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Energy Release on the Surface of a Rapidly Rotating Neutron Star during Disk Accretion: A Thermodynamic Approach.

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...Abstract. The total energy E of a star as a function of its angular momentum J and mass M in the Newtonian theory: $E = E(J, M)$ [in general relativity, the gravitational mass M of a star as a function of its angular momentum J and rest mass m, $M = M(J, m)$], is used to determine the remaining parameters (angular velocity, equatorial radius, chemical potential, etc.) in the case of rigid rotation. Expressions are derived for the energy release during accretion onto a cool (with constant entropy), rapidly rotating neutron star (NS) in the Newtonian theory and in general relativity. A separate analysis is performed for the cases where the NS equatorial radius is larger and smaller than the radius of the marginally stable orbit in the disk plane. An approximate formula is proposed for the NS equatorial radius for an arbitrary equation of state, which matches the exact one at $J = 0$.

Keywords: neutron stars, luminosity, disk accretion, X-ray bursters

Introduction

The change in mass and angular momentum of a cool neutron star (NS) during accretion leads to a transformation of its equilibrium state. Below, we consider NSs with weak magnetic fields that do not affect the accretion dynamics; i.e., our results refer to the accretion pattern in low-mass X-ray binaries and, in particular, in X-ray bursters. Of crucial importance is the question of what part of the energy released during accretion dissipates near and outside the stellar surface (in the accretion disk, in the boundary layer, in the spread layer, in the settling zone as matter is compressed under the weight of the newly supplied material, when uniform rotation is established by viscous forces throughout the entire extended stellar atmosphere, in the zone with a surface density of the order of $10^9 g/cm^2$ subject to nuclear burning during X-ray flares) and what part dissipates inside the star in its interior. Observationally, these two zones of energy release differ radically. The energy released near the surface leaves the surface layers (is emitted) in a very short time: from fractions of a millisecond to several tens of seconds, although the complete energy dissipation can last for several hours. By contrast, the energy release in the stellar interior produces radiation with characteristic times exceeding hundreds of thousands and, possibly, millions of years; this is the time it takes for the interiors of neutron stars with weak magnetic fields to cool down (Levenfish et al. 1999). Such radiation can be detected only in highly variable transients when accretion on them virtually ceases.

Here, we disregard energy release in the stellar interior by assuming that the star is cool and that its entropy does not change during accretion and transformation of the NS internal structure. This strong assumption allowed us to obtain a number of general results, which we used previously (Sibgatullin and Sunyaev 2000). Note that this assumption does not hold in the so-called thermal neutron stars, where the spinup via accretion leads to a difference between the angular velocities of the crust and the central liquid superfluid core inside the star. As a result, energy is released inside the star through viscous friction [see Alpar (1999) for a discussion].

A thermodynamic relation between the change in NS total energy (gravitational mass in general relativity) and the change in its angular momentum and mass (rest mass in general relativity) is considered for cool, rigidly rotating NSs with a given equation of state (EOS). This relation and the energy conservation law are used to derive a formula for the energy release on the stellar surface (in the Newtonian approximation and in general relativity) during disk accretion onto a NS rotating with an arbitrary angular velocity. The case where the NS equatorial radius is smaller than the radius of the marginally stable orbit is also considered in terms of general relativity.

For the equilibrium figure of a rotating, incompressible fluid in its own gravitational field (Maclaurin spheroid), we derive explicit formulas for the dependence of disk and surface energy release on spheroid eccentricity. The fraction of disk energy release in the total energy release is shown to be expressed by a simple linear dependence on the ratio of the Maclaurin-spheroid rotation frequency to the Keplerian equatorial particle velocity.

A universal approximate formula for the NS equatorial radius is derived for an arbitrary EOS.

Previously (Sibgatullin and Sunyaev 2000), we considered astrophysical implications of our results and methods for deriving simple approximation formulas. We also analyzed the universal geometric properties of space-time outside rotating bodies.

