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Differential rotation and magnetic fields in stellar interiors

H.C. Spruit[1]

Max-Planck-Institut für Astrophysik, Postfach 1523, D-85740 Garching bei München, Germany

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Abstract. The processes contributing to the evolution of an initially weak magnetic field in a differentially rotating star are reviewed. These include rotational smoothing (akin to convective expulsion) and a list of about 5 instabilities, among them magnetorotational instability, byoyancy instability, and pinch-type instabilities. The important effects of thermal and magnetic diffusion on these instabilities are analyzed in some detail. One of the first instabilities to set in is Tayler instability. It becomes important in modifying the field configuration before magnetic buoyancy-driven instabilities set in. The evolution of an initially strong field remains a more open question, including the old problem whether dynamically stable magnetic equilibria exist in stars.

Key words: magnetohydrodynamics – instabilities – stars: magnetic fields – stars: rotation – Sun: rotation

1. Introduction

A number of different processes cause stars to rotate differentially. Stars with convective envelopes experience braking of their rotation by a magnetic stellar wind (Schatzman 1962). This torque acts on the convective envelope, and slows it down relative to the radiative core. Secondly, in an evolving star the core contracts and spins up, while the envelope expands and slows down. If core and envelope were to conserve their angular momentum separately, the core of a giant would end up rotating some 10^5 times faster than its envelope. Finally, the Eddington-Sweet circulation in a rotating star causes differential rotation by transport of angular momentum. In a steady state, such that dissipation and driving of the differential rotation balance each other, the rotation decreases outward by some 30% between the core and the surface (Zahn 1992).

Opposing this differential rotation are friction processes due to hydrodynamical instabilities or magnetohydrodynamical processes. The strength of such friction determines how much rotation can be left in the end products of stellar evolution, white dwarfs and neutron stars. If they are effective enough to maintain an approximately uniform rotation in giants, for example, the observed rotation of white dwarfs and pulsars is not a leftover of the rotation of their progenitors (Spruit and Phinney 1998, Spruit 1998). Explosion mechanisms of supernovae that rely on rotation (Bisnovatyi-Kogan 1970, LeBlanc and Wilson 1970, Meier et al. 1976, Rampp et al. 1998), and explanations of the morphology of planetary nebulae and objects like η Carinae in terms of stellar rotation (Heger and Langer 1998) also depend crucially on the degree of differential rotation that can persist in a stellar interior. The relevant hydrodynamical processes have been classified (Zahn 1974, 1983). Their net effect on angular momentum transport is somewhat uncertain and in stellar evolution calculations is usually parametrized with arbitrary adjustable coefficients.

An important test case for such parametrizations is the internal rotation of the Sun, which has been measured with remarkable precision though helioseismology (Corbard et al. 1997, Schou et al. 1998). The most striking result of these measurements is the near-uniform rotation of the radiative core of the Sun. Over a depth range of only $0.05R_\odot$, the differential rotation in the convective envelope (30% faster at the equator than at the poles) changes to a state of uniform rotation in the core. Though the measurements are less accurate in the inner core ($r/R_\odot < 0.3$) and near the rotation axis, no deviation from uniform rotation has yet been detected in the core.

It has been known for a long time that such a low degree of differential rotation is incompatible with the currently known hydrodynamic transport mechanisms of angular momentum. Circulation is ineffective for a slow rotator like the Sun. The known hydrodynamic instabilities are also unlikely to contribute, since the conditions for their occurrence are not satisfied in a core rotating as slowly and uniformly as the present Sun (Spruit et al. 1983).

It is therefore natural to look for magnetohydrodynamic mechanisms of angular momentum transport in stably stratified layers of stars. This may sound like a daunting prospect, given the reputation of magnetic fields for complexity and lack of firm results. The situation in the context of the present problem is in fact quite promising. Over the past 4 decades, a substantial MHD literature bearing on the problem has developed. In this paper I first review the various contributing processes, conclude that Tayler (pinch-type) instability is of particular relevance. Then I show that the effects of magnetic and thermal diffusion are quite important, and how they can be quantified using Acheson's (1978) dispersion relation.

1.1. statement of the problem

Suppose we start with a differentially rotating star with rotation rate $\Omega(r)$ and field configuration $\mathbf{B}_0$. We wish to know how the field and the rotation evolve in time, and to answer questions like: is the field able to make the star rotate uniformly? If so on what time scale? What kind of field configuration remains at large time? Though these questions can not be answered fully because of a few missing pieces of theory, it turns out that a fairly detailed picture can be drawn of the magnetohydrodynamics of an initially weak field in a differentially rotating star. In most of the analysis, I will assume that there is no external torque on the star, and that its internal structure is not evolving. These assumptions are made for convenience and clarity in delineating the processes involved; they can be easily relaxed.

A number of different MHD processes are involved in the problem stated. I introduce them by starting with a number of unrealistic simplifying assumptions and then relaxing these. The evolution of the field and the rotation depends on the strength of the initial field $\mathbf{B}_0$. In most of the following, I assume the initial field to be weak, in the sense that the Alfvén travel time through the stars is much longer than the rotation period. In terms of the Alfvén frequency Ω_A,

$$\Omega_\mathrm{A} = \frac{\bar{B}}{R(4\pi\bar{\rho})^{1/2}}, \tag{1}$$

where $\bar{\rho}$ is the mean density and $\bar{B}$ a mean field strength, the assumption is that $\Omega_\mathrm{A} \ll \Omega$. For the current Sun, this is satisfied if the field is small compared to a megaGauss, which in all likelihood is the case. For early type stars, with their larger rotation rates, the assumption is even more likely to be justified. It is less well justified, however, in the cores of giants, if these corotate with their envelopes.

2. Winding-up of weak fields (no diffusion, v_ϕ only)

2.1. winding-up

Start with ignoring magnetic diffusion, so that the induction equation is

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{2}$$

and by assuming that the only motions to be considered are axisymmetric, purely azimuthal motions,

$$v_\phi = r\Omega(r, \theta). \tag{3}$$

Thus we are ignoring for the moment the motions in the (r, θ) plane due, for example, to instabilities. Also ignored are viscosity and thermal diffusion. Consider first the case of an *axisymmetric* magnetic field (aligned with the rotation axis). In the usual way, an axisymmetric field $\mathbf{B}$ can be written in terms of a stream function ψ:

$$B_r = \frac{1}{r^2 \sin\theta}\partial_\theta \psi, \qquad B_\theta = -\frac{1}{r\sin\theta}\partial_r \psi. \tag{4}$$

This function is constant along field lines. Each value of ψ labels a different magnetic surface (surface generated by rotating a field line around the axis).

The induction equation then has the components

$$\partial_t B_r = \partial_t B_\theta = 0, \qquad \partial_t B_\phi = r\mathbf{B}_\mathrm{p} \cdot \nabla\Omega. \tag{5}$$

Under the assumptions made the poloidal field component $\mathbf{B}_\mathrm{p} = (B_r, B_\theta, 0)$ does not change in time, and the equation for B_ϕ can be integrated in time:

$$B_\phi = N\mathbf{n} \cdot \mathbf{B}_\mathrm{p}, \tag{6}$$

where

$$N(r, \theta, t) = \int_t r|\nabla\Omega| \tag{7}$$

is (modulo a factor 2π), the number of 'differential turns', or the rotational displacement, and

$$\mathbf{n} = \nabla\Omega/|\nabla\Omega| \tag{8}$$

is a unit vector in the direction of the gradient of Ω. The torque per unit surface area,

$$\tau = r\sin\theta B_r B_\phi/4\pi \tag{9}$$

increases linearly with the displacement N, hence the torque causes the rotation to execute a harmonic oscillation. Under the assumptions made, the oscillation remains linear even though the displacement can be very large. This is because of the linear behavior of a pure Alfvén wave at arbitrary amplitude. The oscillation is, in fact, an Alfvén wave traveling along the magnetic surface generated by the poloidal field lines with a given value of the stream function ψ. Its period is the Alfvén travel time $\int \mathrm{d}s/V_\mathrm{Ap}$ along this field line, where V_Ap is the Alfvén speed based on the poloidal field strength (Mestel 1953).

Considering the evolution of the field on a given time scale t_0 (for example the age of the star since its formation), we can define a critical field strength such that the oscillation period $t_\mathrm{A} = R/v_\mathrm{A}$ equals t:

$$B_\mathrm{M} = (4\pi\rho)^{1/2} R/t_0, \tag{10}$$

where ρ is a typical density in the star, and R its radius. For example, take for $t_0 \approx 10^{17}$s the age of the present Sun, and for ρ its mean density ≈ 1. Then $B_\mathrm{M} \approx 2\mu$G (Mestel 1953). That is, if the initial field strength is more than a μGauss or so, the field eventually gets wound up to such an extent that Lorentz forces start affecting the internal rotation of the Sun.

