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The angular function for Compton scattering in mildly and ultra relativistic astrophysical plasmas

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Abstract. Compton scattering of low-frequency radiation by an isotropic distribution of (i) mildly and (ii) ultra relativistic electrons is considered. It is shown that the ensemble-averaged differential cross-section in this case is noticeably different from the Rayleigh phase function. The scattering by an ensemble of ultra-relativistic electrons obeys the law $p = 1 - \cos\alpha$, where α is the scattering angle; hence *photons are preferentially scattered backwards*. This contrasts the forward scattering behaviour in the Klein-Nishina regime. Analytical formulae describing first-order Klein-Nishina and finite-electron-energy corrections to the simple relation above are given for various energy distributions of electrons: monoenergetic, relativistic-Maxwellian, and power-law. A similar formula is also given for the mildly relativistic (with respect to the photon energy and electron temperature) corrections to the Rayleigh angular function. One of the manifestations of the phenomenon under consideration is that hot plasma is more reflective with respect to external low-frequency radiation than cold one, which is important, in particular, for the photon exchange between cold accretion disks and hot atmospheres (coronae or ADAF flows) in the vicinity of relativistic compact objects; and for compact radiosources.

Key words: scattering – radiation mechanisms: non-thermal – plasmas – accretion, accretion disks

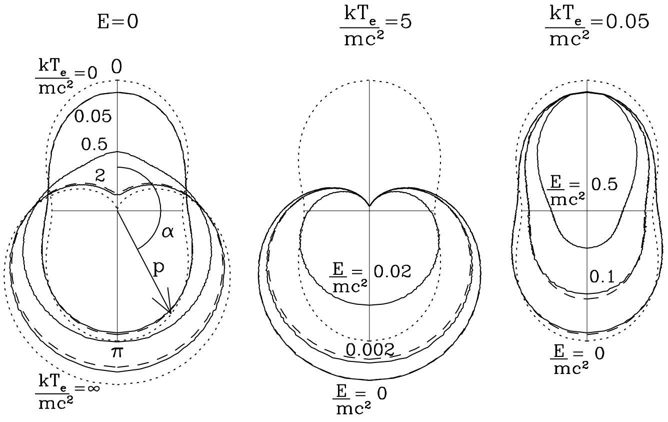

Fig. 1. Angular function (in polar coordinates) for the Compton scattering of photons of energy E by an ensemble of relativistic Maxwellian electrons with temperature $T_{\rm e}$. Solid lines show Monte-Carlo simulation results. Dashed lines represent approximations (when available) given either by Eq. (16) or by Eq. (19). The dotted lines represent two extreme cases: (i) $E = 0$, $T_{\rm e} = 0$, which corresponds to the Rayleigh angular function, and (ii) $E = 0$, $kT_{\rm e} \gg m_{\rm e}c^2$. In the latter case (presented only in the left panel), the scattering obeys the law $p = 1 - \cos\alpha$.

1. Introduction

The problem about the cross-section for scattering of radiation by an individual free electron, i.e. Compton scattering, is classical and its description can be found in any textbook on the interaction of matter and light (see, e.g., Berestetskii et al. 1982). The Compton scattering of low-frequency radiation in a dense plasma for which collective effects are important is also well studied. In particular, formulae that describe the differential cross-section averaged over an ensemble of plasma-bound electrons, or, stated differently, the scattering angular function, are well-known (see, e.g., Bekefi 1966; Zheleznyakov 1996).

In this Letter, we would like to point out the fact, which is seemingly un-noticed in the literature, that the angular function for Compton scattering can be different from the classical Rayleigh function in the case of a low-density relativistic electron gas too. We show below, providing simple analytical formulae, that the differential cross-section averaged over an isotropic ensemble of substantially relativistic electrons (e.g., monoenergetic, Maxwellian, or with a power-law energy spectrum) is backward-oriented, i.e. *photons have a tendency to be scattered backwards*, rather than forwards (Fig. 1). This

Send offprint requests to: S.Y. Sazonov phenomenon results from the combined operation of two effects. One is that a photon is more likely to suffer a scattering from an electron that is moving toward it, rather than away from it (the probability of scattering is proportional to the factor $1-\cos\theta\, v/c$, where v is the electron velocity and θ is the angle at which the photon and electron encounter). The other effect is that photons emerge after scattering preferentially in the direction of the motion of the relativistic electron. The resulting angular function contrasts the forward-oriented Klein-Nishina angular function which corresponds to the case of scattering of high-energy photons on a resting electron.