1 Newtonian treatment

In the Newtonian approximation, the total energy of a NS consists of the gravitational, kinetic, and internal energies:

$$\mathbf{E} = \int_V (-\frac{1}{2}\Phi(\overline{r}) + \frac{1}{2}\Omega^2(x^2+y^2) + u(\rho(\overline{r})))\rho(\overline{r})\, dV \tag{1}$$

where the gravitational potential Φ is

$$\Phi(\overline{r}) = G \int_V \frac{\rho(\overline{r\prime})}{\mid \overline{r} - \overline{r\prime} \mid}\, dV' \tag{2}$$

We take a model of an ideal gas with constant (zero) entropy for the internal energy of superdense matter. The Gibbs identity then yields

$$du = -pd(1/\rho) + Tds = -pd(1/\rho). \tag{3}$$

We have the following obvious expressions for the NS mass and angular momentum:

$$M = \int_V \rho(\overline{r})\, dV, \qquad J = \int_V \rho(\overline{r})(x^2+y^2)\Omega\, dV \tag{4}$$

The following integral holds in a steady equilibrium state:

$$u(\rho) + \frac{p}{\rho} - \Omega^2(x^2+y^2)/2 - \Phi = \mu = \text{const}, \tag{5}$$

since μ has the same value at any point of the star [1], because the equilibrium is isentropic, and because the rotation is rigid [the theorem of Crocco (1937); see also Oswatitsch (1976), Chernyi (1988)]. So, the dynamical equilibrium conditions can also be written as $\nabla\mu = 0$ [see the case of an incompressible fluid in Lamb (1947)].

Let us consider two close equilibrium states with global parameters $\mathbf{E}, M, J, \Omega$ and $\mathbf{E} + \delta\mathbf{E}, M + \delta M, J + \delta J, \Omega + \delta\Omega$.

Theorem 1. The variations of NS mass δM , angular momentum δJ, and total energy $\delta\mathbf{E}$ for two close equilibrium states are related by the thermodynamic relation

$$\delta\mathbf{E} = \Omega\delta J + \mu\delta M \tag{6}$$

Indeed, denote the local displacement of point $\overline{r}$ on the stellar surface along the normal when passing from one equilibrium state to the other by $W(\overline{r})$. For an arbitrary integral $A \equiv \int_V a(\overline{r})\, dV$, we then have

$$\delta A = \int_V \delta a(\overline{r})\, dV + \int_{\partial V} a(\overline{r})W(\overline{r})\, dS \tag{7}$$

In view of Eq. (2) for the gravitational potential, the variation of gravitational energy $\delta\mathbf{E}_{gr}$ is

$$\delta\mathbf{E}_{gr} = -\frac{1}{2}\int_V \delta\rho\Phi\, dV - \frac{1}{2}\int_V \rho\delta\Phi\, dV - \frac{1}{2}\int_{\partial V} \rho W\Phi\, dS.$$

[11] The constant μ should not be confused with the Bernoulli integral i_0, which is constant along a streamline: $\mu = i_0 - [r, \Omega]^2$!

Reversing the order of integration in the second term on the right-hand side, we reduce this expression to

$$\delta\mathbf{E}_{gr} = -\int_V \delta\rho\Phi\, dV - \int_{\partial V} \rho W \Phi\, dS \tag{8}$$

We calculate the variation of internal energy $\delta\mathbf{E}_{in}$ from formula (7) by using the Gibbs identity (3):

$$\delta\mathbf{E}_{in} = \delta \int_V u(\rho)\rho\, dV = \int_V (u(\rho) + \frac{p(\rho)}{\rho})\delta\rho\, dV + \int_{\partial V} \rho W (u(\rho) + \frac{p}{\rho})\, dS \tag{9}$$

In order to simplify the subsequent calculations, we added the surface term $\int_{\partial V} W p\, dS$ to the right-hand part of (9); this term is zero, because the pressure vanishes at the stellar boundary. Clearly, the variation of kinetic energy can be expressed as

$$\delta\mathbf{E}_{\rm c} = \frac{1}{2}\int_V (\delta\rho\Omega + 2\delta\Omega\rho)(x^2+y^2)\Omega\, dV + \frac{1}{2}\int_{\partial V} \Omega^2\rho(\overline{r})(x^2+y^2)W\, dS.$$