2.2. phase mixing

If the initial field is large compared with $B_{\rm M}$, not a very strong condition, the rotation on each magnetic surface oscillates around a mean value given by the total angular momentum on this surface. Since Alfvén waves do not couple across magnetic surfaces, neighboring surfaces oscillate independently, have different periods, and the oscillations on them increasingly get out of phase with each other. The distance between points with oscillation phase differing by π, say, decreases as $1/t$. After a finite time, this distance becomes short enough that magnetic diffusion starts becoming important. Thus, it is necessary to include magnetic diffusion into the picture.

3. Winding up and oscillation with magnetic diffusion (v_ϕ only)

3.1. Rotational smoothing

We now relax the asumption that the initial field is axisymmetric, and also include magnetic diffusion. The induction equation is then

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla^2 \mathbf{B}, \tag{11}$$

where for simplicity I have assumed that the magnetic diffusivity is independent of position. This is a good approximation, since the diffusion effects encountered here are effective only on small length scales. The poloidal field can be decomposed into two components

$$\mathbf{B}_{\rm p} = \mathbf{B}_{\rm a} + \mathbf{B}_{\rm n}, \tag{12}$$

where $\mathbf{B}_{\rm a}$ is the azimuthal average of $\mathbf{B}$, i.e. $\partial_\phi \mathbf{B}_{\rm a} = 0$, and $\mathbf{B}_{\rm n}$ the nonaxisymmetric part, whose azimuthal average $\langle \mathbf{B}_{\rm n} \rangle_\phi$ vanishes. Let these components initially be of similar strength.

If the initial field is weak, the field lines of the initial poloidal field $\mathbf{B}_{\rm p0}$ are wound up tightly before the restoring Lorentz forces become effective. Let

$$q \equiv r|\nabla\Omega|/\Omega \tag{13}$$

be a dimensionless measure of the local rate of differential rotation. If the initial poloidal field is smooth and varying on a length scale of the order of the radius of the star, $q \sim O(1)$. Since the induction equation is linear, the winding up process can be considered separately for the components $\mathbf{B}_{\rm a}$ and $\mathbf{B}_{\rm n}$. Since the azimuthal average of $\mathbf{B}_{\rm n}$ vanishes, it has opposite polarities as a function of azimuth at any given (r, θ), and the azimuthal angle between opposite polarities is π or less. Points with field of opposite polarity at the same θ but separated by a small distance l in the direction of the rotation gradient, will be brought to the same azimuthal angle after a time t given by:

$$t\Omega q l \approx \pi r. \tag{14}$$

The time scale on which these opposite polarities will cancel each other by magnetic diffusion η is

$$t_{\rm d} = l^2/\eta. \tag{15}$$

This cancellation leads to a decay of the poloidal field, governed by the approximate equation

$$\partial_t \ln B_{\rm n} \approx 1/t_{\rm d} = \eta \left(\frac{t\Omega q}{\pi r}\right)^2, \tag{16}$$

which integrates to

$$B_{\rm n} = B_{\rm n0} e^{-(t/t_\Omega)^3}, \tag{17}$$

where the *rotational smoothing time* is

$$t_\Omega = \left(\frac{3r^2\pi^2}{\eta\Omega^2 q^2}\right)^{1/3}. \tag{18}$$

Because of the steep dependence on t, the nonaxisymmetric component of the poloidal field effectively disappears after only a few times t_Ω.

This process has been studied extensively in the context of the kinematic evolution of magnetic fields in convection, and is called convective expulsion there (Zeldovich 1956, Parker 1963, Weiss 1966). In the present context of differentially rotating stars, it was studied by Rädler (1980, 1986). The remarkable effect of differential rotation on a weak field is thus to *expell the nonaxisymmetric field components* from the star on a finite time scale. For example, for an initial Sun rotating at $\Omega_0 = 3\,10^{-5}$ (ten times the present-day rate), $q \approx 1$, with a diffusivity $\eta \approx 10^3$, t_Ω is only $\approx$ 100yr. If nothing else were to happen, the net result of differential rotation would be to make the field axisymmetric on a time scale which is very short compared to its life time.

This, however, applies only if the initial field is sufficiently weak, so that the winding-up can be treated as a kinematic process. A field is weak in this context if magnetic torques, which affect the differential rotation, do not become effective during the rotational smoothing process. This is the case if the initial Alfvén travel time

$$t_{\rm A0} = r(4\pi\rho)^{1/2}/B_0 \tag{19}$$

satisfies

$$t_{\rm A0} > t_\Omega, \tag{20}$$

which translates into a condition on the initial field strength:

$$B_0 < B_1, \tag{21}$$

where

$$B_1 = r(4\pi\rho)^{1/2} \left(\frac{\eta\Omega^2 q^2}{3r^2\pi^2}\right)^{1/3}. \tag{22}$$

For the same initial Sun, $B_1 \approx$ 30G. In terms of the corresponding Alfvén frequency:

$$\frac{\Omega_{\rm A1}}{\Omega} = \left(\frac{q^2}{3\pi^2}\right)^{1/3} \left(\frac{\eta}{\Omega r^2}\right)^{1/3}. \tag{23}$$

If the poloidal field satisfies (21), the nonaxisymmetric field component is smoothed out on the time scale (18), after which there remains an axisymmetric field (Rädler, 1986). For this to be the case, the initial Alfvén travel time has to be long compared with the rotation period by a factor $(\Omega r^2/\eta)^{1/3}$. If the initial field is larger than (22), rotational smoothing does not happen and instead the differential rotation is damped out by the process of phase mixing (all of this still under the assumption that v_ϕ is the only velocity component).

3.2. Phase mixing

Assume first that condition (21) is satisfied, so that rotational smoothing is effective. How does the field then evolve after being axisymmetrized? Since the smoothing process is linear, the axisymmetric and nonaxisymmetric field components evolve independently. The symmetric component continues to be wrapped up after the nonaxisymmetric component has been smoothed away. Eventually, if time permits, the azimuthal field will become strong enough to oppose the wrapping. Phase-mixing then starts acting. Oscillations on neighboring magnetic surfaces get out of phase, and the length scales in the Alfvén wave field become increasingly smaller, until magnetic diffusion becomes effective and damps out the wave [Spruit 1987, Roxburgh 1987 (private communication)]. If the Alfvén travel time varies through the star by a factor $q_\mathrm{A} = \Delta t_\mathrm{A}/t_\mathrm{A}$, the phase difference between magnetic surfaces separated by a distance l is of the order $\Omega_\mathrm{A} t q_\mathrm{A} l/r$. After a time t this phase difference reaches a value of π if the length scale l is

$$\frac{l}{r} = \frac{\pi}{q_\mathrm{A}\Omega_\mathrm{A} t}. \tag{24}$$

The magnetic diffusion time on this length scale is $t_\mathrm{d} = l^2/\eta$. The wave amplitude A gets damped by magnetic diffusion on this time scale,

$$\partial_t \ln A = 1/t_\mathrm{d} = \eta \left(\frac{t\omega_\mathrm{A} q_\mathrm{A}}{\pi r}\right)^2. \tag{25}$$

This is the same equation as in the case of rotational smoothing, but with the wave frequency ω_A replacing the rotation rate Ω. The wave amplitude decays like

$$A = A_0 e^{-(t/t_\mathrm{p})^3}, \tag{26}$$

where the *phase mixing time scale* t_p is

$$t_\mathrm{p} = \left(\frac{3\pi^2 r^2}{\eta \Omega_\mathrm{A}^2 q_\mathrm{A}^2}\right)^{1/3}. \tag{27}$$

When the oscillations have damped out, the rotation rate is constant on each magnetic surface:

$$\Omega = f(\psi), \qquad (t \gtrsim t_\mathrm{p}). \tag{28}$$

Expression (27) for the time scale is very similar to that for rotational smoothing (18), but its consequences are rather different. Rotational smoothing removes the nonaxisymmetric components of the magnetic field, phase mixing damps out the variations of Ω along magnetic surfaces. Detailed calculations of this phase mixing process have been made by Charboneau and McGregor (1993) and Sakurai (1995).

3.3. Stronger initial fields, and interim conclusions

If $B_0 > B_1$ (eq. 22), torques become effective before rotational smoothing can make the field axisymmetric. If the initial field was a not axisymmetric, it will therefore stay nonaxisymmetric. Such a field still has magnetic surfaces however. Alfvén waves and the slow mode continuum travel on these surfaces (Goossens et al. 1985), causing phase mixing just as in the axisymmetric case. These waves will damp out on the time scale t_p. The final field will then be similar to the initial field. But what is the state of rotation at the end of the process? Since the magnetic surfaces are not axisymmetric, and differential motions are damped on these surfaces, the stationary state is one of *uniform rotation*.

This completes the discussion for the case when only the azimuthal equation of motion is taken into account. Depending on the initial field strength, the final state in this picture is either a uniformly rotating magnetic star with a nonaxisymmetric field, or a differentially rotating star with an axisymmetric field, with rotation constant on magnetic surfaces. This is only a preliminary scenario, of course, since we have allowed only for purely azimuthal motions. The axisymmetric poloidal end-state, if it ever were to materialize, would evolve further by magnetic instabilities.