The authors certainly realize that the relativistic modification of the differential cross-section for scattering by an isotropic ensemble of electrons is implicitly taken into account in any Monte-Carlo simulations of spectra and angular distributions of photons scattered in mildly or ultra-relativistic electron gases. In particular, we have verified that results obtained using the Monte-Carlo Comptonization code of Pozdnyakov et al. (1983) and the results of calculations that employ the formulae of this paper are in perfect agreement.

Apart from a general interest, the formulae obtained here are easily applicable to the calculations of the scattering albedos (with respect to low-frequency radiation) of hot atmospheres, such as coronae or outflows above accretion disks, ADAF accretion flows near black holes, layers of spreading matter on the surfaces of accreting neutron stars, intergalactic gas in rich clusters of galaxies, etc. The issue of the modified albedo may also be important in studying clouds of ultra-relativistic electrons in compact radiosources, for which an external radiation may be an additional source of seed photons for inverse-Compton cooling. Our calculations of albedos demonstrate a noticeable reduction of the fraction of incident photons which can propagate inside an optically thick cloud of relativistic electrons. Therefore, less photons are capable of taking part in the Comptonization process, and smaller is the rate at which the electrons lose their energy.

The absence of forward-backward symmetry in the considered regime of scattering also causes the coefficient of spatial diffusion of photons in hot plasma to be different from that in the case of non-relativistic plasma. This affects the shapes of Comptonization spectra and the time delays between soft and hard radiation from variable X-ray sources. A detailed discussion of both albedo and photon-path-lengthening consequences of the results reported here is presented in a separate paper (Sazonov & Sunyaev 2000a).

2. Scattering angular function

2.1. Scattering of low-energy photons in ultra-relativistic plasma

We start from the formula that describes the differential cross-section for the Compton scattering by an electron propagating in the direction $\boldsymbol{\omega}$ with a speed $v=\beta c$:

$$\frac{d\sigma}{d\boldsymbol{\Omega}'}=\frac{3\sigma_{\rm T}}{16\pi\gamma^2}\frac{X}{(1-\beta\boldsymbol{\Omega}\boldsymbol{\omega})^2}\left(\frac{\nu'}{\nu}\right)^2, \tag{1}$$

where

$$\frac{\nu'}{\nu}=\frac{1-\beta\boldsymbol{\Omega}\boldsymbol{\omega}}{1-\beta\boldsymbol{\Omega}'\boldsymbol{\omega}+(h\nu/\gamma mc^2)(1-\mu_s)}, \tag{2}$$

$$\begin{aligned} X \;=\; & 2-\frac{2(1-\mu_s)}{\gamma^2(1-\beta\boldsymbol{\Omega}\boldsymbol{\omega})(1-\beta\boldsymbol{\Omega}'\boldsymbol{\omega})} \\ & +\frac{(1-\mu_s)^2}{\gamma^4(1-\beta\boldsymbol{\Omega}\boldsymbol{\omega})^2(1-\beta\boldsymbol{\Omega}'\boldsymbol{\omega})^2} \\ & +\left(\frac{h\nu}{m_{\rm e}c^2}\right)^2\frac{\nu'}{\nu}\frac{(1-\mu_s)^2}{\gamma^2(1-\beta\boldsymbol{\Omega}\boldsymbol{\omega})(1-\beta\boldsymbol{\Omega}'\boldsymbol{\omega})}, \end{aligned} \tag{3}$$

$\gamma^2=(1-\beta^2)^{-1}$, $(\nu, \boldsymbol{\Omega})$ are the frequency and the direction of propagation of the incident photon, $(\nu', \boldsymbol{\Omega}')$ are the corresponding values for the emergent photon, $\mu_s=\boldsymbol{\Omega}\boldsymbol{\Omega}'$, and $\sigma_{\rm T}$ is the Thomson scattering cross-section (Berestetskii et al. 1982).