As a result, the variation of total energy $\delta\mathbf{E} = \delta\mathbf{E}_{gr} + \delta\mathbf{E}_{in} + \delta\mathbf{E}_c$ is

$$\begin{gathered}\delta\mathbf{E} = \int_V \left\{\delta\rho\left(-\Phi + \tfrac{1}{2}\Omega^2(x^2+y^2) + u(\rho) + p/\rho\right) + \delta\Omega\Omega(x^2+y^2)\right\} dV + \\ \int_{\partial V} \rho W \left(-\Phi + u(\rho) + p/\rho + \tfrac{1}{2}\Omega^2(x^2+y^2)\right)\, dS.\end{gathered}$$

δM, δJ are given, respectively, by

$$\delta M = \int_V \delta\rho(\overline{r})\, dV + \int_{\partial V} \rho(\overline{r})W(\overline{r})\, dS;$$

$$\delta J = \int_V (\Omega\delta\rho(\overline{r}) + \delta\Omega\rho(\overline{r}))(x^2+y^2)\, dV + \int_{\partial V} \Omega\rho(\overline{r})(x^2+y^2)W\, dS. \tag{10}$$

Regrouping the terms, we represent the expression for $\delta\mathbf{E}$ as

$$\begin{gathered}\delta\mathbf{E} = \int_V \left\{\delta\rho\left(-\Phi - \tfrac{1}{2}\Omega^2(x^2+y^2) + u(\rho) + \tfrac{p}{\rho}\right)\right\} dV + \\ \int_V \{\delta\rho\Omega + \delta\Omega\rho\}(x^2+y^2)\Omega\, dV + \int_{\partial V} \Omega\rho(\overline{r})(x^2+y^2)W\, dS + \\ \int_{\partial V} \rho W \left(-\Phi + u(\rho) + \tfrac{p}{\rho} - \tfrac{1}{2}\Omega^2(x^2+y^2)\right)\, dS.\end{gathered} \tag{11}$$

The coefficients in front of $\delta\rho$ in the first and last integrals of Eq. (11) match the constant μ given by (5). Let us now make use of Crocco's theorem and factor this constant outside the integral signs. The angular velocity Ω can also be factored outside the integral signs in the second and third integrals, because the rotation is rigid. Having done these operations and using formula (7) for δM and δJ, we obtain

$$\delta\mathbf{E} = \Omega\delta J + \mu\delta M, \tag{12}$$

hence,

$$\Omega = \frac{\partial\mathbf{E}}{\partial J}|_M, \quad \mu = \frac{\partial\mathbf{E}}{\partial M}|_J. \tag{12a}$$

We see that the constant μ has the physical meaning of chemical potential here (Landau and Lifshitz 1976).

Theorem 1 is conceptually associated with the following remarkable variational principle of general relativity by Hartle and Sharp (1967) (see also Bardeen 1970): the true mass and angular-momentum distributions differ from their virtual (possible) distributions with a fixed rest mass and a fixed total angular momentum in that they give a conditional extremum to the gravitational mass (energy in the Newtonian approximation). In this case, the angular velocity Ω and the constant μ act as the Lagrangian factors.

Corollary 1. *When a star loses its angular momentum and energy by the radiation of electromagnetic or gravitational waves, the rates of change in its total energy and angular momentum are related by* $\dot{\mathbf{E}} = \Omega \dot{J}$.

This equality follows from (12), because the change in NS baryonic mass is zero during wave emission. Corollary 1 was proved by Ostriker and Gunn (1969) when considering the fluxes of angular momentum and energy of electromagnetic or gravitational radiation in the wave zone. It has important applications for radio pulsars: the quasi-equilibrium evolution of the NS structure when it loses its angular momentum does not lead to any heating of the stellar matter.

Denote the fluxes of angular momentum and energy on the NS by $\dot{M}l$ and $\dot{M}e$, respectively, where l and e have the meaning of the specific angular momentum and specific energy brought by the accreting matter.