4. Magnetic instabilities

The fluid motions, taken to be purely azimuthal in the above, are now unconstrained, and the evolution of the field becomes much more interesting since magnetic instabilities can take place.

Consider first the case where the initial field is weak, such that differential rotation has time to produce a predominantly azimuthal field by rotational smoothing (section 3.1). Stronger initial fields are taken up again in section 7.

4.1. Magnetic shear instability

In a differentially rotating object with angular velocity decreasing outward, a weak field is generically unstable to magnetorotational instability (Velikhov 1959, Chandrasekhar 1960, Fricke 1969, Acheson 1978, Balbus and Hawley 1991, 1992, Balbus 1995). This is a linear instability in which the magnetic field mediates the release of free energy in differential rotation. The growing magnetic perturbations couple the fluid at different radii, the differential rotation acting on them converts the free energy into magnetic and kinetic energy on smaller scales. The instability acts when the rotation rate decreases, but angular momentum increases outward. In other cases, i.e. when angular momentum (J) decreases, or angular velocity increases outward, ordinary hydrodynamic shear instabilities are known to exist. These cases are less relevant for differentially rotating stars or accretion disks, however. Whether purely *hydrodynamic* instabilities exist for Ω decreasing and J increasing with radius is still controversial. The possibility that this case might actually be hydrodynamically *stable* should be taken serious, however, given that instabilities have so far been found neither theoretically, nor numerically, nor experimentally. A physical argument for stability, not constituting a proof, has been given by Balbus et al. (1996).

Where magnetic shear instability is present, it generates a magnetic form of turbulence. In stars, a strongly stabilizing factor is the stratification. Except in regions close to convective instability, this limits the instability to cases with strong differential rotation, and a rotation rate near the maximum value. Apart from these extreme situations, magnetic shear instability is limited to displacements on horizontal surfaces which redistribute angular momentum over such a surface if the rotation rate decreases with cylindrical radius (Kato 1992, Balbus 1995).

On sufficiently small length scales, however, thermal diffusion can strongly reduce the stabilizing buoyancy effect of the thermal stratification. As in the case of purely hydrodynamic instabilities (Zahn 1974, 1983), one would therefore expect that *i)* linear magnetic shear instability should reappear at large wavenumbers when thermal diffusion is taken into account, and *ii)* angular momentum transport by some form of small scale magnetic turbulence will take place. The first of these was already demonstrated to be the case in the classical and very detailed analysis by Acheson (1978). He showed that the necessary and sufficient condition for linear instability is

$$-\mathrm{d}\ln\Omega/\mathrm{d}\ln r > \frac{N^2}{2\Omega^2}\frac{\eta}{\kappa}, \tag{29}$$

where N is the buoyancy frequency and κ the thermal diffusivity. For conditions in stellar interiors, $\kappa \gg \eta$. This is because the magnetic diffusion involves the random walk of electrons, which have much shorter mean free paths than the photons that carry the heat. Thus magnetic shear instability is possible with weak rotation gradients $|\mathrm{d}\ln\Omega/\mathrm{d}\ln r| \ll 1$, provided that the rotation rate is not too slow compared with the buoyancy frequency. For a star rotating as slowly as the Sun, however, a fairly strong gradient in rotation rate is still needed for instability. For the present Sun, with $N \sim 10^3$, $\Omega = 3\,10^{-6}$, $\eta/\kappa \sim 5\,10^{-5}$, instability would require $q \sim O(1)$. This is a much larger degree of differential rotation than allowed by current observations (Schou et al. 1998) of the Sun's interior rotation.

How much angular momentum transport this is likely to produce is not certain at the moment. The issue is discussed in Spruit and Balbus (1998). Several lines of argument presented there suggest that a modestly effective transport of angular momentum is possible. A simple, perhaps convincing, argument starts by assuming that the magnetic turbulence has the effect of enhancing the magnetic diffusion of large scale fields. The amplitude at which this turbulence saturates is then found by requiring that instability is just possible according to condition (29), with the magnetic diffusivity η replaced by an effective diffusivity η_e, from which one gets

$$\eta_e = 2q\Omega^2\kappa/N^2. \tag{30}$$

The fluid motions in the turbulence are of slow-mode and Alfvénic type, so that the kinetic energy is of the same order as the energy in the magnetic field. If ν_e is the effective viscosity due to the fluid motions, the magnetic Prandtl number $P_e = \nu_e/\eta_e$ is then of order unity. Numerical simulations suggest $P_e \approx 0.1$ (Hawley et al. 1995, Brandenburg et al. 1995, Matsumoto et al. 1995). With the assumptions made, (29) now yields the effective viscosity expected:

$$\nu_e = 2P_e q\kappa\frac{\Omega^2}{N^2}. \tag{31}$$

Since the induction equation is linear, the winding-up, rotational smoothing and phase mixing processes discussed above all just proceed as before, the only difference being that the magnetic diffusion has to be replaced by the effective value (30). Quantitatively, the difference is not very large. For the initial Sun assumed above, the effective diffusivity in the presence of magnetic shear instability would have been only 10–100 times larger than the microscopic value. With this effective diffusivity, the rotational smoothing time (18) decreases only by a factor of order unity.

The net effect of magnetic shear instability therefore may be small. Even though the instability is not suppressed by stable thermal stratification, its effects are modest. This conclusion, however, is preliminary, as it depends on an estimate of its nonlinear development. Numerical simulations should be able to settle this question more definitively.

4.2. Instabilities of an azimuthal magnetic field

During the rotational smoothing and phase mixing processes in an initially weak field, the azimuthal field is much larger than the poloidal component. An azimuthal field has its own instabilities. Magnetic shear instability operate also on such an azimuthal field, but is more properly regarded as a form of shear instability, extracting its energy from differential rotation (as discussed above). Other forms of instability exist, which draw their energy from the magnetic field itself. They are of two distinct types. Perhaps the most well-known is magnetic buoyancy or Parker instability (Newcomb xx, Parker 1966), due to a vertical gradient in the field strength. This is discussed in section 5.3. The second is a pinch-type instability.

5. Tayler instability

I start with the second form of instability, which derives its energy from (nearly) horizontal interchanges. The classical result in this context is that of Tayler (1973), who considered adiabatic perturbations ($\eta = \kappa = 0$) in a stratified, nonrotating star. By an energy method he showed that *every purely azimuthal field* $B_\phi(r, \theta)$ in a stably stratified (nonrotating) star is *unstable on* *an Alfvén time scale*, no matter how weak the magnetic field. This form of instability is closely related to many forms of instability in pinches and torus configurations (e.g. Tayler 1957). Its behavior under the strongly stratified conditions in stellar interiors is sufficiently distinct, however, that I find it convenient to refer to it by the separate name of 'Tayler instability'.

5.1. Previous results

The most unstable motions are $m = 0$ and $m = 1$ modes with nearly horizontal displacements. The displacements take place at nearly constant total pressure, i.e. the Eulerian perturbation $\delta(P + B^2/8\pi)$ vanishes. They are essentially local in the r and θ directions: when the instability condition is satisfied at a point (r, θ), unstable perturbations can be found that are confined to a small neighborhood of this point. In spherical coordinates (Goossens et al. 1981; Tayler's was derived in cylindrical coordinates), the necessary and sufficient conditions for instability are

$$\partial_\theta \ln(B^2 \sin\theta \cos\theta) > 0, \qquad (m = 1) \tag{32}$$

$$\cos\theta\, \partial_\theta \ln(B^2/\sin^2\theta) > 0, \qquad (m = 0) \tag{33}$$

From the induction equation it is evident that azimuthal fields created by winding up of a poloidal field must have

$$B \sim \theta \qquad (\theta \ll 1) \tag{34}$$

near the pole. Hence there is always a region near the pole where condition (32) is satisfied, and the field is unstable. Away from the pole, the most unstable modes can be either $m = 1$ or $m = 0$. As an example, consider the field resulting from the winding up of an initially uniform field parallel to the rotation axis, when rotation is a function of r only. This field has

$$B_\phi \sim \sin\theta \cos\theta. \tag{35}$$

The region $\pi/4 < \theta < 3\pi/4$ is stable for $m = 1$ modes, the polar caps are unstable, and the axisymmetric modes are stable. The growth rate is of the order (Tayler 1973, Goossens et al. 1981)

$$\sigma \approx \Omega_\mathrm{A} = V_\mathrm{A}/r \qquad (\Omega \ll \Omega_\mathrm{A}). \tag{36}$$

The unstable displacements in Tayler instability are sketched in fig. 1.

When rotation is included, the energy method does not work, and somewhat weaker results are obtained. Nevertheless, Pitts and Tayler (1986) showed that rotation does not by itself remove the instability. There are still unstable $m = 1$ modes, though their growth rate is reduced. When rotation is rapid compared with the Alfvén travel time, as I have assumed above, the growth rate found by Pitts and Tayler is of the order

$$\sigma \approx \Omega_\mathrm{A}^2/\Omega \qquad (\Omega \gg \Omega_\mathrm{A}). \tag{37}$$

This dependence on the rotation rate is typical for instabilities of an interchange type. It represents the fact that the Coriolis force, being perpendicular to the velocity, does not enter into the energy budget of the perturbations, though it can still affect the shape of the unstable modes and their growth rates.

