Analysis of relativistic hydrodynamics in conservation form

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Abstract. Formulations of Eulerian general relativistic ideal hydrodynamics in conservation form are analyzed in some detail, with particular emphasis to geometric source terms. Simple linear transformations of the equations are introduced and the associated equivalence class is exploited for the optimization of such sources. A significant reduction of their complexity is readily possible in generic spacetimes. The local characteristic structure of the standard member of the equivalence class is analyzed for a general equation of state (EOS). This extends previous results restricted to the polytropic case. The properties of all other members of the class, in particular specialized forms employing Killing symmetries, are derivable from the standard form. Special classes of EOS are identified for both spacelike and null foliations, which lead to explicit inversion of the state vector and computational savings. The entire approach is equally applicable to spacelike or lightlike foliations and presents a complete proposal for numerical relativistic hydrodynamics on stationary or dynamic geometries.

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1. Introduction

Relativistic hydrodynamics (RHD) is a basic building block in current efforts in numerical relativistic astrophysics and relativity. Those programmes provide much needed theoretical support to observational efforts focusing on extreme astrophysical systems. The development of numerical RHD has followed mostly on the steps of numerical *non*-relativistic hydrodynamics, whose technological significance has prompted the generation over the past decades of a large amount of mathematical and algorithmic known-how. This knowledge has been historically transferred effectively to the relativistic case, the prime example being artificial viscosity techniques [1, 2], which have been adopted and advocated by the pioneering work of Wilson [3].

More recent techniques are based on deeper mathematical understanding of non-linear conservation laws [4, 5, 6]. The relativistic Euler equations are also readily analyzed in conservation form (see [7] for a modern review of related mathematical aspects). Numerical formulations of conservative RHD were first presented in [8] for the one-dimensional case. A multi-dimensional extension, using an explicit "3+1" decomposition of spacetime, and valid for a general equation of state (EOS), was given in [9]. In [10] a covariant formulation was presented, adapted to polytropic EOS and a specific numerical solution procedure (Roe solver). An alternative formulation, restricted to the special relativistic case was given in [11]. A wide collection of numerical applications based on those approaches is reviewed in [12].

Adopting the point of view of non-linear conservation laws liberates the analysis from the need to adhere to Newtonian fluid dynamical concepts. This seems to be a natural approach when using RHD in general relativistic studies, as is exemplified by the ease with which one can extend RHD methods e.g.,to lightlike foliations of spacetimes - a highly non-Newtonian concept. In [13] we introduced a new covariant approach, significantly simpler than [10], but similarly restricted to perfect fluid EOS. This formulation was extensively tested numerically, in spherical symmetry, in particular in the context of dynamical black hole accretion [13]. The main new results in the present paper extend this formulation to a general EOS and introduce a simple linear analysis of the RHD equations which allows the tailoring of geometric source terms to situations of interest. The discussion unifies and clarifies apparently disconnected choices the characterize the literature mentioned above and will hopefully assist the future development of the field.

The paper is organized as follows: In Section 2 we introduce the useful notion of linearly equivalent conservation laws and in Section 3 we show how the use of the associated transformations allows to tailor the geometrical source structure. In particular we develop there a form of the equations suitable for spacetimes with exact or approximate Killing symmetries. In Section 4 the characteristic structure of the standard form of the RHD equations is derived for a general EOS. The inversion of the state vector is discussed in some detail. Some classes of EOS which lead to explicit inversions are outlined for the algebraically distinct cases of spacelike and null foliations.

Numerical applications will be presented elsewhere. Some discussion of related issues is included in Section 5.

Geometrized units ($G = c = 1$) are used throughout. The metric sign conventions follow [14]. Spacetime indices are denoted by small Greek letters and run from zero to three. Small Latin indices denote hypersurface coordinates and run from one to three. Boldface letters (e.g., $\mathbf{F}$) denote vectors in the fluid state space, which has dimension five. Those dimensions are labeled from zero to four, coinciding with the spacetime dimension for values up to three. Partial derivatives with respect to a coordinate x are denoted as $F_{,x}$ and the summation convention is used.