Theorem 2. *When the angular momentum* $\dot{M}l$ *and energy* $\dot{M}e$, *are transferred to a neutron star by accreting particles in unit time, the following energy is released in the star with constant entropy in unit time:*

$$L_s = \dot{M}(e - \Omega\, l - \mu), \tag{13}$$

where μ *is the chemical potential of the cool star.*

Indeed, according to theorem 1, we have

$$d\mathbf{E} = \Omega dJ + \mu dM.$$

On the other hand, it follows from the energy conservation law that

$$d\mathbf{E} = dMe - L_s dt,$$

where L_s is the rate of energy release (stellar luminosity) during accretion.

¿From the law of conservation of angular momentum, we have

$$dJ = \dot{M}ldt. \tag{14}$$

By equating the expressions for dE from theorem 1 and from the energy conservation law and using (14) for dJ, we reach the conclusion of theorem 2.

Note. For the constant μ, we may choose its value at the stellar equator: $\mu = -\frac{1}{2}\Omega^2 R^2 - \Phi_e$, where Φ_e is the gravitational potential at the stellar equator, and R is the equatorial radius. Here, we make use of the fact that the enthalpy vanishes on the stellar surface. This choice of μ allows us to determine the equatorial radius, the most important NS parameter [see formulas (32) and (33)].

Corollary 1. *When a thin accretion disk is adjacent to the stellar equator and when the star transforms into an equilibrium state with a new mass and angular momentum in time* dt, *the energy* $\frac{1}{2}dM(\Omega_K - \Omega)^2 R^2$ *is released on the stellar surface, where* dM *is the amount* *of baryonic mass accreted onto the NS in time dt; Ω_K and Ω are the Keplerian velocity at the NS equator and its angular velocity, respectively.*

Indeed, in this case, $e = v^2/2 - \Phi_e$; $\quad v = R\Omega_K, l = R^2\Omega_K$, and expression (13) for L_s turns into a full square. In particular, it follows from corollary 1 that the energy release for counterrotation at $\Omega = -0.5\Omega_K$ is a factor of 9 larger than the energy release for corotation at $\Omega = 0.5\Omega_K$!

The local justification here is as follows. Falling on the stellar equator, a particle of mass m increases the NS moment of inertia. An additional work is done, which is equal to the difference between the particle angular momenta in the Keplerian orbit, $mR^2\Omega_K$, and on the stellar surface, $mR^2\Omega$, multiplied by the NS angular velocity Ω. This work is equal, with an opposite sign, to the additional (rotational) energy that must be added to the difference between the particle kinetic energies in the Keplerian orbit, $mR^2\Omega_K^2/2$, and on the stellar surface,$mR^2\Omega^2/2$. Therefore, when a particle approaches the NS from a thin accretion disk and decelerates in a narrow boundary layer at the equator from the Keplerian angular velocity Ω_K to the NS rotation velocity Ω, it releases the energy $1/2m(\Omega_K - \Omega)^2R^2$, giving up its angular momentum and part of its energy to the star. The formula for the energy release in this form was found by Kluzniak (1987) and independently by Shakura and Sunyaev (1987). The various local derivations of this formula were discussed by Kley (1991), Popham and Narayan (1995), and Sibgatullin and Sunyaev (1998).

It should be noted that, having accreted at the equator, the matter cannot remain there for long. It must spread in some way over the surface while changing its angular momentum and energy. Therein lies the inconsistency of the local approach, which disregards the subsequent redistribution of accreting matter over the star. The close match between the energy release given by (13) and the energy released when particles decelerate from the velocity $R\Omega_K$ to the velocity $R\Omega$ on the stellar surface is fairly unexpected.

