + \delta V\frac{dV}{dR} = 2nR\Omega\delta\Omega - R\delta\Omega_{kn}^2 + \frac{\delta\rho}{\rho^2}\frac{dp}{dR} - \frac{1}{\rho}\frac{d}{dR}\delta p,$$

$$\delta p = c_s^2\delta\rho + 2\rho c_s\delta c_s, \tag{13}$$

$$\alpha c_s^2\frac{d}{dR}\delta\Omega + 2\alpha c_s\frac{d\Omega}{dR}\delta c_s = \left(\Omega + \frac{2a}{R^2}\int \Omega R dR\right)(\Omega_{kn}\delta V + V\delta\Omega_{kn}) + V\Omega_{kn}\left(\delta\Omega + \frac{2a}{R^2}\int \delta\Omega R dR\right),$$

$$\frac{\rho V}{\gamma-1}\frac{dc_s^2}{dR}\left[\frac{\delta\rho}{\rho} + \frac{\delta V}{V} + 2\left(\frac{dc_s^2}{dR}\right)^{-1}\frac{d}{dR}(c_s\delta c_s)\right] - c_s^2 V\frac{d\rho}{dR}\left[2\frac{\delta c_s}{c_s} + \frac{\delta V}{V} + \left(\frac{d\rho}{dR}\right)^{-1}\frac{d}{dR}\delta\rho\right]$$

$$= \frac{f\alpha R^2 c_s^2\rho}{\Omega_k}\left(\frac{d\Omega}{dR}\right)^2\left[\frac{\delta\rho}{\rho} + \frac{2\delta c_s}{c_s} - \frac{R_g}{R} + 2\left(\frac{d\Omega}{dR}\right)^{-1}\frac{d}{dR}\delta\Omega\right].$$

Here again as in the equilibrium case, for self similar solutions, the assumption $j << \delta(\Omega R^2 + 2a\int\Omega R dR)$ is maintained. Thus again using self similarity of the perturbed solutions through

$$\delta V = V_{01}\left(\frac{R_g}{R}\right)^{3/2}, \ \delta c_s = c_{s01}\left(\frac{R_g}{R}\right)^{3/2}, \ \delta\Omega = \Omega_{01}\left(\frac{R_g}{R}\right)^{5/2}, \ \delta\rho = \rho_{01}\left(\frac{R_g}{R}\right)^{5/2} \tag{14}$$

and substituting these in (13) one gets the set of algebraic equations:

$$\frac{\rho_{01}}{\rho_0} + \frac{V_{01}}{V_0} + \frac{c_{s01}}{c_{s0}} = 1, \tag{15}$$

$$\frac{Ag}{9\alpha^2}\left(Ag - (1+4a)\right)\frac{V_{01}}{V_0} + \frac{2n\epsilon' Ag}{9\alpha^2}\frac{\Omega_{01}}{\Omega_0} + \frac{6Ag(1+4a)}{9\alpha^2}\frac{c_{s01}}{c_{s0}} - 1 + \frac{Ag(1+4a)}{9\alpha^2} = 0, \tag{16}$$

$$-\frac{3}{2}\frac{V_{01}}{V_0} + \left(\frac{5}{2} - \frac{3}{2}\frac{(1-4a)}{(1+4a)}\right)\frac{\Omega_{01}}{\Omega_0} + 3\frac{c_{s01}}{c_{s0}} = \frac{3}{2}, \tag{17}$$

$$(\gamma-3)\frac{V_{01}}{V_0} + 10\left(\frac{5}{3} - \gamma\right)\frac{\Omega_{01}}{\Omega_0} - 2(\gamma+1)\frac{c_{s01}}{c_{s0}} = 7 - 5\gamma. \tag{18}$$

[1] There are a couple of minor typographical errors in Bhatt & Prasanna 2000, which can be corrected by taking these equation with a=0