Consider now an isotropic distribution of electrons of energy $\gamma m_{\rm e}c^2$ (which corresponds to a given speed β). We wish to average the cross-section given by Eq. (1) over this velocity distribution and thereby to find the scattering angular function:

$$p(\gamma,\mu_s)=\frac{1}{\sigma_{\rm T}}\int\frac{d\sigma}{d\boldsymbol{\Omega}'}(1-\beta\boldsymbol{\Omega}\boldsymbol{\omega})d\boldsymbol{\omega}. \tag{4}$$

Here the factor $1-\beta\boldsymbol{\Omega}\boldsymbol{\omega}$ takes into account the relative velocity of the photon and electron along the direction of the former's motion (see, e.g., Landau & Lifshits 1975; Gould 1971). The factor $\sigma_{\rm T}^{-1}$ has been introduced to make p dimensionless and hence similar to the phase function that is used in the theory of radiative transfer for describing scattering processes. According to the definition (4), the mean photon free path can be found as

$$\lambda=\frac{1}{N_{\rm e}\sigma_{\rm T}\int p(\gamma,\mu_s)\,d\boldsymbol{\Omega}'/4\pi}=\frac{1}{N_{\rm e}\sigma_T\int p(\gamma,\mu_s)\,d\mu_s/2}, \tag{5}$$

where $N_{\rm e}$ is the electron number density.

We shall first consider the limiting case where $\gamma\gg 1$, $\gamma h\nu/m_{\rm e}c^2\ll 1$, i.e. the electrons are ultra-relativistic, and the photons are low-energy. The treatment can then be done in the Thomson limit, and, in a first approximation, the terms proportional to $h\nu/m_{\rm e}c^2$ in Eqs. (2) and (3) can be ignored. The angular function will therefore be given by the integral

$$\begin{aligned} p \;=\; & \frac{3}{16\pi}\int\left[\frac{2(1-\beta\boldsymbol{\Omega}\boldsymbol{\omega})}{\gamma^2(1-\beta\boldsymbol{\Omega}'\boldsymbol{\omega})^2}-\frac{2(1-\mu_s)}{\gamma^4(1-\beta\boldsymbol{\Omega}'\omega)^3}\right. \\ & \left.+\frac{(1-\mu_s)^2}{\gamma^6(1-\beta\boldsymbol{\Omega}\boldsymbol{\omega})(1-\beta\boldsymbol{\Omega}'\omega)^4}\right]d\boldsymbol{\omega}. \end{aligned} \tag{6}$$

The main contribution to this integral is provided by the electrons with $\boldsymbol{\omega}$ approaching $\boldsymbol{\Omega}'$, which is the result of Doppler aberration. Therefore, we may put in Eq. (6) $\boldsymbol{\Omega\omega} \approx \mu_s$ and $1-\beta\boldsymbol{\Omega\omega} \approx 1-\mu_s$. The following integral over $\mu' = \boldsymbol{\Omega}'\boldsymbol{\omega}$ then arises:

$$p = \frac{3(1-\mu_s)}{8}\int_{-1}^{1}\left[\frac{2}{\gamma^2(1-\beta\mu')^2}-\frac{2}{\gamma^4(1-\beta\mu')^3}+\frac{1}{\gamma^6(1-\beta\mu')^4}\right]d\mu', \quad (7)$$

The integration leads to a very simple result:

$$p(\gamma \gg 1, \mu_s) = 1-\mu_s. \quad (8)$$

The angular function (8) is presented in the left panel of Fig. 1. It has a simple, apple-like shape (in polar coordinates), which is the result of the combined action of two effects, namely the effect of selection of electrons according to the incidence angle by photons and the Doppler aberration effect, as discussed in the Introduction.

Using formula (5), we can find the mean photon free path that corresponds to the angular function given by Eq. (8): $\lambda = (N_{\rm e}\sigma_{\rm T})^{-1}$, as it should be in the Thomson limit.

It is not difficult to include Klein-Nishina corrections in the expression (8). We shall do that to the first order of accuracy. We assume as before that $\gamma h\nu/m_{\rm e}c^2 \ll 1$. In this limit, $h\nu/\gamma m_{\rm e}c^2 \ll (1-\beta)$, and, therefore, we can expand the ratio ν'/ν given by Eq. (2) as follows:

$$\left(\frac{\nu'}{\nu}\right)^2 = \left(\frac{1-\beta\boldsymbol{\Omega\omega}}{1-\beta\boldsymbol{\Omega}'\boldsymbol{\omega}}\right)^2\left[1-2\frac{h\nu(1-\mu_s)}{\gamma m_{\rm e}c^2(1-\beta\boldsymbol{\Omega}'\boldsymbol{\omega})}+...\right]. \quad (9)$$

The expression in square brackets in the equation above implies that corrections need to be made to the three terms in Eq. (6), namely three additional terms proportional to $h\nu/m_{\rm e}c^2$ appear. Note that the fourth term in the expression X (Eq. [3]) leads to a correction to the angular function of the order of $(\gamma h\nu/m_{\rm e}c^2)^2$, which will be neglected here. The final result (the calculation is very similar to the one that had led to Eq. [8]) is

$$p(\gamma,\nu,\mu_s) = 1-\mu_s-2\gamma\frac{h\nu}{m_{\rm e}c^2}(1-\mu_s)^2. \quad (10)$$