Here, we make an attempt to solve the problem by taking into account the restructuring of a cool star during accretion. This global approach leads to formula (13). Thus, (13) holds not only for a narrow boundary layer near the equator (Popham and Sunyaev 2000). Our derivation of corollary 1 from theorem 1 suggests that the same energy is released by a particle in the spread layer (Inogamov and Sunyaev 1999), although the situation in the spread layer differs radically from the problem with a thin equatorial boundary layer. In the surface layer, the matter heats up, the thermal energy radiates away in bright latitudinal rings, and other complex physical processes accompanying the spread of matter over an equipotential of a rapidly rotating star take place. However, this all affects the instantaneous equilibrium of a neutron star only through global changes in its baryonic mass and angular momentum. It is pertinent to recall here that, according to Huygens's theorem (see, e.g., Tassoul 1978), the work done by gravitational and centrifugal forces on a particle to move it over the surface of a rigidly rotating liquid body in its own gravitational field is zero.

Corollary 2. Since the kinetic energy $\dot{M}v^2/2$ of the decelerated matter is released during spherical accretion onto a nonrotating cool star at rate $\dot{M}$ in its surface layer in unit time, there is no volume energy release in this case either.

This assertion follows from theorem 2, because, in this case, $l = \Omega = 0$, $\Phi = \Phi_e = GM/R$, and the total energy release given by (13) matches the surface one.

Disk Energy Release. Passing from one Keplerian orbit to another with a smaller radius, an accreting particle loses its energy through viscous friction, which radiates away. The total energy radiated away by one particle from an infinitely distant point before it approaches a Keplerian circular orbit on the stellar surface is equal to the particle energy in this orbit, with an opposite sign. Therefore, the following formula holds for the energy release of the entire disk:

$$L_d = -\dot{M}(R^2\Omega_K^2/2 - \Phi_e). \tag{15}$$

Setting the sum of centrifugal and gravitational forces equal to zero yields

$$R\Omega_K^2 = -\frac{d\Phi}{dr}|_{r=R}. \tag{16}$$

The formula for the disk energy release follows from (15) and (16):

$$L_d = \frac{1}{2R}\dot{M}\frac{d(r^2\Phi)}{dr}|_{r=R}. \tag{17}$$

If there is no rotation, then $L_d = L_s = \dot{M}GM/2R$ (Shakura and Sunyaev 1973).

In order to estimate the effect of NS rotation on the disk energy release, let us consider an example in which the stellar matter is modeled by an ideal, incompressible fluid. In this case, the axisymmetric equilibrium figure is a Maclaurin spheroid. The Dirichlet formula (see, e.g., Lamb 1947) can be used for the gravitational potential of a homogeneous ellipsoid, and the following expression can be derived for L_d from (17) (below, e denotes the spheroid eccentricity, $e \equiv \sqrt{1 - c^2/a^2}$, c, a to be the smaller and larger semiaxes):

$$L_d = \dot{M}\frac{3GM}{2a}\left(\frac{e^2-1}{e^3}\arcsin e + \frac{\sqrt{1-e^e}}{e^2}\right). \tag{18}$$

The Keplerian equatorial rotation frequency can be calculated from (16):

$$f_K = \sqrt{G\rho/2\pi}B(e); \quad B(e) \equiv \sqrt{\frac{\sqrt{1-e^2}}{e^3}\arcsin e - \frac{1-e^2}{e^2}}. \tag{19}$$

The formula for the rotation frequency is (Lamb 1947)

$$f = \sqrt{G\rho g(e)/2\pi}; \quad g(e) \equiv \frac{\sqrt{1-e^2}}{e^3}(3-2e^2)\arcsin e - 3\frac{1-e^2}{e^2}. \tag{20}$$

According to corollary 1 of theorem 2 and formulas (19) and (20), the energy release on the NS surface is

$$L_s = \frac{1}{2}\dot{M}(\Omega_K - \Omega)^2R^2 = \dot{M}\frac{3GM}{4R}\left(\sqrt{\frac{\arcsin e}{e^3} - \frac{\sqrt{1-e^2}}{e^2}} - \sqrt{\frac{3-2e^2}{e^3}\arcsin e - 3\frac{\sqrt{1-e^2}}{e^2}}\right)^2. \tag{21}$$

For the case of counterrotation one should change the sign before the second therm in round squares of expression (21).