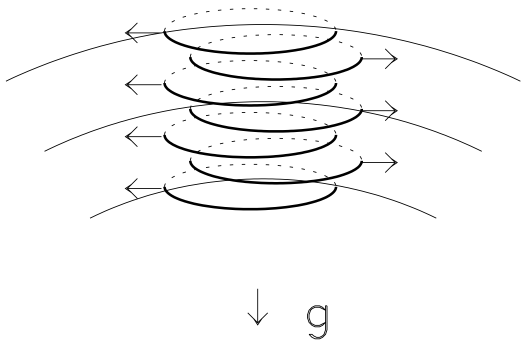

Fig. 1. Unstable displacements in an azimuthal field near the pole. Shown is the $m = 1$ mode, which occurs under the widest range of conditions. The displacements are along horizontal surfaces (indicated by arcs)

The growth rate (37) is still fast compared with the rotational smoothing and phase mixing time scales, so that the effects of the instability are likely to significantly change the evolution of an initially weak field as sketched in section 2.

5.2. Effects of rotation and diffusion

The published results deal mostly with the stability and growth rates considering the star as a whole, using an energy method. Since Tayler's result shows that the essence of the instability is a local interchange-type process (see also Appendix 2 in Tayler 1973), it is natural to study the instability by a modal analysis, in a local approximation. This also has the advantage that the effects of rotation, magnetic and thermal diffusion can be included, which is not possible with an energy method.

The result of such a local approximation, for a purely azimuthal field, has been given by Acheson (1978). His dispersion relation includes the effects of rotation, viscosity, thermal and magnetic diffusion, and is rather complicated. The special case of an azimuthal field at the equator has been analysed in detail by Acheson (1978, 1979). Tayler instability, however, disappears at the equator, and shows its most characteristic behavior at the poles.

In the following I derive the properties of the instability by heuristic arguments. In the Appendix, I derive these results more systematically from the dispersion relation. I also show there that Acheson's local approximation agrees, in cases where a comparison is possible, with all results based on the more rigorous energy method. This establishes the validity of his local analysis for the case of Tayler instability. The Appendix also discusses a few subtleties that are not captured by the heuristic derivations.

A useful simplification is to ignore viscosity. In a stellar interior in which radiation is not the dominant source of viscosity, it is smaller than the next larger diffusion process, magnetic diffusion, by 1–2 orders of magnitude[1]. Use cylindrical coordinates (ϖ, ϕ, z) at the pole, where the rotation axis is in the z direction. The magnetic field is $\mathbf{B} = (0, B(\varpi), 0)$. Define an Alfvén frequency $\omega_{\rm A}$:

$$\omega_{\rm A} = \frac{B}{(4\pi\rho)^{1/2}\varpi}, \tag{38}$$

(which is a function of ϖ). Since the analysis is local, the field is characterized completely by two numbers, the local value B of the field strength, and its radial gradient

$$p \equiv {\rm d}\ln B/{\rm d}\ln\varpi. \tag{39}$$

The vertical gradient of the field is neglected here. Instabilities associated with such a gradient are discussed below in section 5.3. The rotation rate is assumed to be uniform in the present analysis. A gradient ${\rm d}\Omega/{\rm d}z$ is present and causes its own forms of instability, but these have been covered already above in the discussion of magnetic shear instability.

There are several different time scales involved in the problem. The largest frequency is the buoyancy frequency N, reflecting the strong effect of stratification. The next lower frequency is the rotation rate Ω. For the weak fields that turn out to be most relevant, the Alfvén frequency $\omega_{\rm A}$ is small compared to Ω. In this case we have the ordering

$$N \gg \Omega \gg \omega_{\rm A}. \tag{40}$$

The perturbations are of the form

$$e^{i(lr+m\phi+nz)+\sigma t}, \tag{41}$$

and are essentially incompressive[2], div$\mathbf{v}$ = 0. In the absence of diffusion ($\kappa = \eta = 0$), the growth rate of the instability is maximized for $n \to 0$ (Pitts and Tayler 1986). In this limit, the instability is essentially confined to horizontal surfaces. The growth rate, which in the absence of rotation is of the order $\omega_{\rm A}$, is reduced by the Coriolis forces to a value of the order

$$\sigma \sim \omega_{\rm A}^2/\Omega, \qquad (\Omega \gg \omega_{\rm A}) \tag{42}$$

but the instability condition itself does not depend on Ω. This behavior is generic for interchange-type instabilities in the presence of rotation, as discussed above. The instability conditions are (Tayler 1957, see Appendix)

$$p > \frac{m^2}{2} - 1 \quad (m \neq 0), \qquad \text{and} \qquad p > 1 \quad (m = 0). \tag{43}$$

The $m = 1$ mode thus occurs under the widest range of conditions (usually, its growth rate is also largest if several modes are unstable). In the following, I restrict the discussion to $m = 1$.

5.2.1. Magnetic diffusion only

Consider first the case where magnetic diffusion is included but thermal diffusion is neglected. This would be appropriate for situations where the dominant contribution to the buoyancy frequency is due to a composition gradient. In order to avoid doing work against the stable stratification, the unstable displacements must be nearly horizontal, $v_z/v_{\rm p} \sim l/n \ll 1$. For displacements of amplitude ξ, the work done per unit mass against the stable stratification is $\frac{1}{2}\xi^2(l/n)^2N^2$. The energy gained from the field configuration is $\frac{1}{2}\omega_{\rm A}^2\xi^2$. For instability, the field must be strong enough, such that $\omega_{\rm A}^2 > (l^2/n^2)N^2$. For a given vertical wavenumber, this is most easily satisfied at the longest possible horizontal wavelength, which is of the order r, the (spherical) radius. The vertical wavenumber thus has to satisfy

$$n^2 > \frac{N^2}{\omega_{\rm A}^2 r^2}. \tag{44}$$

At large wavenumbers, however, diffusion starts affecting the perturbations. The condition that the rate at which they decay by magnetic diffusion does not exceed the growth rate yields

$$n^2\eta < \sigma, \tag{45}$$

where σ is the growth rate in the absence of stratification and diffusion. For our ordering $\Omega \gg \omega_{\rm A}$, σ is given by (42). Wavenumbers for which both conditions (44,45) are satisfied then exist only if $p > -\frac{1}{2}$ and

$$\omega_{\rm A} > \Omega(\frac{N}{\Omega})^{1/2}(\frac{\eta}{r^2\Omega})^{1/4}. \qquad (\Omega \gg \omega_{\rm A},\ \kappa = 0) \tag{46}$$

This condition applies to the most unstable mode, $m = 1$, and is always satisfied in some region near the poles. For example, in the outer core of the present Sun ($r \sim 5\,10^{10}$, $N \sim 10^{-3}$, $\eta \sim 2\,10^3$, $\kappa \sim 4\,10^7$, $\Omega = 3\,10^{-6}$), the minimum azimuthal field strength needed for instability, $B = \omega_{\rm A} r(4\pi\rho)^{1/2}$, would be of the order 1000G.

5.2.2. Magnetic and thermal diffusion

Condition (46) overestimates the field strength required for instability if the stratification is due to a thermal gradient. At the short wavelengths where magnetic diffusion plays a role, thermal diffusion strongly reduces the stabilizing temperature perturbations. As in the case of hydrodynamic instabilities in a stable thermal stratification (Zahn 1974), this effect can be taken into account by replacing N with an effective buoyancy frequency $\tilde{N}$,

$$\tilde{N}^2 = N^2/(1 + \tau/\tau_{\rm T}), \tag{47}$$

where $\tau_{\rm T} = (n^2\kappa)^{-1}$ is the thermal diffusion time at wavenumber n, and τ the time scale of the process under consideration. In our case τ is the adiabatic instability time scale σ^{-1} (eq. 42). For these time scales, the temperature perturbations are reduced by the factor in brackets in (47). Since $\kappa \gg \eta$, this factor is

[1] The effects of viscosity may need to be reconsidered, however, for the interiors of massive stars, where radiation pressure is important.

[2] Small expansions and contractions occur in order to make the total pressure perturbation vanish for the most unstable modes. These disappear in the limit $v_{\rm A}/c_{\rm s} \approx \omega_{\rm A}/N \ll 1$.

approximately $\tau/\tau_{\rm T}$. The same argument as that leading to (46) then yields as conditions for instability (cf. Appendix eq A29) $p > -\frac{1}{2}$ and

$$\frac{\omega_{\rm A}}{\Omega} > \left(\frac{N}{\Omega}\right)^{1/2} \left(\frac{\eta}{\kappa}\right)^{1/4} \left(\frac{\eta}{r^2 N}\right)^{1/4}. \ (\Omega \gg \omega_{\rm A},\ \kappa \gg \eta) \quad (48)$$

With this condition the critical field for instability in the present Sun is now about 100G, and the typical growth time (from 42) is of the order 10^5 yr (for conditions significantly exceeding marginal). The wavenumber at marginal stability is $nr = 10^3$, corresponding to a wavelength of 3000km.

