The cosmological lens equation and the equivalent single-plane gravitational lens

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Abstract

The gravitational lens equation resulting from a single (non-linear) mass concentration (the main lens) plus inhomogeneities of the large-scale structure is shown to be strictly equivalent to the single-plane gravitational lens equation without the cosmological perturbations. The deflection potential (and, by applying the Poisson equation, also the mass distribution) of the equivalent single-plane lens is derived. If the main lens is described by elliptical isopotential curves plus a shear term, the equivalent single-plane lens will be of the same form. Due to the equivalence shown, the determination of the Hubble constant from time delay measurements is affected by the same mass-sheet invariance transformation as for the single-plane lens. If the lens strength is fixed (e.g., by measuring the velocity dispersion of stars in the main lens), the determination of H_0 is affected by inhomogeneous matter between us and the lens. The orientation of the mass distribution relative to the image positions is the same for the cosmological lens situation and the single-plane case. In particular this implies that cosmic shear cannot account for a misalignment of the observed galaxy orientation relative to the best-fitting lens model.

1 Introduction

Light from distant sources propagates through the matter inhomogeneities of our Universe; light bundles are deflected and distorted during their propagation (e.g., Gunn 1967a,b). This gravitational light deflection can be used to infer statistical properties of the large-scale matter distribution in the Universe, as was pointed out by Miralda-Escudé (1991), Blandford et al. (1991), Kaiser (1992, 1996), Villumsen (1996), Bernardeau, van Waerbeke & Mellier (1997), and Jain & Seljak (1997), by measuring the mean ellipticity of the images of distant galaxies; assuming that galaxies are intrinsically randomly oriented, any net ellipticity is then attributed to the propagation. Estimates of the expected effect have both been done numerically (e.g., Jaroszyński, Park & Paczyński 1990, Jaroszyński 1991, 1992, Bartelmann & Schneider 1991, Wambsganss et al. 1996) and analytically, using the linear or fully non-linear evolution of the power spectrum of density fluctuations (Seljak 1996, Jain & Seljak 1997, and references therein).

This cosmic shear acts of course also on light bundles corresponding to gravitational lens systems such as multiply-imaged QSOs. Since the cosmic shear is a stochastic process, its value in the direction of a gravitational lens system is unknown and has to be determined from lens modeling. By that – as we shall see below – the cosmic shear cannot be distinguished from locally generated shear, i.e., by a group of galaxies or a cluster close to the main lens. In addition to the shear, the large-scale matter inhomogeneities can also produce a convergence (which can have either sign). This convergence which is indistinguishable from a convergence caused by the local environment of the main lens, affects the determination of the Hubble constant (Kayser & Refsdal 1983, Gorenstein, Falco & Shapiro 1988). The primary effect here is that matter inhomogeneities between us and the lens change the angular-diameter distance to the lens, which essentially is the quantity determined from a measurement of the time delay (Narayan 1991).

Surpi, Harari & Frieman (1996) and Bar-Kana (1996, hereafter B-K96) have investigated the effect of large-scale structure on multiply imaged sources. In B-K96, the effect of cosmic shear added to a given gravitational lens was considered and shown to affect the observable image positions and flux ratios. However, since the mass model for the gravitational lens is obtained by fitting observable image properties with model predictions, the addition of a cosmic shear implies that the adopted lens model is modified such as to reproduce the observables best. Therefore if one wants to investigate the effect of cosmic shear on multiple image lens systems, the combined effect of adding cosmic shear and modifying the lens model accordingly must be studied.

The present paper considers some aspects of B-K96 in more detail. The main results to be shown are as follows: (1) B-K96 derives a lens equation including cosmic shear which is *formally* identical to the (multiple-plane) generalized quadrupole gravitational lens equation (Kovner 1987; Schneider, Ehlers & Falco 1992; hereafter SEF, Chap. 9). B-K96 shows that the analogue of the 'telescope matrix' (Kovner 1987) is symmetric if the effects of the large-scale matter inhomogeneities are considered to first order. It will be shown below that this matrix is manifestly symmetric to all orders. This is not unimportant: whereas the rms value of the cosmic shear is well below 10%, it obeys non-Gaussian statistics, and considerably higher values are probably not rare. (2) The symmetry of this matrix is then used to show that the cosmic lens equation is fully equivalent to a single-plane gravitational lens equation, and the equivalent single-plane matter distribution (or, equivalently, the single-plane deflection potential) is obtained. (3) If a family of lens models is considered where the main lens consists of a linear term (a shear matrix) plus a main lens with elliptical potential, then the equivalent single-plane lens is contained in this family of lens models. Note that this kind of lens models has been applied to observed lens systems, and it appears that an (external) shear in addition to the ellipticity of the lens galaxy is needed in most cases where enough observational constraints are available (e.g., four-image systems) – see Keeton, Kochanek & Seljak (1997), Witt & Mao (1997). (4) The observed image positions relative to the observable major axis of the matter distribution in the main lens is the same in the cosmic lens situation and in the equivalent single-plane situation, in contrast to the possibility indicated by Blandford & Kundić (1996) who suspected possible misalignment of the images with respect to the observed orientation of the galaxy.