Using (18 - 21), we can obtain the following unexpectedly simple relation to estimate the fraction of radiation from the disk in the NS total radiation:

$$L_d/(L_s + L_d) = 0.5(1 + f/f_K). \tag{22}$$

This formula is valid both for the positive values of f (the case of corotation) and for negative ones (the case of counterrotation).

Note that the eccentricity $e \to 1$ as $f \to f_K$ for an incompressible fluid. Therefore, the spheroid asymptotically transforms into a plane disk. However, large angular velocities cannot be reached: Maclaurin spheroids become unstable to quadrupole perturbations at $e > 0.8127$ (at $f > 0.4326\sqrt{G\rho/2\pi}$) (Lamb 1947; Chandrasekhar 1973).

According to (18) and (21), the total luminosity $L_s + L_d$ in the stability range of Maclaurin spheroids satisfies the inequalities

$$0.0485 < \frac{L_s + L_d}{\dot{M}c^2}\left(\frac{M}{1.4M_\odot}\right)^{-2/3}\left(\frac{\rho}{10^{14}\mathrm{g/cm}^3}\right)^{-1/3} < 0.11$$

in the case of corotation and the inequalities

$$0.11 < \frac{L_s + L_d}{\dot{M}c^2}\left(\frac{M}{1.4M_\odot}\right)^{-2/3}\left(\frac{\rho}{10^{14}\mathrm{g/cm}^3}\right)^{-1/3} < 0.1953$$

in the case of counterrotation. The speed of light (constant) was introduced here for convenience of comparing the results in the Newtonian approximation and in general relativity (Sibgatullin and Sunyaev 2000).

For the critical $e = 0.8127$, the ratio of angular velocities Ω/Ω_K is 0.602. It thus follows from formula (22) (in the stability range of Maclaurin spheroids) for spheroid and disk corotation that $0.199 < L_s/(L_s + L_d) < 0.5$. For a nonrotating star as was shown by Shakura and Sunyaev 1973 the ratio $L_s/(L_s + L_d)$ is equal 0.5. The inequality $0.5 < L_s/(L_s + L_d) < 0.801$ holds for disk and spheroid counterrotation.

2 Treatment in General Relativity.

The metric of stationary, axisymmetric spaces invariant to the change $t, \phi \Rightarrow -t,\ -\phi$ can be written in cylindrical $r,\ z,\ \phi$ coordinates as

$$ds^2 = a^2\, dt^2 - b^2\,(d\phi - \omega dt)^2 - e^{2\sigma}\,(dr^2 + dz^2), \tag{23}$$

where $a,\ b,\ \sigma$ are the sought-for functions of r and z. The gravitational-mass functional in the steady-state, axisymmetric case is (Hartle and Sharp 1967)

$$Mc^2 = c^4 \int_\infty \frac{R}{16\,\pi G}\sqrt{-g}\, d\phi dr\, dz + c^{-1}\int_V T_0^0\sqrt{-g} d\phi,\, dr\, dz, \tag{24}$$

where V and ∂V are the region occupied by the star and its boundary, respectively. The integral of the scalar curvature is extended to the entire three-dimensional space:

$$T_0^0 = (p+\epsilon)\frac{(a^2 + \omega\Omega b^2)}{a^2 - (\omega - \Omega)^2 b^2} - p; \quad \sqrt{-g} = a\, b\, e^{2\sigma}.$$

The NS nuclear overcompressed matter is an ideal isentropic gas with a given internal energy density, $\epsilon = \rho u(\rho)$, with $d\epsilon = (p+\epsilon)d\rho/\rho$. According to formula (24), the mass functional for a steady-state, axisymmetric star is determined by the functions $\rho,\ a,\ b,$ and σ, of independent variables r and z and by the constant Ω. Let us consider two close equilibrium states A and $A+\delta A$. Subtracting the mass functional in state A from the mass functional in state $A+\delta A$ yields

$$\delta M = 2\pi \int \left(\delta\rho\frac{\delta M}{\delta\rho} + \delta\sigma\frac{\delta M}{\delta\sigma} + \delta a\frac{\delta M}{\delta a} + \delta b\frac{\delta M}{y\delta b}\right) dr\, dz + \delta\Omega\frac{\partial M}{\partial\Omega} + c^{-3}\int_{\partial V} T_0^0 W\, \sqrt{-g}\, dl.$$

The functions of the form $\frac{\delta M}{\delta a}$ are variational derivatives of the functional M with respect to the corresponding function $a(r,z)$.