5.2.3. Effect of a composition gradient

The situation is more complicated if the stratification is due to the combined effects of a thermal gradient and a gradient in composition. If μ is the mean atomic weight per particle, the buoyancy frequency is given by (e.g. Kippenhahn and Weigert, 1990):

$$N^2 = N_{\rm T}^2 + N_\mu^2, \quad (49)$$

where the thermal and compositional contributions $N_{\rm T}$ and M_μ are given by

$$N_{\rm T}^2 = \frac{g\delta}{H}(\nabla - \nabla_{\rm a}); \qquad N_\mu^2 = \frac{g\phi}{H}\nabla_\mu \quad (50)$$

with

$$\nabla = \frac{{\rm d}\ln T}{{\rm d}\ln P}, \quad \nabla_{\rm a} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{S,\mu}, \quad \nabla_\mu = \frac{{\rm d}\ln\mu}{{\rm d}\ln P}. \quad (51)$$

$$\delta = \left(\frac{\partial\ln\rho}{\partial\ln T}\right)_{P,\mu}, \qquad \phi = \left(\frac{\partial\ln\rho}{\partial\ln\mu}\right)_{P,T}. \quad (52)$$

Here straight derivatives measure the variation of the physical variables with depth in the stratification, partials are thermodynamic derivatives of the equation of state, and S is the entropy. For an ideal gas, ($P = \rho\mathcal{R}T/\mu$), $\phi = \delta = 1$, hence

$$N_{\rm T}^2 = \frac{g}{H}(\nabla - \nabla_{\rm a}); \qquad N_\mu^2 = g\frac{{\rm d}\ln\mu}{{\rm d}z}. \quad (53)$$

For adiabatic perturbations, the buoyancy frequency in the formulas above is just replaced by (50). For nonadiabatic perturbations, we need to take into account that inhomogeneities in composition and temperature diffuse at different rates. The diffusivity κ_μ of inhomogeneities in μ is of the same order as the viscosity, since both quantities scale with the mean free path and velocity of the ions. We can then neglect κ_μ compared with the magnetic diffusivity. The same arguments as those leading to eq (46) and (48) then gives the instability conditions $p > -\frac{1}{2}$ and

$$\frac{\omega_{\rm A}}{\Omega} > \left(\frac{N_{\rm T}^2}{\Omega^2}\frac{\eta}{\kappa} + \frac{N_\mu^2}{\Omega^2}\right)^{1/4} \left(\frac{\eta}{r^2\Omega}\right)^{1/4}. \quad (54)$$

$(\Omega \gg \omega_{\rm A},\ \kappa \gg \eta)$

5.3. *Magnetic buoyancy instability*

If the magnetic field strength increases in the direction of gravity, the gas is supported against gravity in part by the magnetic pressure. If the gradient is sufficiently strong, the free energy in the field gradient can be released by buckling of the field lines. As with all instabilities where displacements in the vertical are necessary, the stratification provides a strong stabilizing force. Like in the other instabilities, however, this stabilizing force is reduced by thermal diffusion on sufficiently small scales. To demonstrate the properties of the instability it is sufficient to consider the situation at the equator. This case has been analysed in detail by Acheson (1978). Ignoring viscosity, the condition for instability is (Acheson's eq. 7.27):

$$-\frac{\Omega^2}{\omega_{\rm A}^2}\frac{{\rm d}\ln\Omega}{{\rm d}\ln r} - \left(\frac{r}{H} - 2\right)\frac{{\rm d}\ln B}{{\rm d}\ln r} > \frac{\eta}{\kappa}\frac{\gamma N^2}{\omega_{\rm A}^2}, \quad (55)$$

where $\omega_{\rm A} = V_{\rm A}/r$, H the pressure scale height, and γ the ratio of specific heats. The first term describes magnetic shear instability, discussed above in section 4.1. By assuming uniform rotation, we get:

$$-\left(\frac{r}{H} - 2\right)p > \frac{\eta}{\kappa}\frac{\gamma N^2}{\omega_{\rm A}^2}, \quad (56)$$

where $p = {\rm d}\ln B/{\rm d}\ln r$. This condition is independent of the rotation rate. The growth rates, however, are reduced by rapid rotation. Compared with the nonrotating case, they are smaller by a factor σ/Ω, where σ is the growth rate in the absence of rotation.

Near the center of the star, $r \to 0$ and $H \to \infty$, so that the second term in the bracket dominates. Instability is then possible if the field strength *increases* outward. This is obviously not the normal buoyancy instability. It is, in fact, the same pinch-type instability as the Tayler instability that appears near the poles, and is associated there with the horizontal gradient of the field strength. These have been discussed in section 5. The directions of the rotation axis and stratification are different in the present case, but in the absence of rotation and gravity the instability would be the same.

Hence we can concentrate on here the case $H \ll r$. In terms of the Alfvén frequency, the instability condition can be written as

$$\omega_{\rm A}^2 > \frac{\gamma}{-p}\frac{\eta}{\kappa}\frac{H}{r}N^2. \quad (57)$$

For a smooth field gradient, $-p \sim 1$, the critical Alfvén frequency for instability is of the order (subscript $_{\rm b}$ for buoyancy):

$$\omega_{\rm Ab} \approx (\frac{\eta H}{\kappa r})^{1/2}N. \quad (58)$$

Comparing this with condition (48) for diffusive Tayler instability, with its critical Alfvén frequency $\omega_{\rm AT}$, we have

$$\frac{\omega_{\rm Ab}}{\omega_{\rm AT}} \approx (\frac{N}{\Omega})^{1/2}(\frac{\eta}{\kappa})^{1/4}(\frac{H}{r})^{1/2}(\frac{\Omega r^2}{\eta})^{1/4}. \quad (59)$$

This is minimized for the largest possible rotation rate, $\Omega \sim N$. The dominant factor is the last one, which is quite large since the magnetic diffusivity is so small for stellar length and time scales. One finds that $\omega_{\rm Ab}/\omega_{\rm AT} \gg 1$ for main sequence stars, white dwarfs and neutron stars. For the present Sun, $\omega_{\rm Ab}/\omega_{\rm AT} \approx 10^3$.

Buoyancy instability thus requires much stronger fields than Tayler instability. A field wound up by differential rotation into an azimuthal field therefore becomes unstable to Tayler instability first. If Tayler instability is able to limit the growth of the toroidal field, a slowly wound-up field will settle at a value near the marginal conditions for Tayler instability, and it is unlikely that buoyancy instability will become important.

The stability conditions discussed are summarized in Fig. 2

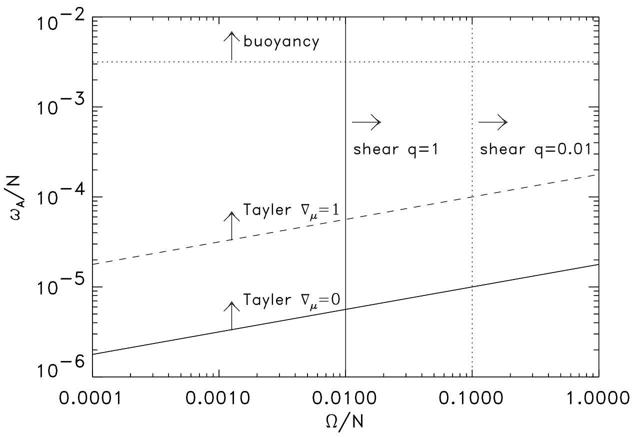

Fig. 2. Stability conditions for an azimuthal magnetic field as functions of field strength and rotation rate. $\omega_{\rm A}$ is the Alfvén frequency $V_{\rm A}/r$, N the buoyancy frequency, q is the differential rotation $\mathrm{d}\ln\Omega \mathrm{d}\ln r$. Assumed stellar parameters are $\eta/\kappa = 10^{-4}$, $r^2N/\eta = 10^{15}$. With increasing field strength, the first instability to appear is Tayler instability. Magnetic shear instability appears above a minimum rotation rate.

6. poloidal field instabilities

Though azimuthal fields are conceptually attractive as a natural result of differential rotation, the possibility of purely poloidal fields ($B_\phi = 0$) or general mixed poloidal-toroidal fields must also be considered as possible initial field configurations. As discussed further in the next section, few results exist on the stability of mixed poloidal-toroidal fields.