If the electron energy is not too high, say $\gamma \lesssim 10$, corrections of the form γ^{-n} to the angular function become important. These can be found as follows. On introducing spherical coordinates with the polar axis pointing along $\boldsymbol{\Omega}'$, we obtain

$$\mu = \boldsymbol{\Omega\omega} = \mu'\mu_s+(1-\mu'^2)^{1/2}(1-\mu_s^2)^{1/2}\sin\phi. \quad (11)$$

Integration of the first and second bracketed terms in Eq. (6) then reduces to

$$\int\left[\frac{2(1-\beta\mu'\mu_s)}{\gamma^2(1-\beta\mu')^2}-\frac{2(1-\mu_s)}{\gamma^4(1-\beta\mu')^3}\right]d\mu', \quad (12)$$

whereas the third term can be expanded as follows:

$$\int\frac{(1-\mu_s)^2}{\gamma^6(1-\beta\mu'\mu_s)(1-\beta\mu')^4}\left[1+\frac{\beta(1-\mu'^2)^{1/2}(1-\mu_s^2)^{1/2}}{1-\beta\mu'\mu_s}\sin\phi+\frac{\beta^2(1-\mu'^2)(1-\mu_s^2)}{(1-\beta\mu'\mu_s)^2}\sin^2\phi+...\right]d\mu' d\phi. \quad (13)$$

Implementing the straightforward integrations in Eqs. (12) and (13) and keeping only the leading terms (of order γ^{-2}), we derive

$$p(\gamma,\mu_s) = 1-\mu_s+\frac{-1+3\ln 4\gamma^2}{4\gamma^2}\mu_s. \quad (14)$$

We can finally combine Eqs. (10) and (14) to obtain

$$p(\gamma,\nu,\mu_s) = 1-\mu_s-2\gamma\frac{h\nu}{m_{\rm e}c^2}(1-\mu_s)^2+\frac{-1+3\ln 4\gamma^2}{4\gamma^2}\mu_s. \quad (15)$$

We implemented a series of Monte-Carlo simulations to determine the parameter range of applicability of formula (15): $\gamma \gtrsim 2$, $\gamma h\nu \lesssim 0.02 m_{\rm e}c^2$.

In the case of electrons obeying a relativistic Maxwellian distribution, which is described by the function $dN_{\rm e} \propto \gamma(\gamma^2-1)^{1/2}\exp\left(-\gamma m_{\rm e}c^2/kT_{\rm e}\right)d\gamma$, Eq. (15) should be convolved with this function. The result is

$$p(T_{\rm e},\nu,\mu_s) = 1-\mu_s-6\frac{h\nu}{m_{\rm e}c^2}\frac{kT_{\rm e}}{m_{\rm e}c^2}(1-\mu_s)^2+\frac{-1+3\ln 4+6\Gamma(0,m_{\rm e}c^2/kT_{\rm e})}{8}\left(\frac{m_{\rm e}c^2}{kT_{\rm e}}\right)^2\mu_s, \quad (16)$$

where $T_{\rm e}$ is the temperature of the electrons and $\Gamma(\alpha,z) = \int_z^\infty x^{\alpha-1}e^{-x}dx$ is the incomplete Gamma function. Formula (16) is a good approximation if $kT_{\rm e} \gtrsim 2m_{\rm e}c^2$, $h\nu kT_{\rm e} \lesssim 0.01(m_{\rm e}c^2)^2$. It is interesting to compare the first of these conditions with the corresponding constraint on the applicability of the approximation (15) (see text following that equation). Evidently, the relatively poor convergence of the series (16) is due to the contribution of the low-energy tail of the Maxwellian distribution, i.e. electrons with $\gamma \lesssim \langle\gamma\rangle = 3kT_{\rm e}/m_{\rm e}c^2$.

Various examples of the angular function corresponding to the scattering on high-temperature electrons, as resulted from Monte-Carlo simulations or calculated from Eq. (16), are presented in Fig. 1. One can see that Klein-Nishina corrections, which are described in the first approximation by the second term on the right-hand side of Eq. (16), act to reduce the probability of scattering in all directions. This reduction reaches a maximum at $\mu_s \to -1$ (backward scattering) and monotonically diminishes to become vanishing at $\mu_s = 1$ (forward scattering). The effect of temperature corrections, which are, to the first order, described by the last term of Eq. (16), is that as the temperature decreases, progressively more photons become scattered forwards — see the pattern corresponding to $kT_{\rm e} = 0.5 m_{\rm e}c^2$ in the left panel of Fig. 1 (the result of a Monte-Carlo simulation is shown).