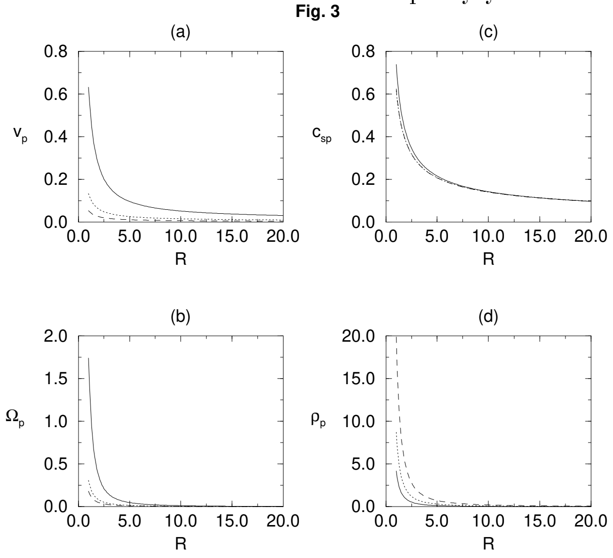

Fig. 3. Same as Fig. 1 but with the addition of perturbative solutions arising from quasi-Newtonian potential; $a = 0$ (———), $a = 0.5$ (........) and $a = 1.5$ (– – – – –).

Since it is a consistent set of inhomogeneous equations, one can solve for the perturbations and thus get the set of solutions incorporating the quasi Newtonian potential, to be

$$\begin{aligned} \rho_p &= \rho_0 \left(\frac{R_g}{R}\right)^{3/2} \left[1 + \frac{\rho_{01}}{\rho_0}\frac{R_g}{R}\right], \\ V_p &= V_0 \left(\frac{R_g}{R}\right)^{1/2} \left[1 + \frac{V_{01}}{V_0}\frac{R_g}{R}\right], \\ c_{sp} &= c_{s0} \left(\frac{R_g}{R}\right)^{1/2} \left[1 + \frac{c_{s01}}{c_{s0}}\frac{R_g}{R}\right], \\ \Omega_p &= \Omega_0 \left(\frac{R_g}{R}\right)^{3/2} \left[1 + \frac{\Omega_{01}}{\Omega_0}\frac{R_g}{R}\right]. \end{aligned} \tag{19}$$

Figs. (3)-(5) present these solutions for the physical quantities as a function of R with two of the three parameters a, γ, α fixed, while the other is varied. While the changes as a function of a remains almost the same as in the non-perturbed case (Fig. 3), there appear some changes as a function of the viscosity (α) and gas parameter (γ). As may be noted in Fig. 4a the radial velocity shows a jump in the profile as γ changes from 1.1 to 1.6, while the other three do not show such a jump (Fig. 4b,c,d). On the other hand as a depicted in Fig. 5, their behaviour as a function of α for fixed $\gamma = 1.5$ and $a = 0.5$ is quite interesting. While V_p increases with α (Fig. 5a) and ρ_p decreases (Fig. 5d), Ω_p and c_p show no change at all with varying α (Fig. 5b,c). In order to understand these features, we have presented the plots of the ratios of perturbed to the unperturbed quantities (V_{01}/V_0, Ω_{01}/Ω_0, c_{s01}/c_{s0}, ρ_{01}/ρ_0) in Figs. (6) and (7) one as a function of γ and the other as a function of α for the cases $a = 0, 0.5, 1.5$. As is evident from Fig. 6 the singularity in the perturbations has shifted to lower values of γ as a changes from 0 to 1.5. For $a = 0$, the location of the singularity in the γ space is at $\gamma = 1.55$, a result which was obtained earlier by Bhatt & Prasanna (2000). Figure 7 shows the perturbation in