2. Equivalence classes of conservation laws

The general form of an N-dimensional system of conservation laws, expressed in a coordinate system $x^\mu = (x^0, x^j)$, where x^j parametrize the hypersurfaces of constant time x^0, is

$$\mathbf{U}_{,x^0} + \mathbf{F}^j{}_{,x^j} = \mathbf{S}\,, \tag{1}$$

where $\mathbf{U}$ is an N-vector describing the state of the system, the flux vectors $\mathbf{F}^j(x^\mu, \mathbf{U})$ control the time rate of change of the system state within an elementary volume, and possible (conservation violating) source terms are denoted by $\mathbf{S}(x^\mu, \mathbf{U})$. The independence of the source vector on the state vector derivatives is worth stressing.

We introduce a $N \times N$ dimensional square matrix $\mathbf{G}(x^\mu)$, which is invertible ($det(\mathbf{G}) \neq 0$). We consider the new state variables $\bar{\mathbf{U}} = \mathbf{G}\mathbf{U}$, satisfying the equations

$$\bar{\mathbf{U}}_{,x^0} + \bar{\mathbf{F}}^j{}_{,x^j} = \bar{\mathbf{S}} + \mathbf{G}_{\,,x^0}\mathbf{G}^{-1}\bar{\mathbf{U}} + \mathbf{G}_{\,,x^j}\mathbf{G}^{-1}\bar{\mathbf{F}}^j\,, \tag{2}$$

where $\bar{\mathbf{F}} = \mathbf{G}\mathbf{F}$ and $\bar{\mathbf{S}} = \mathbf{G}\mathbf{S}$. The linear transformation introduced by $\mathbf{G}$ leaves the characteristic structure of the original system intact. This can be seen immediately by writing the homogeneous version of system (1) in quasi-linear form,

$$\mathbf{U}_{,x^0} + \mathbf{B}^j\mathbf{U}_{,x^j} = 0\,, \tag{3}$$

where

$$\mathbf{B}^j = \frac{\partial \mathbf{F}^j}{\partial \mathbf{U}}\,, \tag{4}$$

are the Jacobians of the flux vectors with respect to the state vector. The eigenvalues of each Jacobian are then determined by the equation

$$det(\mathbf{B}^j - \lambda^j\mathbf{I}) = 0\,, \tag{5}$$

where λ^j denotes an eigenvalue in the direction j and $\mathbf{I}$ denotes the unit matrix. The corresponding right and left eigenvectors, $\mathbf{r}^j$ and $\mathbf{l}^j$, are determined by the equations

$$(\mathbf{B}^j - \lambda^j\mathbf{I})\mathbf{r}^j = 0, \tag{6}$$

$$\mathbf{l}^j(\mathbf{B}^j - \lambda^j\mathbf{I}) = 0\,. \tag{7}$$

It follows then from elementary properties of matrices and determinants that the transformed system (2) has along the j-th direction the same set of eigenvalues λ^j as

the original system. The new right eigenvectors given by $\bar{\mathbf{r}} = \mathbf{G}\mathbf{r}$, while the new left eigenvectors are given by $\bar{\mathbf{l}} = \mathbf{G}^{-1}\mathbf{l}$.

Smooth solutions $\mathbf{U}(x^\mu)$ of the system of equations (1) obviously lead to smooth solutions $\bar{\mathbf{U}}(x^\mu)$ of the transformed system. The same is true for weak solutions, as can easily be seen (in the one-dimensional case) from the definition of such solutions [6] as those satisfying the integral relation:

$$\int_0^{+\infty} dx^0 \int_{-\infty}^{+\infty} dx^1 \left[\phi_{,x^0}\mathbf{U} + \phi_{,x^1}\mathbf{F}^1\right] = \int_{-\infty}^{+\infty} dx^1 \phi(0, x^1)\mathbf{U}\,, \tag{8}$$

where $\phi(x^\mu)$ are continuously differentiable test functions of compact support. Pre-multiplying this system of integral conservation laws by $\mathbf{G}$ immediately shows that weak solutions are also transformed properly. Hence from the point of view of a hyperbolic set of partial differential equations, all such linearly related systems are equivalent.

Proceeding to the effects of the linear transformation on the source terms, we note that the point-wise properties of $\mathbf{G}$ simply reshuffle the source components among the equations, but the spacetime dependence of $\mathbf{G}$ modifies the RHS of the equations in a non-trivial way. Genuine source terms, e.g., such as those introduced in Newtonian hydrodynamics to model combustion processes, depend only on the state vector and cannot be essentially modified in this manner. In contrast, the regular source terms of RHD are highly coordinate dependent and involve a variety of metric derivatives. It becomes immediately apparent that an appropriate choice can maximize the conservation nature of the system.