For purely poloidal fields, strong results are again available. The most important result is that, most likely, all purely poloidal fields in stars are unstable to adiabatic perturbations, in the absence of rotation. This was demonstrated by Markey and Tayler (1973, 1974). These authors considered poloidal fields in which some or all field lines in each meridional plane are closed within the star. Then on each of these planes there is (at least one) point where the field strength vanishes. Near this point the field lines are ellipses centered around the point. The configuration thus closely resembles the configuration of an azimuthal field near the pole, and one expects similar pinch-type instabilities, driven by the curvature of the field lines[3]. The situation differs in the present case in that the direction of the stable stratification is now within the plane of the field lines. Unstable displacements $\boldsymbol{\xi}$ must be nearly incompressive and close to horizontal surfaces, $\mathbf{g}\cdot\boldsymbol{\xi} = 0$. Displacements in latitude, with a small length scale in the azimuthal direction satisfy these requirements. They have the same effect as an $m = 1$ displacement in an azimuthal field near the pole. Markey and Tayer (1973,1974) find that these are indeed the most unstable ones, and that they make all poloidal fields with closed field lines unstable. The growth rate, as expected, is of the order $\omega_{\rm A}$.

A special case occurs when none of the field lines is closed inside the star. All field lines then cross the stellar surface. An example would be a uniform field inside the star, with a dipolar vacuum field outside. The results by Markey and Tayler do not apply to this case, but a simple argument (Flowers and Ruderman 1977) shows that this case is equally unstable. As a trial function for the displacements consider splitting the star in half by a plane containing the axis, and rotating one of the halves over 180°. This changes neither the thermal, nor the gravitational, nor the magnetic energy of the star. It changes the external vacuum field, however. By a suitable choice of the plane, the rotation brings fields of opposite polarity closer together on the surface, which lowers the energy of the external vacuum field. This is most easily visualized for a dipole field. The star is then analogous to a bundle of bar magnets, oriented in parallel. By splitting the bundle in two and rotating one half, the energy of the bar magnets in the field of the other half is reduced by an amount of the order $B^2V/8\pi$, where V is the volume of the star. Note that this energy change is due entirely to the external field energy, the internal field energy does not change. With this value for the change in energy, the growth rate of the instability is of the order $\omega_{\rm A}$. The process can be continued by rotations along other planes, which increases the complexity of the external field (higher multipoles), and reduces its strength. Asymptotically, the external field energy and the growth rate of the instability vanish.

6.1. Effect of rotation

For adiabatic perturbations, the effect of rotation has been studied by Pitts and Tayler (1986). The results are less complete than in the nonrotating case, since the powerful energy method fails for rotating systems. For the cases studied, however, poloidal fields were again found to be unstable. The reason for this can be visualized easily by considering the most unstable displacements of the nonrotating case. These have azimuthal wavenumber $m \gg 1$ and $\xi_\theta \gg \xi_r, \xi_\phi$. The Coriolis force on such displacements is in the azimuthal direction. On neighboring meridional planes, such that the phase of the perturbation differs by π, the Coriolis forces are opposite. These forces can

[3] Note that this is now the third instance of such instabilities. The first time was in Tayler instability at the poles, the second time in the stability of an azimuthal field near the center of a star in section 5.3.

be balanced entirely by the azimuthal pressure perturbation, so that the net effect of the Coriolis force vanishes for high-m perturbations. The instability then proceeds under the same conditions and with the same growth rate as in the nonrotating case. An explicit example of this effect has been given in Spruit and Taam(1990), where poloidal field instability was studied in the very analogous case of a uniformly rotating disk. Numerical simulations of this instability in rotating disks have been made by Stehle (1998).

One should therefore expect rotation to have even less effect on poloidal fields than it has on the azimuthal fields discussed in section 5.

6.2. Effect of magnetic and thermal diffusion

The effect of the diffusivities has not been studied for poloidal fields. Given the nature of the unstable diplacements, which are closely analogous to the $m = 1$ instabilities near the pole in an azimuthal field, we should expect the effects of the diffusivities to be qualitatively the same. In the nonrotating azimuthal field case, diffusion does not stabilize the configuration at any field strength, but only affects the growth rates. Since the effect of rotation on the unstable displacements is small for poloidal fields (see above), it is a good guess that poloidal fields will be unstable at all field strengths, even in the presence of rotation and magnetic diffusion. Demonstration of this may need more detailed study, however.

7. Initial field strong: stable equilibria?

In the above, I have assumed the initial field to be weak, so that a predominantly azimuthal field quickly develops by differential rotation. If the initial field is strong (as measured by condition 22), this is not the case. Instead, in this case, the differential rotation can be regarded as a perturbation of the field which damps out on the phase mixing time scale (27). In doing so, the field settles to a stable equilibrium state, if one exists.

This is a loose end in the story since in spite of extensive work (mainly in the 60's and 70's, see e.g. Tayler 1980, Borra et al. 1982, Mestel 1984), it is still not known how to prove the existence or absence of stable magnetic equilibria in stars, whether they rotate or not. As discussed above, all purely toroidal field configurations are *unstable* above a critical field strength given by the diffusive Tayler instability condition (48). All purely poloidal fields are likely to be unstable as well, as discussed in section 6. As argued by Mestel (1984) and Tayler (1980) this leaves the possibility that special stable configurations might exist with poloidal and toroidal fields of similar strength, but no example of such a field is known.

An indirect, and not completely compelling, argument that stable configurations are possible can be made by appealing to the conservation of magnetic helicity (Moffat, 1989, private communication). If $\mathbf{A}$ is the vector potential of $\mathbf{B}$ (with a suitable gauge), then it can be shown that the integral of $H = \mathbf{B} \cdot \mathbf{A}$ over volume is a conserved quantity in ideal MHD (i.e. in the absence of magnetic diffusion) (Woltjer 1958, see also Taylor, 1974). If a star is born with a field for which H is nonzero, then this field must reach a stable equilibrium configuration. Suppose it is not initially in a stable equilibrium. By putting in a sufficient damping mechanism such as a viscosity, we can make sure that the energy released by the instability is dissipated on some finite time scale. The final state must then be a stable equilibrium at a finite field strength, since an infinitesimal field can not have a finite helicity. This argument is unfortunately not compelling, since it is possible that unstable magnetic fields can evolve in such a way as to develop singularities (current sheets) within an effectively finite time. Once such current sheets develop, reconnection sets in and the helicity is no longer conserved.

While theory fails to give a clear answer to this long standing question (Mestel 1984), we may appeal to observations to argue that stable configurations do in fact exist in stars. The magnetic A stars and the magnetic white dwarfs have strong magnetic fields (of the order 10^4 and 10^7G, respectively), that do not change on time scales of at least decades. Since these stars are also slow rotators, the field would change on an Alfvén time scale, of the order of a year and a day, respectively, if their field configurations were not stable. The A stars have convective cores, in which a steady dynamo might possibly exist to produce the observed stable field. The magnetic white dwarfs do not have a plausible location for such a dynamo. Very cool white dwarfs may form a crystalline lattice in part of the interior, which might be able to anchor a strong field. The known magnetic white dwarfs (Schmidt and Norsworthy 1991, Liebert 1995) however, are not all of this type. We can therefore take the magnetic white dwarfs as a fairly strong argument for the existence of stable field configurations in stars.

Assuming that this is the case, initial field configurations with strength above (22) will evolve into stable configurations on the phase mixing time scale. The final field strength would depend on the degree to which reconnection has taken place during the evolution of the configuration. The star would be uniformly rotating, except for special axisymmetric field configurations aligned with the rotation axis (which are apparently not realized in the observed magnetic A stars and white dwarfs).

8. Discussion

In the above I have reviewed the known processes relevant for magnetic fields in differentially rotating stably stratified stellar interiors. This excludes dynamo processes such as are thought to occur in convective zones. In initially weak fields, (condition 21, about 30G for the Sun) the nonaxisymmetric components are smoothed out by *rotational expulsion*, producing an axisymmetric field. Differential rotation is subject to damping by phase mixing. This results in a uniformly rotating star if sufficient time is available, compared with time scale on which the internal structure evolves and compared with the spin down timescale due to external torques.

The azimuthal field that results from winding-up of an initially weak field is subject to instabilities. In the stably stratified environment of a stellar interior they are of two types. There are the Parker (or magnetic buoyancy) and Tayler (or stratificationmodified pinch-type) instabilities, both driven by the magnetic field energy in the toroidal field. They are of particular relevance for fields produced by differential rotation, through the winding-up of a weak initial field.

In addition to these instabilities driven by the free energy in the magnetic field, there is magnetic shear instability, which feeds on differential rotation. It occurs in any field configuration, and occurs already in very weak fields. Due to effects of stratification and magnetic diffusion, however, the rotation gradient has to be significant for it to occur, especially in slowy rotating stars. Poloidal fields configurations have similar instabilities.

It turns out that Tayler instability (pinch-type instability in the presence of a stabilizing stratification) is of particular relevance. In an azimuthal field, it generically occurs in a region near the pole, in the form of an $m = 1$ displacement of the field lines along horizontal surfaces. This instability is of a local interchange type, and the effects of rapid rotation and magnetic and thermal diffusion can be included in the analysis. I find that Acheson's (1978) dispersion relation can be applied to this form of instability. The results of section 5 and the Appendix show that Tayler instabilities probably are more relevant than the better known buoyancy (or Parker-) instabilities. In stars with magnetic fields that have been wound up by differential rotation, they set in at the lowest field strength.