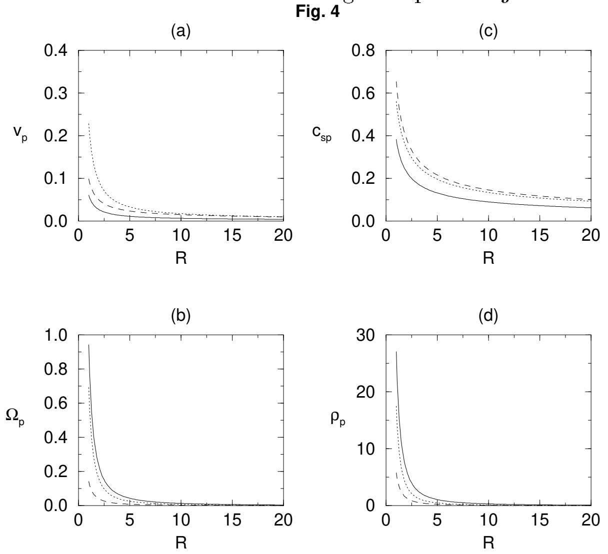

Fig. 4. Same as Fig. 3 for fixed $\alpha = 0.3$ and $a = 0.5$, but with different γs; $\gamma = 1.1$ (———), $\gamma = 1.4$ (........) and $\gamma = 1.6$ (– – – – –).

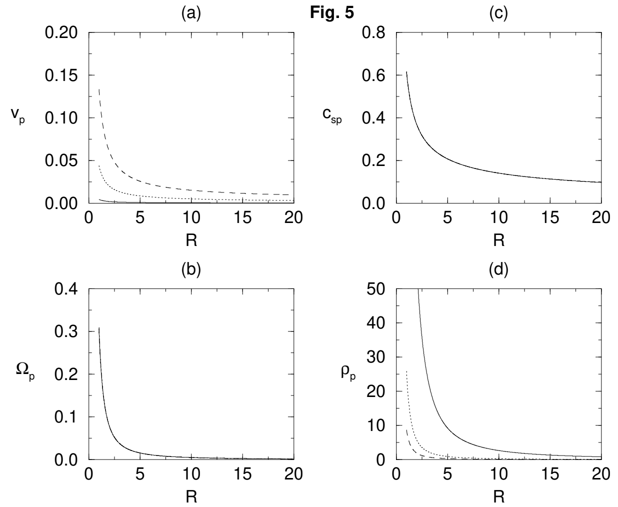

Fig. 5. Same as Fig. 3 for fixed $\gamma = 1.5$ and $a = 0.5$, but varying α; $\alpha = 0.01$ (———), $\alpha = 0.1$ (........) and $\alpha = 0.3$ (– – – – –).

the α-space, whereas for $a = 0$ the perturbations grows as the flow gets nearer to the compact object, (a result known earlier), with the inclusion of Coriolis terms $a \neq 0$, the perturbations are well confined and further the ratio decreases as the flow approaches the central gravitating source. Fig. 8 depicts the plots of the ratios of the perturbed to unperturbed physical quantities for $a = 0, -0.1, -0.2$ as a function of γ for fixed $\alpha = 0.3$. It can be clearly seen that there appears no singular behaviour for $a \neq 0$ unlike the case $a = 0$.

Fig. 6

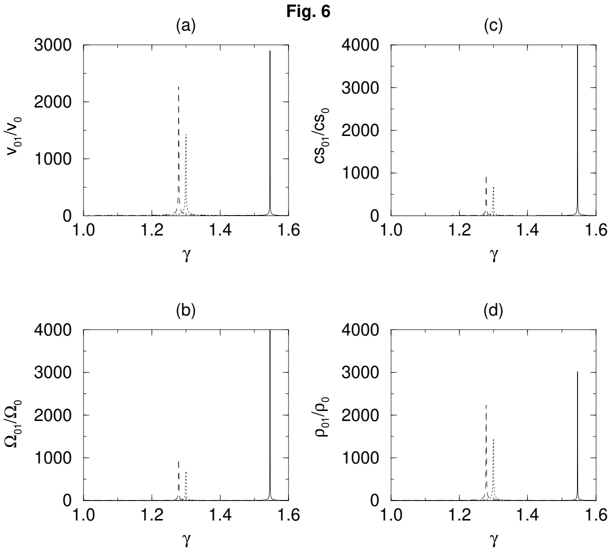

Fig. 6. Plots of the ratios of perturbed to the unperturbed parameters as a function γ, for fixed $\alpha = 0.3$ and different values of a; $a = 0$ (———), $a = 0.5$ (........) and $a = 1.5$ (– – – – –).