3. Relativistic fluid dynamics

3.1. Reductions of the equations

The above discussion is applicable to a variety of relativistic systems but we specialize here to RHD. We hence assume a matter current and stress energy tensor corresponding to a perfect fluid, i.e., $J^\mu = \rho u^\mu$, $T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$. In these definitions, u^μ is the fluid four velocity, which is constrained by the normalization condition $u^\mu u_\mu = -1$, ρ is the rest mass density, p is the pressure, ε is the specific internal energy and $h = 1 + \varepsilon + p/\rho$ is the specific enthalpy. The pressure is determined by a general two-parameter family EOS, $p = p(\rho, \varepsilon)$. The relativistic conservation equations in covariant form are given by

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{9}$$

$$\nabla_\mu J^\mu \;\; = 0\,. \tag{10}$$

3.1.1. The standard form of RHD Upon introducing a coordinate system, (x^0, x^i), one obtains a simple form for the continuity equation

$$(\sqrt{-g}J^\mu)_{,\mu} = 0\,. \tag{11}$$

The standard form of the divergence of the stress-energy tensor can be written as

$$(\sqrt{-g}\,T^{\mu\nu})_{,\mu} = -\Gamma^\nu_{\mu\lambda}(\sqrt{-g}\,T^{\mu\lambda})\,, \tag{12}$$

where

$$\Gamma^{\nu}_{\mu\lambda} = \frac{1}{2} g^{\nu\rho}(g_{\mu\rho,\lambda} + g_{\lambda\rho,\mu} - g_{\mu\lambda,\rho}) \,, \tag{13}$$

are the usual Christoffel symbols of the second kind. The set of equations (11,12) form a system of non-linear coupled conservation laws which has been the basis for the formulations proposed in [10, 11], and a "3+1" decomposition of the same equations forms the basis of [9]. We hence call this set of equations the *standard form* and will analyze it further in the sequel. Linear equivalence with many other forms makes our analysis applicable to a wider set.

The state vector is given by

$$\mathbf{U}^{\mu} = \sqrt{-g}\, T^{0\mu} = \sqrt{-g}\, (\rho h u^0 u^{\mu} + p g^{0\mu}) \,, \tag{14}$$

$$\mathbf{U}^{4} = \sqrt{-g}\, J^{0} = \sqrt{-g}\, \rho u^{0} \,, \tag{15}$$

while the flux vectors are given by

$$\mathbf{F}^{j\mu} = \sqrt{-g}\, T^{j\mu} = \sqrt{-g}\, (\rho h u^{\mu} u^{j} + p g^{\mu j}) \,, \tag{16}$$

$$\mathbf{F}^{j4} = \sqrt{-g}\, J^{j} = \sqrt{-g}\, \rho u^{j} \,, \tag{17}$$

and the geometric source terms are (in this case $\mathbf{F}^{j0} = \mathbf{U}^{0}$)

$$\mathbf{S}^{\mu} = \, - \, (\Gamma^{\mu}_{00} \mathbf{U}^{0} + 2\Gamma^{\mu}_{k0} \mathbf{U}^{k} + \Gamma^{\mu}_{kl} \mathbf{F}^{kl}) \,, \tag{18}$$

$$\mathbf{S}^{4} = 0 \,. \tag{19}$$

We note that in RHD the source terms depend on the coordinates, the state vector *and* the flux vector. The dependence of the latter on the state vector is implicit, as will be elaborated on later.

3.1.2. The Killing form of the RHD equations It is well known that whereas the continuity equation (10) is a true conservation law, the divergence of the stress-energy tensor will lead in general to true conservation only when a spacetime symmetry is present [15]. Spacetimes with exact symmetries (e.g., the Kerr spacetime) or approximate ones (e.g., quasistationary binary configurations with helicoidal symmetry [16]) are of wide interest in fluid-dynamical studies. It is hence of some importance to identify ways to maximize conservation in this context.