Since the metric coefficients and their first derivatives are continuous at the boundary of the region ∂V, the surface terms from variations of the gravitational mass reduce to the integral over a distant surface at pseudo-Euclidean infinity, where they vanish. The variational derivatives of M with respect to the metric coefficients are zero by virtue of the Einstein equations.

Using the equalities $u^0 u_0 + u^\phi u_\phi = 1$, and $u^\phi = \Omega u^0$, one could derive the following Hartle-Sharp formula at fixed metric coefficients:

$$\delta u^0 = -u_\phi (u^0)^2 \delta\Omega. \tag{25}$$

If we vary the gravitational-mass functional (24) at given metric coefficients, then we obtain

$$\delta M c^2 = 2\pi c^{-1} \delta \int_V T_0^0 \sqrt{-g}\, dr\, dz = 2\pi c^{-1} \delta \int_V (\epsilon - T_\phi^0 \Omega)\sqrt{-g} dr\, dz. \tag{26}$$

We now use the expressions for rest mass m and angular momentum J (Hartle and Sharp 1967)

$$m = 2\pi \int_V \rho u^0 \sqrt{-g}\, dr\, dz,$$

$$J = -2c^{-1}\pi \int_V T_\phi^0 \sqrt{-g} dr\, dz$$

and the relativistic integral of dynamical equilibrium conditions

$$\frac{p+\epsilon}{(\rho c u^0)} = \mu = \text{const}.$$

In view of (25), the following formula holds:

$$\delta\epsilon = \frac{p+\epsilon}{\rho}\delta\rho = \frac{p+\epsilon}{\rho u^0}(\delta(\rho u^0) - \rho \delta u^0) = \frac{p+\epsilon}{\rho u^0}\delta(\rho u^0) + T_\phi^0 \delta\Omega.$$

The quantity δM can then be easily represented after reducing similar terms in (26) as

$$\delta M\, c^2 = \Omega\, \delta J + \mu \delta\, m \tag{27}$$

hence

$$\Omega = \frac{\partial M c^2}{\partial J}|_m, \quad \mu = \frac{\partial M\, c^2}{\partial m}|_J \tag{28}$$

Theorem 3. *For any two close isentropic equilibrium states of a NS, the variations in gravitational mass, angular momentum, and rest mass are related by (28).*

Energy Release during Disk Accretion onto a NS in General Relativity. Let us now use theorem 3 to calculate the NS radiation and evolution from equilibrium states. The constant in the relativistic case is also convenient to estimate from its value at the stellar equator:

$$\mu = c/u^0 = c\sqrt{a^2 - (\omega - \Omega)^2 b^2}, \tag{29}$$

the metric coefficients (23) are taken at the stellar equator.

In general relativity, the NS equatorial radius can be larger and smaller than the radius of the marginally stable orbit, which is equal to three gravitational radii in the absence of rotation. In the former case, we denote the energy and angular momentum of a test particle of unit mass in a Keplerian orbit with the NS radius by e and l, respectively, take into account the law of conservation of angular momentum $dJ = ldm$, and equate the expressions for the change in gravitational mass from theorem 3 and from the law of conservation of energy:

$$(\Omega l + \mu)\, dm = dm\, e - L_s dt.$$

We use the equalities $e = c\,(u_K)_0, \quad l = c\,(u_K)_\phi, \quad \mu = c/u^0$, and $u^\phi = \Omega u^0$, and express L_s as

$$L_s = \dot{M}(e - \Omega l - \mu) = \mu(\mathbf{u} \cdot \mathbf{u}_K - 1)\dot{M}, \qquad \frac{dm}{dt} = \dot{M}. \tag{30}$$