The mean photon free path that corresponds to the angular function (16), according to Eq. (5), is given by

$$\frac{1}{\lambda(T_{\rm e},\nu)} = N_{\rm e}\sigma_{\rm T}\left(1-8\frac{h\nu}{m_{\rm e}c^2}\frac{kT_{\rm e}}{m_{\rm e}c^2}\right). \quad (17)$$

This expression is equivalent to Eq. (2.17) of Pozdnyakov et al. (1983).

In radio galaxies, the relativistic electrons have a power-law energy spectrum $dN_{\rm e} \propto \gamma^{-\alpha} d\gamma$ with a low-energy cut-off $\gamma > \gamma_{\rm min}$. The angular function in this case is

$$p(\alpha, \gamma_{\rm min}, \nu, \mu_s) = 1 - \mu_s - \frac{2(\alpha-1)}{\alpha-2}\gamma_{\rm min}\frac{h\nu}{m_{\rm e}c^2}(1-\mu_s)^2 + \frac{\alpha-1}{4(\alpha+1)}\left(-1+\frac{6}{\alpha+1}+3\ln 4\gamma_{\rm min}^2\right)\frac{1}{\gamma_{\rm min}^2}\mu_s \quad (18)$$

This formula is applicable if $\gamma_{\rm min} \gg 1$, $\alpha > 2$ and $\gamma_{\rm min} h\nu/m_{\rm e}c^2 \ll 1$.

2.2. Scattering of photons of mildly relativistic energy in mildly relativistic thermal plasma ($h\nu, kT_{\rm e} \ll m_{\rm e}c^2$)

In this case, the angular function can be approximated by the following formula, which was derived earlier in (Sazonov & Sunyaev 2000b),

$$\begin{aligned} p(T_{\rm e}, \nu, \mu_s) = \frac{3}{4}\Bigg[& 1+\mu_s^2 - 2(1-\mu_s)(1+\mu_s^2)\frac{h\nu}{m_{\rm e}c^2} \\ & +2(1-2\mu_s-3\mu_s^2+2\mu_s^3)\frac{kT_{\rm e}}{m_{\rm e}c^2} \\ & +(1-\mu_s)^2(4+3\mu_s^2)\left(\frac{h\nu}{m_{\rm e}c^2}\right)^2 \\ & +(1-\mu_s)(-7+14\mu_s+9\mu_s^2-10\mu_s^3)\frac{h\nu}{m_{\rm e}c^2}\frac{kT_{\rm e}}{m_{\rm e}c^2} \\ & +(-7+22\mu_s+9\mu_s^2-38\mu_s^3+20\mu_s^4)\left(\frac{kT_{\rm e}}{m_{\rm e}c^2}\right)^2 + ...\Bigg]. \quad (19) \end{aligned}$$

The main term in this power series, $3(1+\mu_s^2)/4$, is the usual Rayleigh function, which corresponds to the non-relativistic case.

The range of applicability of formula (19) is roughly $kT_{\rm e} \lesssim 0.05 m_{\rm e}c^2$, $h\nu \lesssim 0.1 m_{\rm e}c^2$. Some examples are displayed in the right panel of Fig. 1. The effect of the first-order temperature correction (the term $\propto kT_{\rm e}/m_{\rm e}c^2$ in Eq. [19]) on the angular function is to enhance the number of photons scattered at intermediate angles between 69° and 138° (a maximum of $12kT_{\rm e}/(0.05m_{\rm e}c^2)$ per cent is reached at an angle of 105°) and to suppress scattering both in forward and in backward directions (by $10kT_{\rm e}/(0.05m_{\rm e}c^2)$ per cent at angles 0 and π).

The mean photon free path corresponding to the angular function (19) is given by

$$\frac{1}{\lambda(T_{\rm e},\nu)} = N_{\rm e}\sigma_{\rm T}\left[1-2\frac{h\nu}{m_{\rm e}c^2} - 5\frac{h\nu}{m_{\rm e}c^2}\frac{kT_{\rm e}}{m_{\rm e}c^2} + \frac{26}{5}\left(\frac{h\nu}{m_{\rm e}c^2}\right)^2\right], \quad (20)$$

which is a well-known expression (see, e.g., Pozdnyakov et al. 1983).

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