A spacetime symmetry is captured by a Killing vector ξ_{ν} which satisfies the Killing equation $\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0$. Using all existing symmetries, and complementing any missing vectors by appropriate coordinate basis vectors, we introduce four linearly independent vectors $\xi^{\nu}_{(a)}$, with a running from zero to three. Upon defining the contracted vectors $K^{\mu}_{(a)} = \xi^{\nu}_{(a)} T^{\mu}_{\nu}$, one obtains instead of (9),

$$\nabla_{\mu} K^{\mu}_{(a)} = T^{\mu}_{\nu} \nabla_{\mu} \xi^{\nu}_{(a)} \,, \tag{20}$$

which becomes a true conservation law $\nabla_{\mu} K^{\mu}_{(a)} = 0$, for each $\xi^{\nu}_{(a)}$ that satisfies the Killing equation. If $\xi^{\nu}_{(a)}$ is an *approximate* Killing vector in some direction, then the corresponding current would be approximately conserved $\nabla_{\mu} K^{\mu}_{(a)} \approx 0$ and the related source term would capture the, possibly very small, deviations from pure conservation.

The state vector $\mathbf{U} = \left[\mathbf{U}_{(a)}, \mathbf{U}_4\right]^T$ in this case is given by

$$\mathbf{U}_{(a)} = \sqrt{-g}\,\xi^{\nu}_{(a)} T^{0}{}_{\nu} = \sqrt{-g}\,(\rho h u^0 \xi^{\nu}_{(a)} u_\nu + p \xi^{0}_{(a)})\,, \tag{21}$$

$$\mathbf{U}_4 \;\; = \sqrt{-g}\,J^0 = \sqrt{-g}\,\rho u^0\,, \tag{22}$$

with the flux vectors given by

$$\mathbf{F}^{j}{}_{(a)} = \sqrt{-g}\,\xi^{\nu}_{(a)} T^{j}{}_{\nu} = \sqrt{-g}\,(\rho h u^j \xi^{\nu}_{(a)} u_\nu + p\,\xi^{j}_{(a)})\,, \tag{23}$$

$$\mathbf{F}^{j}{}_{4} \;\; = \sqrt{-g}\,J^j = \sqrt{-g}\,\rho u^j\,, \tag{24}$$

and the geometric source terms are

$$\mathbf{S}_{(a)} = T^{\mu}{}_{\nu} \nabla_\mu \xi^{\nu}_{(a)}\,, \tag{25}$$

$$\mathbf{S}_4 \;\; = 0\,. \tag{26}$$

The linear relation between the standard form of the equations and one adapted to Killing symmetries is given simply by the matrix

$$\mathbf{G} = \left[\begin{array}{cc} \xi_{\nu(a)} & 0 \\ 0 & 1 \end{array}\right]. \tag{27}$$

3.1.3. Optimal source terms for RHD in general spacetimes The considerations in the last paragraph lead naturally to the case also important for *numerical relativity*, namely the case of no symmetries. Upon introducing a tetrad adapted to the coordinates, and using $K^{\mu}_{(a)} = \epsilon^{\nu}_{(a)} T^{\mu}_{\nu}$, the form (20) results to

$$(\sqrt{-g}\,T^{\mu}{}_{\nu})_{\,,\mu} = -\Delta^{\rho}_{\mu\nu}(\sqrt{-g}\,T^{\mu}{}_{\rho})\,, \tag{28}$$

where

$$\Delta^{\rho}_{\mu\nu} = \frac{1}{2} g_{\mu\lambda} g^{\lambda\rho}{}_{,\nu}\,. \tag{29}$$

With the definition of the state vector according to

$$\mathbf{U}_\mu = \sqrt{-g}\,T^{0}{}_{\mu} = \sqrt{-g}\,(\rho h u^0 u_\mu + p\delta^{0}_{\mu})\,, \tag{30}$$

$$\mathbf{U}_4 = \sqrt{-g}\,J^0 = \sqrt{-g}\,\rho u^0\,, \tag{31}$$

the flux vectors

$$\mathbf{F}^{j}{}_{\mu} = \sqrt{-g}\,T^{j}{}_{\mu} = \sqrt{-g}\,(\rho h u^j u_\mu + p\delta^{j}{}_{\mu})\,, \tag{32}$$

$$\mathbf{F}^{j}{}_{4} = \sqrt{-g}\,J^j = \sqrt{-g}\,\rho u^j\,, \tag{33}$$

and the source terms

$$\mathbf{S}_\mu = \; - \;(\Delta^{0}_{0\mu}\mathbf{U}_0 + \Delta^{k}_{0\mu}\mathbf{U}_k + \Delta^{0}_{k\mu}\mathbf{F}^{k}_{0} + \Delta^{k}_{l\mu}\mathbf{F}^{l}_{k})\,, \tag{34}$$

$$\mathbf{S}_0 = 0\,, \tag{35}$$

we obtain a second form for the equations of RHD. Note that for notational economy we have used a lowercase state vector index in the above expressions. With the explicit substitution of the perfect fluid stress-energy tensor, we obtain

$$\mathbf{S}_\mu = \frac{\sqrt{-g}}{2}\rho h u^\lambda u^\rho g_{\lambda\rho,\mu} - p(\sqrt{-g}\,)_{,\mu}\,. \tag{36}$$