One can wonder how complete our 'catalog' of known instabilities is. As discussed above, the situation for nearly azimuthal fields, such as would result from differential rotation, is probably quite satisfactory. The energy method used by Tayler (1973) gives necessary and sufficient conditions for the adiabatic, nonrotating case, and is thus complete for this case. Quite important is that this analysis shows that the instabilities are of a *local* nature (in the r and θ coordinates). This allows a complete study of the effects of rotation, viscosity magnetic and thermal diffusion by Acheson's (1978) approach. Barring the possibility of diffusive non-local instabilities that don't have an equivalent in the nonrotating adiabiatic case, the stability of azimuthal fields can thus be analyzed completely. The same applies essentially to purely poloidal fields, where the energy method again shows the instabilities to be local. A complete study of the effects of rotation and diffusion on these instabilities still has to be done for this case, however.

Another question is, of course, how the magnetic configuration would evolve nonlinearly under these instabilities, in particular how effective they would be at transporting angular momentum. It is conceivable that significant progress in this question can be made with numerical MHD simulations.

The stability of initially strong fields, such that the differential rotation is not strong enough to wind the field into an azimuthal configuration, is as open a question as before. The same applies to cases where the field has gone through phases of winding-up and phase mixing. In the absence of forces that continue to create differential rotation, the phase mixing process eliminates differential rotation, and leaves the field in a configuration of unkown stability. If such a field is to be stable, it is clear that it can neither be a purely poloidal nor a purely toroidal field since these are unstable on short time scales. The possibility that a mixed poloidal-toroidal configuration can be stable on long time scales can not be excluded (Tayler 1980, Mestel 1984). The magnetic white dwarfs are fairly strong observational evidence that such configurations do in fact exist.

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A. Appendix: Tayler instability with rotation and diffusion

We start with the local dispersion relation in Acheson (1978). The goals of the analysis are to show that this relation yields the correct results for Tayler instability in the known cases, and to extend these to include rotation, magnetic and thermal diffusion.

The perturbations are of the form

$$e^{i(l\varpi+m\phi+nz-\omega t)}. \tag{A1}$$

The field is axisymmetric and purely azimuthal. Its strength B is taken to be a function of cylindrical radius ϖ only. If B depends on z as well, additional forms of instability are possible (buoyancy or Parker instability). This is discussed in section 5.3. At the pole of a uniformly rotating star, where gravity is parallel to the rotation axis, and ignoring viscosity, the dispersion relation reduces to

$$\begin{aligned}&\frac{V_{\rm A}^2}{\varpi}\left[2\Omega m+\omega\left\{2+\frac{l}{n}\frac{\varpi}{H}\frac{\omega+i\kappa s^2}{\omega\gamma+i\kappa s^2}\right\}\right]\left[\partial_h F-\frac{\omega}{\omega\gamma+i\kappa s^2}\partial_h E\right]\\
&+\left[\frac{s^2}{n^2}\left(\omega-\frac{\omega_{\rm A}^2}{\omega+i\eta s^2}\right)+\frac{l}{n}\frac{g}{\omega\gamma+i\kappa s^2}\partial_h E\right]\left[\omega(\omega+i\eta s^2)-\omega_{\rm A}^2\right.\\
&\left.+\frac{V_{\rm A}^2}{c_{\rm s}^2}\frac{\omega+i\kappa s^2}{\omega+i\kappa s^2/\gamma}\omega^2\right]-\left[2\Omega+\frac{mV_{\rm A}^2/\varpi^2}{\omega+i\eta s^2}(p+1)\right]\\
&\times\left[2\Omega\left\{\omega+i\eta s^2+\frac{V_{\rm A}^2}{c_{\rm s}^2}\frac{\omega+i\kappa s^2}{\omega+i\kappa s^2/\gamma}\omega\right\}+\right.\\
&\left.\frac{mV_{\rm A}^2}{\varpi^2}\left\{\frac{2}{\varpi}+\frac{l}{n}\frac{\varpi}{H}\frac{\omega+i\kappa s^2}{\omega\gamma+i\kappa s^2}\right\}\right]=0,\end{aligned} \tag{A2}$$

where

$$s^2=n^2+l^2, \qquad \omega_{\rm A}=mV_{\rm A}/\varpi, \tag{A3}$$

$$\partial_h=\partial_\varpi-(l/n)\partial_z, \qquad c_{\rm s}^2=\gamma P/\rho, \tag{A4}$$

$$E=\ln\frac{P}{\rho^\gamma}, \qquad F=\ln\frac{B}{\rho\varpi}, \qquad p=\frac{{\rm d}\ln B}{{\rm d}\ln\varpi}, \tag{A5}$$

and H the pressure scale height. The magnetic field strengths relevant here are weak in the sense $V_{\rm A}\ll c_{\rm s}$, i.e. the magnetic pressure is negligible compared to the gas pressure. The terms of order $v_{\rm A}/c_{\rm s}$ on the third and fourth lines can then be neglected, and the gradient of the density in the ϖ-direction can be neglected. Under the same assumption, the expected growth rates, of the order $\omega_{\rm A}$ or smaller, are small compared with the buoyancy frequency N, hence instability is possible only for $l/n\sim\omega_{\rm A}/N\ll 1$. The terms with factor l/n in the first and fourth lines can thus be neglected. The same assumptions also imply that $s/n\approx 1$. Thus we have

$$\varpi\partial_h F\approx {\rm d}\ln(B/\varpi){\rm d}\ln\varpi=p-1, \tag{A6}$$

$$\partial_h E\approx-\frac{l}{n}\partial_z\ln(P/\rho^\gamma)=-\frac{l}{n}\frac{N^2\gamma}{g}. \tag{A7}$$

The term involving E on the first line can be neglected, but not the term on the second line, since it is multiplied by g. For $\omega\sim\omega_{\rm A}$, this term is of the same order as the first term on the second line. Eq. (A2) thus reduces to

$$\begin{aligned}&\frac{V_{\rm A}^2}{\varpi^2}(2\Omega m+2\omega)(p-1)+\\
&\left[\omega-\frac{\omega_{\rm A}^2}{\omega+i\eta s^2}-\frac{l^2}{n^2}\frac{N^2}{\omega+i\kappa s^2/\gamma}\right]\left[\omega(\omega+i\eta s^2)-\omega_{\rm A}^2\right]-\\
&\left[2\Omega+\frac{mV_{\rm A}^2/\varpi^2}{\omega+i\eta s^2}(p+1)\right]\left[2\Omega(\omega+i\eta s^2)+2\frac{mV_{\rm A}^2}{\varpi^2}\right]=0.\end{aligned} \tag{A8}$$

To see that the stability conditions implied by this relation are the same, in the adiabatic case as, those found by Tayler (1973), Goossens et al. (1981), and Pitts and Tayler (1986), set $\eta=\kappa=0$, so that

$$\begin{aligned}&\frac{V_{\rm A}^2}{\varpi^2}(2\Omega\omega m+2\omega^2)(p-1)+\left[\omega^2-\omega_{\rm A}^2-\frac{l^2}{n^2}N^2\right](\omega^2-\omega_{\rm A}^2)\\
&-\left[2\Omega\omega+\frac{mV_{\rm A}^2}{\varpi^2}(p+1)\right]\left[2\Omega\omega+2\frac{mV_{\rm A}^2}{\varpi^2}\right]=0.\end{aligned} \tag{A9}$$

Consider first the case $\Omega=0$. Then (A9) is a quadratic equation in ω^2. Instability exists if there are wavenumbers for which $\omega^2<0$. The sufficient and necessary condition for instability is that the constant term in the quadratic is positive. The cases $m=0$ and $m\neq 0$ have to be considered separately. For both cases, the range of instability is maximized in the limit in which the vertical wavenumber $n\to\infty$, and the instability conditions are

$$p>1 \qquad (m=0), \qquad p>\frac{m^2}{2}-1 \qquad (m\neq 0). \tag{A10}$$

These are the conditions found by Tayler (1973) and Goossens et al. (1981), if we take into account that their results are valid on the entire sphere, whereas we have considered only the situation near the pole.

For an azimuthal field resulting from the winding up of a a radial field component by differential rotation, $B \sim \varpi$ near the pole, so that the $m = 0$ mode is marginally stable. The higher m's require a steeper field gradient than $m = 1$, hence $m = 1$ is the most unstable mode, at least in some region around the pole. From now on I ignore the $m = 0$ mode.