Fig. 7

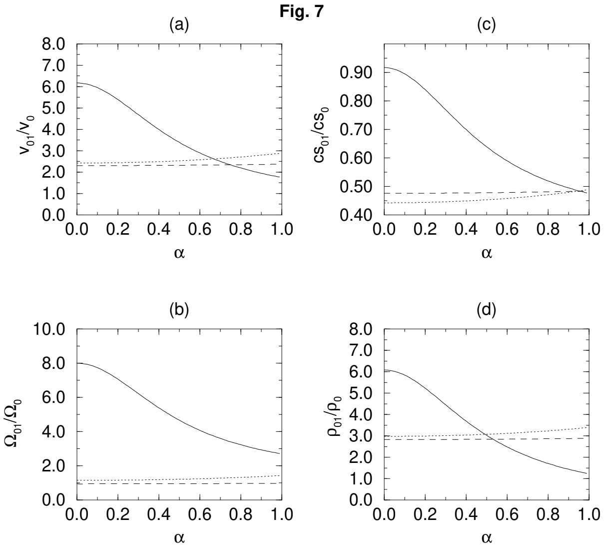

Fig. 7. Plots same as in Fig. 5 for fixed $\gamma = 1.5$ as a function of α and different values of a; $a = 0$ (———), $a = 0.5$ (........) and $a = 1.5$ (– – – – –).

5. Discussion

The introduction of the Coriolis force and a direct coupling of the two angular velocities seem to bring in some features which could be helpful for understanding the parameter space available for model building. At the outset one can see that the reality of the solutions, immediately requires $(1 + 4a) > 0$, meaning that the counter-rotating flows have to have a minimum speed, at least greater than four times the angular velocity of the central source, to be in equilibrium. This may not be very surprising, as the influence of the central rotation of 'frame dragging' would imply that any counter-rotating body has to overcome this barrier. Though in principle 'frame dragging' is purely a general relativistic effect, it appears as though the Coriolis term in Newtonian physics mimics a

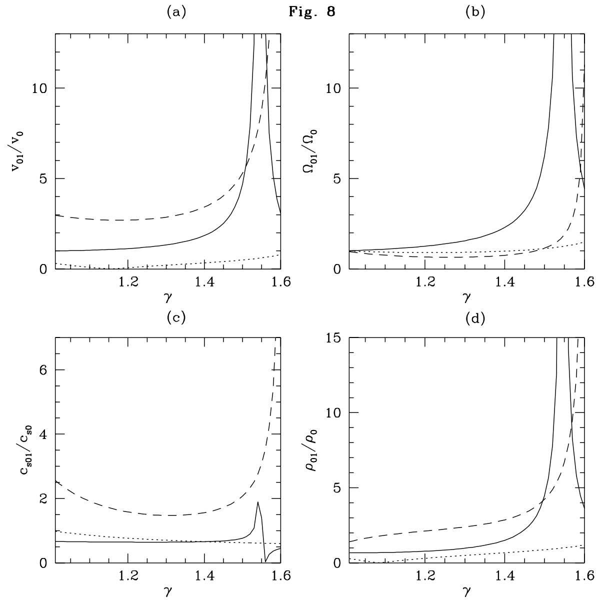

Fig. 8. Plots similar as Fig. 6 but for a negative; $a = 0$ (———), $a = -0.1$ (........) and $a = -0.2$ (– – – – –).

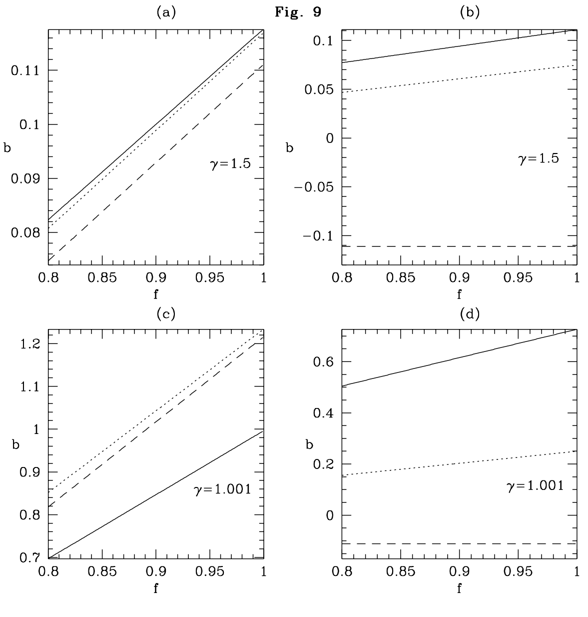

Fig. 9. Bernoulli parameter b as a function of f for: (a),(c) $a = 0$ (———), $a = 0.25$ (........) and $a = 0.5$ (– – – – –); (b),(d) $a = -0.1$ (———), $a = -0.2$ (........) and $a = -0.25$ (– – – – –).