The source terms in this case involve considerably fewer summations of metric derivatives compared to the standard form (18). The mixed form $\nabla_\mu T^\mu{}_\nu = 0$ has been used before as the starting point for non-conservative approaches to the RHD equations [3]. We point out here that the the related source simplification should benefit conservative formulations as well. The relation of the two forms is of course a lowering of the free spacetime index, which is captured by a linear transformation of the form

$$\mathbf{G} = \begin{bmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{bmatrix} . \tag{37}$$

We also include here the source terms in the notation of the "3+1" spacetime decomposition [14], which is a commonly adopted starting point for developing algorithms for dynamical evolutions of spacetimes. Assuming a foliation with spacelike surfaces having a unit normal n^μ, the four-metric is decomposed as

$$g_{\mu\nu} = \gamma_{\mu\nu} - n_\mu n_\nu \, , \tag{38}$$

where γ_{ij} is the 3-metric of the hypersurfaces. Upon introducing the lapse function N and the spacelike shift vector N^i [15], the evolution proceeds along the vector field $t^\mu = N n^\mu + N^\mu$ and the components of the metric read explicitly

$$g_{00} = -N^2 + \gamma_{ij} N^i N^j, \tag{39}$$

$$g_{0j} = \gamma_{ij} N^j, \tag{40}$$

$$g_{ij} = \gamma_{ij} \, . \tag{41}$$

The state vector we use differs from the usual one (see, e.g., [3, 17]) in that it does not explicitly use the above decomposition. For example, $\mathbf{U}^0 = T^{00}$ in contrast to $\rho_H = T^{\mu\nu} n_\mu n_\nu$. The state vector variables hence lose their usual meaning as observables in the instantaneous Eulerian rest frame. In a generic spacetime this frame is of no special significance and simply reflects the particular choice of lapse and shift vector. In compensation, our choice of state vector is valid in the absence of a spacelike foliation.

The volume element now reduces to $\sqrt{-g} = N\sqrt{\gamma}$, where γ is the determinant of the 3-metric, and the source terms for equation (28) have the following explicit form:

$$\begin{aligned} \mathbf{S}_\mu = & -\sqrt{\gamma} N^2 N_{,\nu} (u^0)^2 \rho h \\ & + \frac{1}{2} N \sqrt{\gamma} \rho h \gamma_{ij,\nu} \left[(u^0)^2 N^i N^j + u^i u^j + 2 u^0 u^i N^j \right] \\ & + N \sqrt{\gamma} u^0 \rho h \gamma_{ij} N^i{}_{,\nu} \left[u^0 N^j + u^j \right] . \end{aligned} \tag{42}$$

4. Further analysis of the RHD equations

In contrast to non-relativistic hydrodynamics, the relativistic theory exhibits a non-linear algebraic coupling of all equations (including the continuity equation) through the velocity normalization condition. This feature has wide ranging implications for the structure of the theory, and in particular:

- The computation of fluxes from the state vector. In non-relativistic hydrodynamics, the fluxes can be written as explicit functions of the state vector. In RHD both fluxes and state vector are more properly seen as algebraic functions of suitable "primitive" variables, as is obvious e.g., from equations (14-17).
- The analysis of the local characteristic structure. The implicit dependence of the fluxes on the state vector in RHD requires, in turn, that the analysis of the Jacobians of the fluxes with respect to the state vector be done through a set of intermediate variables.