2 Light propagation in an inhomogeneous Universe

In this section we closely follow the paper by Seitz, Schneider & Ehlers (1994; hereafter SSE); the reader is referred to this paper for details.

2.1 General propagation equations

Consider a fiducial light ray ending at the observer at redshift $z = 0$. The corresponding null geodesic is denoted by $\gamma_0^\mu(\lambda)$, where λ is an affine parameter along the ray. We choose λ such that $\lambda = 0$ at the observer, and that locally λ coincides with the proper distance. Consider a neighbouring light ray $\gamma^\mu(\lambda; \boldsymbol{\theta})$ which at the observer propagates at an angle $\boldsymbol{\theta}$ relative to the central ray. The transverse component of the separation vector $\xi^\mu(\lambda; \boldsymbol{\theta}) = \gamma^\mu(\lambda; \boldsymbol{\theta}) - \gamma_0^\mu(\lambda)$ is (essentially) a two-dimensional vector which shall be denoted by $\boldsymbol{\xi}(\lambda; \boldsymbol{\theta})$ (for an exact definition, see SSE). Provided $\boldsymbol{\theta}$ is sufficiently small, $\boldsymbol{\xi}(\lambda; \boldsymbol{\theta})$ will depend linearly on $\boldsymbol{\theta}$; it satisfies the Jacobi differential equation

$$\boldsymbol{\xi}''(\lambda; \boldsymbol{\theta}) = \mathcal{T}(\lambda)\boldsymbol{\xi}(\lambda; \boldsymbol{\theta}) \quad , \tag{2.1}$$

where $\mathcal{T}(\lambda)$ is the optical tidal matrix evaluated at the position $\gamma_0^\mu(\lambda)$ of the fiducial ray at the affine parameter λ. The optical tidal matrix $\mathcal{T}$ is symmetric and is calculated from the local curvature tensor. Specifically, the trace of $\mathcal{T}$ is determined by the local mass density, whereas the trace-free part of $\mathcal{T}$ depends on the Weyl curvature, i.e., the tidal gravitational field; see SSE for the general definition of the optical tidal matrix. A prime denotes differentiation with respect to λ. Equivalently, we can write

$$\boldsymbol{\xi}(\lambda; \boldsymbol{\theta}) = \mathcal{D}(\lambda)\boldsymbol{\theta} \quad , \tag{2.2}$$

which after insertion into (2.1) yields an equation for the matrix $\mathcal{D}$,

$$\mathcal{D}''(\lambda) = \mathcal{T}(\lambda)\,\mathcal{D}(\lambda) \quad . \tag{2.3}$$

From the requirement that λ agrees locally with the proper distance, the initial conditions for $\boldsymbol{\xi}$ are $\boldsymbol{\xi}(0) = \mathbf{0}$, $\boldsymbol{\xi}'(0) = \boldsymbol{\theta}$, or equivalently, $\mathcal{D}(0) = 0$, $\mathcal{D}'(0) = \mathcal{I}$, where $\mathcal{I}$ denotes the unit matrix.