similar effect. An important outcome of the analysis is that the rotational effect seems to stabilise the flows for the usually considered gas and viscosity parameters γ and α as depicted by the perturbations of the self similar solution.

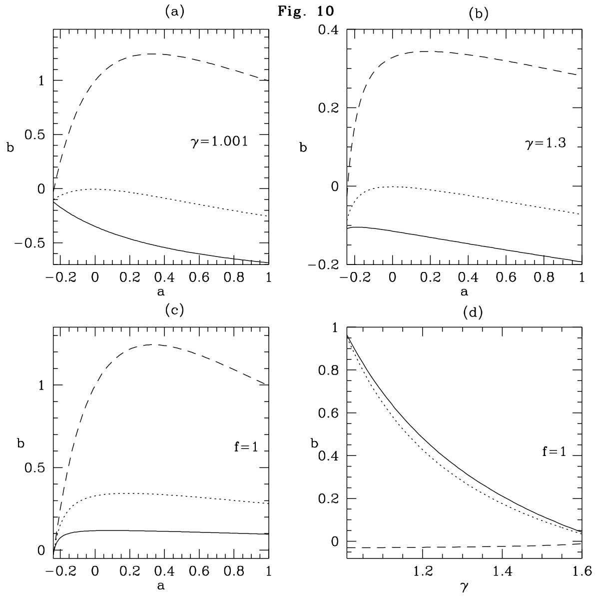

Fig. 10. Plots of b: (a),(b) as a function of a; $f = 0.1$ (———), $f = 0.33$ (........) and $f = 1$ (– – – – –); (c) as a function of a; $\gamma = 1.5$ (———), $\gamma = 1.3$ (........) and $\gamma = 1.001$ (– – – – –); (d) as a function of γ; $a = 0$ (———), $a = 1$ (........) and $a = -0.24$ (– – – – –).

Equation (7) shows that the solution satisfies the relation

$$\frac{V^2}{2} + n\Omega^2 R^2 - \Omega_K^2 R^2 + \frac{5}{2} c_s^2 = 0. \tag{20}$$

Using this in the computation of the normalised Bernoulli parameter 'b' as defined by Narayan & Yi (1994)

$$b = \frac{1}{V_K^2}\left(\frac{V^2}{2} + \frac{\Omega^2 R^2}{2} - \Omega_K^2 R^2 + \frac{\gamma}{\gamma - 1} c_s^2\right) \tag{21}$$

one finds after using Eqns. (11) and (12)

$$b = \frac{Ag\epsilon'}{19\alpha^2}\left[1 - 2n + 3f\,(1 + 4a)\right] \tag{22}$$

$$b \approx \left[\frac{(1 - 2n + 3f\,(1 + 4a))}{\left(2n + \frac{5f}{\epsilon}\,(1 + 4a)\right)}\right]. \tag{23}$$

Figs. (9) and (10) show the behaviour of 'b' for different ranges of parameters α, γ, f. For $1 < \gamma < \frac{5}{3}$, b is positive for $f \geq 1/3$, for $a > -0.1$ and as a approaches -0.2, b tends to become negative even for values of $f > 1/3$. As a tends to the value -0.25, b is negative for all values of f and γ. Thus while the co-rotating fluid can have energy transfer only outwards for $f > 1/3$, the counter-rotating fluid can have it in either direction depending upon its angular velocity as compared to that of the central accreting source. For advection dominated flows ($f = 1$) b is mostly positive for $a > -0.2$. However b changes sign for co-rotating flows at $a = 2 + \sqrt{5}$ and for counter-rotating flows at

$a = 2 - \sqrt{5}$. Hence in principle if the energy transfer inwards has to be effective, the co-rotating flow has to have very low angular velocity $\Omega < \omega/(2+\sqrt{5})$, whereas the counter-rotating flow has to have very large angular velocity $\Omega > \omega/(2-\sqrt{5})$.