A unique set of such intermediate variables employed in both procedures is commonly used in the literature without explicit statement. There are few constraints on the choice of primitive variables $\mathbf{w}$. An appropriate choice influences the algebraic difficulty of analyzing the characteristic structure of the system, a key ingredient of state-of-the-art Riemann solver based numerical schemes for non-linear conservation laws [6]. The rest mass density ρ is a common candidate in all proposed sets [11, 9]. With our choice of representation (e.g., equations (12) or (28)), the most appropriate velocity variable is u^i, although a lower index would lead to simpler calculations in the latter case. The set must be completed with the choice of an additional thermodynamical variable. Both the specific internal energy [9] and pressure [11] have been used. The choice of pressure leads to slightly simpler analysis. A reasonable compromise between simplicity and maintaining continuity with past work is hence the choice $\mathbf{w} = (\rho, u^i, \varepsilon)$.

4.1. Characteristic structure

Using the intermediate variables $\mathbf{w}$ the conservation law (1) is rewritten as a quasi-linear system

$$\mathbf{A}^0 \mathbf{w}_{,x^0} + \mathbf{A}^j \mathbf{w}_{,x^j} = 0\,, \tag{43}$$

where

$$\mathbf{A}^0 = \frac{\partial \mathbf{U}}{\partial \mathbf{w}}, \quad \mathbf{A}^j = \frac{\partial \mathbf{F}^j}{\partial \mathbf{w}}\,. \tag{44}$$

Hence, upon introducing the intermediate eigenvalues $\hat{\lambda}^j$

$$det(\mathbf{A}^j - \hat{\lambda}^j \mathbf{A}^0) = 0\,, \tag{45}$$

and the intermediate right and left eigenvectors,

$$(\mathbf{A}^j - \hat{\lambda}^j \mathbf{A}^0)\hat{\mathbf{r}}^j = 0 \tag{46}$$

$$\hat{\mathbf{l}}_j(\mathbf{A}^j - \hat{\lambda}^j \mathbf{A}^0) = 0\,, \tag{47}$$

elementary algebra establishes that $\hat{\lambda}^j = \lambda^j$ and the right eigenvectors of the matrix $\mathbf{B}^j$ are given by $\mathbf{r}^j = \mathbf{A}^0 \hat{\mathbf{r}}^j$. The left eigenvectors of this matrix are given simply by $\hat{\mathbf{l}}_j = \mathbf{l}_j$.

The $\mathbf{A}^\mu$ matrices for the equations of RHD are

$$\mathbf{A}^0 = \begin{bmatrix} Yu^0u^0 + \kappa g^{00} & 2\rho h \mu_i u^0 & Zu^0u^0 + \chi g^{00} \\ Yu^k u^0 + \kappa g^{k0} & \rho h(\delta^k_i u^0 + u^k \mu_i) & Zu^k u^0 + \chi g^{k0} \\ 0 & \rho \mu_i & u^0 \end{bmatrix}\,, \tag{48}$$

$$\mathbf{A}^j = \begin{bmatrix} Yu^0u^j + \kappa g^{0j} & \rho h(\mu_i u^j + u^0\delta_i^j) & Zu^0u^j + \chi g^{0j} \\ Yu^k u^j + \kappa g^{kj} & \rho h(\delta_i^k u^j + u^k\delta_i^j) & Zu^k u^j + \chi g^{kj} \\ 0 & \rho\delta_i^j & u^j \end{bmatrix}, \tag{49}$$

where $Y = \rho + \kappa$, $Z = 1 + \varepsilon + \chi$ and

$$\mu_i = \frac{\partial u^0}{\partial u^i} = -\frac{u_i}{u_0}\,, \quad \chi = \frac{\partial p}{\partial \rho}\,, \quad \kappa = \frac{\partial p}{\partial \varepsilon}\,. \tag{50}$$

We choose a coordinate direction, which we label '1'. The other two coordinate directions are then denoted by $A = (2, 3)$.

The matrix $\mathbf{A}^1 - \lambda^1\mathbf{A}^0$ has *eigenvalues*

$$\lambda_0^1 = v^1 \quad \text{(triple)}\,, \tag{51}$$

and

$$\lambda_\pm^1 = \frac{1}{1 - c_s^2(1+L)}\left[-Mc_s^2 + v^1(1-c_s^2) \pm c_s\sqrt{D}\right], \tag{52}$$

where

$$D = c_s^2(M^2 - LN) + (1 - c_s^2)(N - 2Mv^1 + L(v^1)^2)\,, \tag{53}$$

with $v^1 = \frac{u^1}{u^0}$. The local sound speed is denoted by c_s and satisfies

$$hc_s^2 = \chi + \frac{p}{\rho^2}\kappa\,, \tag{54}$$

and the following shorthand notation was also used:

$$L = \frac{g^{00}}{(u^0)^2}\,, M = \frac{g^{01}}{(u^0)^2}\,, N = \frac{g^{11}}{(u^0)^2}\,. \tag{55}$$