The case of arbitrary rotation rate is a bit complicated, so I consider only the limiting cases $\Omega = 0$ and $\Omega/\omega_A \to \infty$. For $\Omega = 0$, the dispersion relation can be made real by the substitution

$$\omega = i\sigma. \tag{A11}$$

Let $K \equiv \kappa s^2/\gamma$ and $H \equiv \eta s^2$. Multiplying (A8) by $m^2(\sigma + K)(\sigma + H)$, one gets a third degree polynomial equation in σ:

$$-\omega_A^2(p-1)\sigma(\sigma+H)(\sigma+K) + \frac{m^2}{2}[\sigma(\sigma+H)(\sigma+K)+$$
$$\omega_A^2(\sigma+K) + \frac{l^2}{n^2}N^2(\sigma+H)]$$
$$\times[\sigma(\sigma+H)+\omega_A^2] - \omega_A^2(\sigma+K)(p+1) = 0. \ (\Omega = 0) \tag{A12}$$

The system can in principle have both monotonic and oscillatory (overstable) instabilities, and eq. (A12) would have to be checked for both possibilities. For systems with real coefficients like (A12), experience from double diffusive systems shows that overstability results if the destabilizing agent diffuses faster than the stabilizing agent, while monotonic instability results if the destabilizing agent has the lower diffusivity. In the present case the destabilizing agent is the magnetic field, which has a lower diffusivity than the stabilizing thermal stratification, so we expect monotonic instability. Marginal stability then corresponds to $\sigma = 0$, and the condition for instability is that the constant term in the polynomial be negative. This yields

$$p > \frac{m^2}{2} - 1 + \frac{l^2}{n^2}\frac{\gamma N^2}{\omega_A^2}\frac{\eta}{\kappa}. \tag{A13}$$

For a smooth field gradient, $p \sim O(1)$, instability is possible only if the last term does not exceed order unity:

$$n^2 > l^2\frac{N^2}{\omega_A^2}\frac{\eta}{\kappa}. \tag{A14}$$

This can be achieved, for arbitrarily low field strength, by taking the radial wavenumber n sufficiently large. There is no critical field strength for instability. Magnetic diffusion does have an effect on the growth rates, however. Closer inspection of eq. (A12) shows that the maximum growth rate as a function of n is of the order of the adiabatic rate ω_A only if a critical field strength ω_{Ac} is exceeded, given by:

$$\omega_{Ac}^3 \approx \frac{\eta^2}{r^2\kappa}N^2, \tag{A15}$$

where we have taken $l \sim 1/r$ as the lowest possible horizontal wavenumber. If this satisfied, the radial wavenumber at maximum growth rate is of the order $n \sim (\omega_A/\eta)^{1/2}$. If the field strength is less than given by (A15), the maximum growth rate is reduced:

$$\sigma \sim \omega_A \quad (\omega_A \gg \omega_{Ac}), \qquad \sigma \sim \omega_A^2\frac{r^2}{\eta} \quad (\omega_A \ll \omega_{Ac}). \tag{A16}$$

Next consider the opposite limiting case, $\Omega \gg \omega_A$. In this limit, one finds that the frequency scales as $\omega \sim \omega_A^2/\Omega$. Writing

$$\omega = \alpha\omega_A^2/\Omega, \qquad h = \frac{\eta\Omega}{\omega_A^2}, \qquad k = \frac{\kappa\Omega}{\gamma\omega_A^2}, \tag{A17}$$

and neglecting higher order terms in ω_A/Ω, relation (A8) reduces to

$$m(p-1)(\alpha+in^2h)(\alpha+in^2k)+$$
$$\frac{m^2}{2}[\alpha+in^2k+\frac{l^2N^2}{n^2\omega_A^2}(\alpha+in^2h)]-$$
$$[2m(\alpha+in^2h)+p+1](\alpha+in^2k)[m(\alpha+in^2h)+1] = 0. \tag{A18}$$

Consider first the adiabatic case, $h = k = 0$. The dispersion relation (A18) is then quadratic in α and one finds that the necessary and sufficient condition for instability is

$$p > 1 + \frac{m^2}{2}. \qquad (m \neq 0,\ \Omega \gg \omega_A). \tag{A19}$$

This condition is significantly more restrictive than nonrotating condition (A10). For a field $B \sim \varpi$, such as would result from differential rotation near the pole, the condition predicts stability. Since the fields we envisage are just of this type, one would conclude stabilily, at least in the interesting region near the pole that critical for driving the instability. This was noted by Pitts and Tayler (1986), who also found that a sufficiently large gradient p would again be unstable in the rapidly rotating case. It turns out, as shown next, that the instability condition for the rapidly rotating case is relaxed again when the effects of thermal and magnetic diffusion are taken into account.

Returning to eq. (A18) one easily verifies that there is no direct instability (α imaginary), since the coefficients are complex due to the combined effects of rotation and diffusion. Thus we have to check the stability of oscillatory modes. The stability boundary for marginally stable oscillations is found by requiring the dispersion relation to have a solution with α real. The case with both diffusivities present is rather complicated, so I specialize further to the cases $\kappa = 0$ and $\kappa/\eta \to \infty$. The first limit is appropriate for cases where the stratification N^2 is due to a composition gradient instead of the thermal gradient, because the ions making up such a gradient diffuse only slowly. The latter case applies when composition gradients can be ignored.

For the case $k = 0$ the real and imaginary parts of (A18) yield

$$Re: \quad -2(m\alpha+1)^2 + 2 + \tfrac{m^2}{2} - (p+1) + \tfrac{m^2}{2}\tfrac{l^2N^2}{n^2\omega_A^2} + 2m^2n^4h^2 = 0,$$

$$Im: \qquad -2m^2\alpha^2 - 2m\alpha + \tfrac{m^2}{4}\tfrac{l^2N^2}{n^2\omega_{\rm A}^2} = 0, \qquad (\kappa = 0,\ \Omega \gg \omega_{\rm A})$$

Eliminating α between the real and imaginary parts yields

$$\tfrac{1}{2}[(1+\tfrac{m^2}{2}\tfrac{l^2N^2}{n^2\omega_{\rm A}^2})^{1/2}-1]^2 + \tfrac{m^2}{2} - (p+1) + 2m^2n^4h^2 = 0. \quad (A20)$$

The terms involving the wavenumber n are both positive. The last one increases monotonically with wave number, the first with the inverse of the wavenumber. Instability, if it exists, is therefore restricted to a finite range in wavenumbers. At the high wavenumber end the instability is cut short by magnetic diffusion (last term), at low wavenumbers by the stable stratification (first term). Instability exists if there are wavenumbers for which eq. (A20) has solutions. Necessary and sufficient for this to be the case is that the sum of the first and last terms be less than $p+1-m^2/2$. This yields rather complicated expressions. Instead of the exact conditions, a sufficient condition for instability which is also sufficiently accurate as a necessary condition is found by noting that $[(1+x^2)^{1/2}-1] \le x^2$, hence instability is guaranteed if there are n's for which both

$$\frac{m^2}{4}\frac{l^2N^2}{n^2\omega_{\rm A}^2} < \tfrac{1}{2}(p+1-\tfrac{m^2}{2}) \quad (A21)$$

and

$$2m^2n^4h^2 < \tfrac{1}{2}(p+1-\tfrac{m^2}{2}). \quad (A22)$$

This is possible only if the adiabatic instability condition $p+1-m^2/2>0$ is satisfied, hence magnetic diffusion restricts the range of instability (as opposed to other double diffusive systems, where diffusion can be destabilizing). Both conditions can be satisfied if both $p > \frac{m^2}{2}-1$ and

$$\omega_{\rm A}^4 > \frac{|m|^3}{a^{3/2}} l^2N^2\eta\Omega, \qquad (\kappa = 0,\ \Omega \gg \omega_{\rm A}) \quad (A23)$$

where

$$a = p+1-m^2/2. \quad (A24)$$

With $m=1$, $l \sim 1/r$, and $a \sim 1$, this is equivalent to the heuristic result of section 5.2 (eq 46).

In the opposite limit $\kappa \gg \eta$, the real and imaginary parts of (A18) yield

$$2-2(m\alpha+1)^2 + f^2 + \tfrac{m^2}{2} - (p+1) + g^2 = 0, \qquad m\alpha = -2g^2/(f^2+2g^2), \quad (A25)$$

where

$$f^2 = \frac{m^2}{2}\frac{l^2N^2}{n^2\omega_{\rm A}^2}\frac{\eta}{\kappa}, \qquad g^2 = 2m^2n^4h^2. \quad (A26)$$

The second of these implies that

$$2-2(m\alpha+1)^2 > 0. \quad (A27)$$

Necessary for instability is again that $a = p+1-m^2/2 > 0$. Supposing that this is satisfied, a sufficient for instability is that there exist wavenumbers for which

$$2-2(m\alpha+1)^2 < \frac{1}{3}a, \quad \text{and} \quad f^2 < \frac{1}{3}a, \quad \text{and} \quad g^2 < \frac{1}{3}a. \quad (A28)$$

This is the case if both $p > \frac{m^2}{2}-1$ and

$$\omega_{\rm A}^4 \gtrsim \frac{|m|^3}{a^{3/2}} l^2N^2\frac{\eta}{\kappa}\eta\Omega, \qquad (\eta \ll \kappa,\ \Omega \gg \omega_{\rm A}). \quad (A29)$$

This is a slightly imprecise condition, since I have not determined the exact numerical factor in front of the RHS. With $m=1$, $l \sim 1/r$, and $a \sim 1$, this condition is equivalent to the heuristic result of section 5.2 (eq 48).

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