In conclusion it may be seen that introducing rotational effects into a Newtonian description of the accretion flow, through the Coriolis term, has indeed enlarged the parameter space of the self similar solutions, making them viable for model building scenarios, particularly treating both co and counter rotating flows. A simple coupling of the angular velocities has distinctly shown the possibility of energy transfer both inwards and outwards depending on the effective angular velocity of the fluid flow. More detailed studies, like looking for a global solution and inclusion of other effects like convection, may perhaps reveal interesting features which could influence the 'energy budget' of accretion dynamics.

Appendix A:

The equations of motion for a perfect fluid disk in a given background spacetime are obtained through the conservation laws

$$(\rho + p)\left(u^i_{;j} u^j\right) + p\left(u^i u^j + g^{ij}\right) = 0. \tag{A.1}$$

Introducing the 3 velocity $u^\alpha = u^t v^\alpha$ and simplifying one gets for stationary axisymmetric spacetimes

$$(\rho+p)(u^t)^2\left[v^\beta \frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{00} + 2v^\mu\left(\Gamma^\alpha_{0\mu} - v^\alpha \Gamma^0_{0\mu}\right) + \left(\Gamma^\alpha_{\mu\nu} - \Gamma^0_{\mu\nu} v^\alpha\right) v^\mu v^\nu\right] = \left(v^\alpha g^{0\beta} - g^{\alpha\beta}\right) p_\beta . \tag{A.2}$$

In the Hartle-Thorne (1968) exterior metric of a slowly rotating star (taken up to monopole corrections only) in the equatorial plane one has

$$g_{tt} = -\left(1 - \frac{2M}{r} - \frac{2J^2}{r^4}\right), \quad g_{rr} = \left(1 - \frac{2M}{r} + \frac{2J^2}{r^4}\right)^{-1}, \quad g_{\phi\phi} = r^2, \quad g_{\phi t} = -2J/r. \tag{A.3}$$

Using these in A.2, one can get the equations in the equatorial plane with $\theta = \pi/2$,

$$(\rho+p)(u^t)^2\{v^r \frac{\partial v^r}{\partial r} + \left(1 - \frac{2M}{r} + \frac{2J^2}{r^4}\right)\left(\frac{M}{r^2} - r(v^\phi)^2 - \frac{2Jv^\phi}{r^2} + \frac{4J^2}{r^5}\right) + \left(1 - \frac{2M}{r} + \frac{2J^2}{r^4}\right)^{-1} (v^r)^2\left(\frac{3M}{r^2} - \frac{2Jv^\phi}{r^2}\right)\} = -\left(1 - \frac{2M}{r} + \frac{2J^2}{r^4}\right)\frac{\partial p}{\partial r} \tag{A.4}$$

and

$$v^r \frac{\partial v^\phi}{\partial r} + \frac{2Jv^r}{r^4} + \frac{2v^r v^\phi}{r}\left(1 - \frac{3M}{r} - \frac{Jv^\phi}{r} - \frac{2J^2}{r^4}\right)\left(1 - \frac{2M}{r} + \frac{2J^2}{r^4}\right)^{-1} = 0. \tag{A.5}$$

To the lowest order (A.4) and (A.5) give

$$V\frac{dV}{dR}+\frac{GM}{R^2}-R\Omega^2-\omega\Omega R+\omega^2 R=-\frac{1}{\rho}\frac{dp}{dR} \tag{A.6}$$

and

$$V\frac{d}{dR}(\Omega R^2)+\omega RV=0 \tag{A.7}$$

wherein, we have used, $v^r=V$, $v^\phi=\Omega$, $r=R$, $\frac{2J}{r^2}=\omega$ and $p/c^2<<\rho$. The difference that occurs between (A.6) and (A.7) and equations written in the rotating coordinates (without viscosity) is perhaps due to the fact that the metric (A.3) used above is not an exact solution of the field equations.

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