A complete set of linearly independent *right-eigenvectors* ($\mathbf{r}^1 = \mathbf{A}^0\hat{\mathbf{r}}^1$), is given by

$$\mathbf{r}_{0,1} = \left[u^\mu, \frac{1}{\alpha}\right]^T, \tag{56}$$

$$\mathbf{r}_{0,A} = \left[\delta_A^\mu(1 + u^B u_B), 0\right]^T + u_A\left[u^0, u^1, 0, 0, \frac{1}{h} - \frac{1}{\alpha}\right]^T, \tag{57}$$

$$\mathbf{r}_\pm = \left[u^\mu + \frac{\Lambda_\pm}{(u^0)^2}(u^1 g^{0\mu} - u^0 g^{1\mu}), \frac{1}{h}\right]^T, \tag{58}$$

with the definitions

$$\alpha \equiv 1 + \varepsilon - \frac{\chi}{\kappa}\rho\,, \tag{59}$$

$$\Lambda_\pm \equiv \frac{c_s^2}{(v^1 - \lambda_\pm)(1 - c_s^2) - c_s^2(M - \lambda_\pm L)}\,. \tag{60}$$

We note that the $\mathbf{r}_\pm$ eigenvectors are unique up to normalization, whereas the $\mathbf{r}_{0,1}, \mathbf{r}_{0,A}$ vectors can be any set spanning the degenerate subspace. For a perfect fluid EOS these expressions coincide with the ones reported in [13]. Note that in that restricted case $\alpha = 1$. The spectral decomposition given above applies to a chosen direction j. Since j is arbitrary, to obtain similar expressions for the remaining

directions, it suffices to specialize them accordingly, e.g., obtain the eigenvalues from expressions (51) and (52) with substitution of the desired direction, and permutation of the corresponding eigenvectors. Complete sets of eigenvectors for other versions of the equations are obtained by a straightforward multiplication with the corresponding **G** matrix and are not reproduced here.

4.2. Inverting the state vector

The general statement of the recovery of the primitives fields is to find values for e.g., $\mathbf{w} = (\rho, \varepsilon, u^i)$, given a set of conserved variables **U**. This procedure is part of any solution algorithm that uses the conservation form of RHD, i.e., it is not related to the characteristic decomposition of the system.

We focus on the standard form of the equations, in which case $\mathbf{U} = \sqrt{-g}\,(T^{00}, T^{0i}, J^0)$. At first sight, this requires the inversion of the system of six non-linear algebraic equations

$$T^{00} = \frac{1}{\rho}(\frac{p}{\rho} + 1 + \varepsilon)(J^0)^2 + pg^{00}\,, \tag{61a}$$

$$T^{0i} = (\frac{p}{\rho} + 1 + \varepsilon)J^0 u^i + pg^{0i}\,, \tag{61b}$$

$$-\rho^2 = g_{00}(J^0)^2 + 2\rho g_{0i} J^0 u^i + \rho^2 g_{ij} u^i u^j, \tag{61c}$$

$$p = p(\rho, \varepsilon)\,. \tag{61d}$$

We point out that even in the most general case the size of the non-linear system can be reduced, with the elimination of the velocity from the unknowns. We introduce the tensor $S^{\mu\sigma} = g_{\nu\rho} T^{\mu\nu} T^{\rho\sigma}$. Inspection of the S^{00} component of this tensor shows immediately that it is only a function of conserved variables. Together with T^{00} and the EOS, we have a reduced system which reads

$$S^{00} = (\frac{p}{\rho} + 1 + \varepsilon)(\frac{p}{\rho} - 1 - \varepsilon)(J^0)^2 + p^2 g^{00}\,, \tag{62a}$$

$$T^{00} = \frac{1}{\rho}(\frac{p}{\rho} + 1 + \varepsilon)(J^0)^2 + pg^{00}\,, \tag{62b}$$

$$p = p(\rho, \varepsilon)\,. \tag{62c}$$

The use of a general EOS which may be only available in tabulated form implies that the reduced system (62*a*-62*c*) is in general to be inverted numerically with an iteration procedure. A description of a typical iterative procedure (applied to equations (61*a*-61*d* in the special relativistic limit) is given in [18], where estimated values for (ε, v^i, ρ) are used to start a non-linear iteration of the system. In [10] several other procedures are discussed for the special case of a polytropic gas.