Suppose the matter distribution in the Universe is characterized by a homogeneous matter density $\bar\rho(z) = (1+z)^3\Omega_0\rho_{\rm cr}$, with critical density $\rho_{\rm cr} = 3H_0^2/(8\pi G)$; in that case, the optical tidal matrix $\mathcal{T}$ is proportional to the unit matrix and is given by

$$\mathcal{T}(\lambda) = -\frac{3}{2}\left(\frac{H_0}{c}\right)^2 \Omega_0\,(1+z)^5\,\mathcal{I} \quad . \tag{2.4}$$

The relation between the affine parameter and the redshift z reads

$$\mathrm{d}\lambda = \frac{c}{H_0}\frac{\mathrm{d}z}{(1+z)^3\sqrt{1+z\Omega_0 - \Omega_\Lambda\left[1-(1+z)^{-2}\right]}} \quad , \tag{2.5}$$

where $\Omega_\Lambda = \Lambda/(3H_0^2)$ is the density parameter associated with the cosmological constant Λ. In this case of a homogeneous Universe, the matrix $\mathcal{D}(\lambda)$ remains proportional to the unit matrix, $\mathcal{D}(\lambda) = D(\lambda)\,\mathcal{I}$, which implies an isotropic propagation of light bundles;

i.e., the cross-section of initially circular light bundles remains circular. The angular-diameter-distance D as a function of λ or z can then be obtained from the differential equation (2.3) for the $\mathcal{D}$, using (2.4).

Adding density perturbations $\delta\rho$ to the Universe, the optical tidal matrix changes to

$$\mathcal{T}_{ij}(\lambda) = -\frac{3}{2}\left(\frac{H_0}{c}\right)^2 \Omega_0(1+z)^5\,\delta_{ij} - \frac{(1+z)^2}{c^2}\left(2\Phi_{,ij} + \delta_{ij}\Phi_{,33}\right) \quad , \tag{2.6}$$

where the local coordinates are such that the light ray propagates in the x_3-direction. The gravitational potential Φ is related to the density perturbations via the Poisson equation $\nabla^2\Phi = 4\pi G\delta\rho$ (where the differential operator is taken with respect to local proper coordinates) which can be justified provided the density perturbations have typical scales much smaller than the radius of curvature of the Universe, i.e., if that scale is much less than the Hubble radius c/H_0. In general, the matrix $\mathcal{D}(\lambda)$ now attains shear components, and initially circular light bundles obtain elliptical cross sections.

Finally, we add to the density perturbations a single non-linear gravitational lens at redshift $z_{\rm d}$, described by its surface mass density $\Sigma(\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is the proper distance vector in the lens plane, i.e., a plane perpendicular to the light rays under consideration. If the fiducial light ray is chosen such that it traverses the lens plane at $\boldsymbol{\xi} = \mathbf{0}$, then the separation vector $\boldsymbol{\xi}(\lambda)$ becomes

$$\boldsymbol{\xi}(\lambda) = \begin{cases} \mathcal{D}(\lambda)\boldsymbol{\theta} & \text{for } \lambda \le \lambda_{\rm d} \\ \mathcal{D}(\lambda)\boldsymbol{\theta} - \hat{\mathcal{D}}(\lambda)\left[\boldsymbol{\alpha}(\boldsymbol{\xi}(\lambda_{\rm d})) - \boldsymbol{\alpha}(\mathbf{0})\right] & \text{for } \lambda > \lambda_{\rm d} \end{cases} \quad , \tag{2.7}$$

where $\boldsymbol{\alpha}(\boldsymbol{\xi})$ is the deflection angle a light ray undergoes at position $\boldsymbol{\xi}$,

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = \frac{4G}{c^2}\int_{\mathbb{R}^2} \mathrm{d}^2\xi'\,\Sigma(\boldsymbol{\xi}')\,\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{\left|\boldsymbol{\xi}-\boldsymbol{\xi}'\right|^2} \quad , \tag{2.8}$$

and $\hat{\mathcal{D}}(\lambda)$ is a 2×2-matrix. Inserting the ansatz (2.7) into (2.1) shows that it is a solution of the propagation equation (2.1) for all $\lambda \ne \lambda_{\rm d}$, provided that $\hat{\mathcal{D}}(\lambda)$ also satisfies the differential equation (2.3). Eq.(2.7) is derived in Sect. 4.4 of SSE, where it is also shown that the requirement that the local change of direction of the light ray in the lens plane agrees with the deflection angle implies that the initial conditions for $\hat{\mathcal{D}}$ are $\hat{\mathcal{D}}(\lambda_{\rm d}) = 0$, $\hat{\mathcal{D}}'(\lambda_{\rm d}) = (1+z_{\rm d})\mathcal{I}$.

2.2 The cosmological lens equation

In the case of an isolated mass concentration acting as a gravitational lens, there always exists at least one point where the deflection angle vanishes; this can be easily shown with the Poincaré's index theorem (see, e.g., Sect. 5.4.1 of SEF). For notational convenience, and without loss of generality, we shall assume that the origin in the lens plane – and thus the fiducial ray γ_0^μ – is chosen such that $\boldsymbol{\alpha}(\mathbf{0}) = \mathbf{0}$.