Here, we introduced the designation $\cdot$ for a scalar product of the 4-velocity vector of a particle rotating with the NS angular velocity and the 4-velocity vector of a particle rotating in a Keplerian orbit at the same point on the stellar equator. Expression (30) for L_s is always positive and becomes zero if the NS angular velocity reaches the Keplerian one at the stellar equator. In our previous paper (Sibgatullin and Sunyaev 2000), we gave approximation formulas for the dependences of M, e, l, and f_K, on Kerr parameter and rest mass for particles in equatorial Keplerian orbits for a NS with EOS A and FPS. We used (28) to calculate the NS angular velocity and chemical potential. For stars with EOS A and FPS and with a gravitational mass of 1.4 $M_\odot$, we derived approximation formulas for the dependences of total luminosity $L_s + L_d$ and $L_s/(L_s + L_d)$ ratio on NS angular velocity by using (30).

The case where the NS radius is smaller than the radius of the marginally stable orbit is characteristic of general relativity alone. In this case, the gravitational field does an additional work on the spiraling-in particles in the gap; no stable Keplerian orbits exist within the marginally stable orbit. Thus, the rate of energy release is

$$L_s = \dot{M}(e_* - \Omega l_* - \mu) = \mu(\mathbf{u} \cdot \mathbf{u}_* - 1)\dot{M}, \tag{31}$$

where $\mathbf{u}_*$ denotes the 4-velocity vector of the particle that fell on the stellar surface from a circular marginally stable orbit with the energy e_* and angular momentum l_* corresponding to this orbit. In paper Sibgatullin and Sunyaev (2000), we gave universal (valid for an arbitrary equation of state) approximation formulas for the dependences of e_*, l_*, R_*, and f_K^* on Kerr parameter and on dimensionless quadrupole coefficient b for the particles falling on the surface from circular marginally stable orbits. We constructed plots of total luminosity against angular velocity for a NS with gravitational mass $M = 1.4M_\odot$ by using (27) and (30).

Note the fundamental difference between our formulas (30), (31) for the luminosity and formulas (16), (18) from Thampan and Datta (1998). These authors calculated the energy release produced by particles during accretion as a difference between the particle energy in the stable circular equatorial orbit nearest the body and the energy of the particles lying on the equator and corotating with the body.

Using Theorem 3 to Calculate the NS Angular Velocity and Equatorial Radius. As the NS parameters, we choose its rest mass and Kerr parameter j (dimensionless angular momentum cJ/GM^2). The methods of constructing the function $M(j, m)$ were developed previously (Sibgatullin and Sunyaev 2000).

According to (28), the NS angular velocity at given rest mass m and Kerr parameter j can be determined from the known function $M(j,m)$ by using the formula

$$\Omega GM/c^3 = \frac{M_{,j}\,|_m}{M + 2jM_{,j}\,|_m}.$$

Another important formula that relates the constant μ to a derivative of the gravitational mass with respect to the rest mass at constant angular momentum follows from (28). Taking (29) for μ, we obtain

$$c\frac{MM_{,m}\,|_j}{M + 2jM_{,j}\,|_m} = \sqrt{a^2 - b^2\,(\Omega - \omega)^2}, \qquad R = b, \tag{32}$$

where R is the geometric equatorial radius of the star (the equator length divided by 2@[pi]); the metric coefficients a and b are taken at the NS equator.

In the static case, the NS radius can be determined from (32) by using the Schwarzschild metric ($\omega = \Omega = 0, a^2 = c^2\,(1 - \frac{2GM}{Rc^2}$):

$$\frac{Rc^2}{GM} = \frac{2}{1 - M_{,m}^2}. \tag{33}$$

Remarkably, formula (33) [if $M(j,m)$ is substituted in it and differentiated at constant j] closely agrees with the numerical data from Cook et al. (1994) and Stergioulus (1998) for the equatorial radius of a rotating NS, to within a few percent:$R \approx 2GM/(c^2 - c^2 M_{,m}^2\,|_j)$.

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