It is of interest to point out that further reduction of the system to a binomial equation, and hence with an explicit and convenient solution, is possible for special classes of analytic EOS. Assuming that the EOS if of the explicit form $p = \rho F(h)$ allows the further manipulation of (62*a*-62*c*) to obtain

$$(T^{00})^2 S^{00} - h(2F(h) - h)(J^0)^2 (T^{00})^2 - (S^{00} - h(F(h) - h)(J^0)^2)^2 g^{00} = 0\,, \tag{63}$$

which is a non-linear equation for the enthalpy. An explicit relation between conserved and primitive variables, which rests on the ability of solving equation (63), has an impact on the efficiency of the numerical code, as it eliminates an iterative process that is required, at least once per each spacetime point.

It is seen immediately that the metric component g^{00} stands out as having special significance in this algebraic equation. Indeed, as already pointed out in [13] any null foliation (characterized by $g^{00} = 0$) leads to explicit solutions for h in terms of conserved quantities, in the case of a polytropic equation of state $p = (\Gamma - 1)\rho\varepsilon$, where Γ is the adiabatic index. The most general case explicitly reducible to a binomial is $F(h) = \alpha/h + \beta + \gamma h$, where (α, β, γ) are constants characterising the fluid. The polytropic gas is the special case with $\alpha = 0, \gamma = -\beta = (\Gamma - 1)/\Gamma$.

For spacelike foliations ($g^{00} \neq 0$), the situation is slightly different. The choice $F(h) = \alpha/h + \beta + h$ leads to a binomial for h, whereas the choice $F(h) = \alpha/h + \gamma h$ leads to a binomial for h^2. None of those cases includes the polytropic gas, but see the arguments of [19] for an example of the latter case, with $\gamma = -\alpha = 1/4$.

Once the enthalpy is obtained, the other variables follow straightforwardly, e.g., the velocity follows from

$$u^i = \frac{\rho(T^{0i} - pg^{0i})}{(p + \rho + \rho\varepsilon)J^0}\,. \tag{64}$$

5. Summary and concluding remarks

In view of the increasing importance of conservative RHD in numerical applications in relativistic astrophysics and relativity, we explored the corresponding framework in some detail. The algebraic complexity of the systems involved suggests that links between specific manifestations of the equations be established at the outset. This is accomplished with the introduction of simple linear transformations of the equations, which leave the local characteristic structure invariant but have non-trivial impact on the source terms. It is pointed out that for spacetimes with exact or approximate symmetries, but also in the general case, the equations can be modified to capture the conservation property in an optimal way.

The local characteristic structure of the RHD equations in a general spacetime foliation (spacelike or lightlike) is analyzed for a general equation of state (EOS). This extends previous results restricted to the polytropic case. Special classes of EOS are identified for both spacelike and null foliations, which lead to explicit inversion of the state vector and computational savings. In a lightlike foliation, the commonly used polytropic EOS is included in the explicitly invertible cases.

Conservative RHD techniques have been applied only recently to general 3D spacetime evolutions [17]. It seems worthwhile to investigate the benefits of the increased control over source term structure presented here, in situations of current interest, e.g., the study of neutron star binary coalescence.

The presented framework is uniquely suitable for strong-field simulations using

lightlike foliations of the spacetime. Lightlike foliations attached to the exterior of spacelike surfaces have been suggested as an effective way for providing global spacetime solutions [20] (for a review see [21]). The whole approach has been dubbed Cauchy-characteristic matching (CCM). Fluid evolution in the CCM context has already been investigated [22]. The RHD framework proposed here is the natural candidate for providing economical and state of the art numerical fluid evolution in both foliations, as it is form invariant with respect to the slicing. Algorithmically, the only change between spacelike/null domains would be the routine used for the recovery of primitives.

A recent demonstration of long-term stable numerical evolutions of a single black hole based on null coordinates [23] opens the possibility for applications to non-vacuum single black hole environments solely within a lightlike framework. A full implementation of the present formulation in the case of spherical symmetry has been presented [13], illustrating the ease of studying black hole growth through accretion using an ingoing null foliation. Three dimensional implementations for fixed black hole backgrounds in null coordinates are also available and results will be discussed elsewhere.

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