Consider sources at redshift $z_{\rm s}$, corresponding to the affine parameter $\lambda_{\rm s}$. Let $\boldsymbol{\eta} \equiv \boldsymbol{\xi}(\lambda_{\rm s})$, and, as before, $\boldsymbol{\xi} \equiv \boldsymbol{\xi}(\lambda_{\rm d})$. The lens equation in an inhomogeneous Universe then becomes

$$\boldsymbol{\eta} = \mathcal{D}(\lambda_{\rm s})\mathcal{D}^{-1}(\lambda_{\rm d})\boldsymbol{\xi} - \hat{\mathcal{D}}(\lambda_{\rm s})\boldsymbol{\alpha}(\boldsymbol{\xi}) \quad , \tag{2.9}$$

where we have assumed that the lensing effect by the perturbations $\delta\rho$ is sufficiently small so that no caustic points are caused by these perturbations themselves. This implies that the matrices $\mathcal{D}(\lambda)$ and $\hat{\mathcal{D}}(\lambda)$ are nowhere singular and thus can be inverted. Furthermore, for the validity of the lens equation (2.9) it has to be assumed that the large-scale density field causes a light deflection that can be well approximated by a linear behaviour in $\boldsymbol{\theta}$ over the angular range which is of interest for the study of multiple QSOs.

In (2.9), we have written $\boldsymbol{\xi} = \mathcal{D}(\lambda_{\rm d})\boldsymbol{\theta}$ as independent variable; the non-singularity requirement for $\mathcal{D}$ implies that there is a one-to-one relation between $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$. Eq. (2.9) maps a vector $\boldsymbol{\xi}$ from the lens plane into a vector $\boldsymbol{\eta}$ in the source plane, just as in standard gravitational lens theory; the difference between (2.9) and the standard lens equation is that in the latter case, the matrices $\mathcal{D}$ and $\hat{\mathcal{D}}$ are proportional to the unit matrix, i.e., they effectively become scalars. These scalars are the angular diameter-distances, which are uniquely defined in terms of redshifts in a homogeneous Universe. In an inhomogeneous Universe, the matrices $\mathcal{D}$ and $\hat{\mathcal{D}}$ attain shear components, so that the angular diameter-distances are no longer isotropic; moreover, since the density perturbations $\delta\rho$ form a random field, the optical tidal matrix along the direction to a gravitational lens has random components, and so there is basically no hope to obtain enough knowledge about this matter distribution to determine $\mathcal{D}(\lambda_{\rm d})$, $\mathcal{D}(\lambda_{\rm s})$, and $\hat{\mathcal{D}}(\lambda_{\rm s})$ for a given lens system. For notational convenience, we define

$$\mathcal{D}_{\rm d} \equiv \mathcal{D}(\lambda_{\rm d}) \quad ; \quad \mathcal{D}_{\rm s} \equiv \mathcal{D}(\lambda_{\rm s}) \quad ; \quad \mathcal{D}_{\rm ds} \equiv \hat{\mathcal{D}}(\lambda_{\rm s}) \quad . \tag{2.10}$$

Furthermore it should be noted that these matrices are in general not symmetric, so that the cosmic perturbations induce a rotational component to the matrices $\mathcal{D}$ and $\hat{\mathcal{D}}$ (this has been called 'twist' in SSE). However, this twist is unobservable: let $\boldsymbol{\beta} = \mathcal{D}_{\rm ds}^{-1}\boldsymbol{\eta}$; then,

$$\boldsymbol{\beta} = \mathcal{D}_{\rm ds}^{-1}\mathcal{D}_{\rm s}\mathcal{D}_{\rm d}^{-1}\boldsymbol{\xi} - \boldsymbol{\alpha}(\boldsymbol{\xi}) =: C_\xi\boldsymbol{\xi} - \boldsymbol{\alpha}(\boldsymbol{\xi}) \quad . \tag{2.11}$$

As we shall show in the Appendix, the matrix C_ξ multiplying $\boldsymbol{\xi}$ in (2.11) is symmetric, so that the matrix $\partial\boldsymbol{\beta}/\partial\boldsymbol{\xi}$ is symmetric. The symmetry of C_ξ is in complete analogy to the symmetry of the 'telescope matrix' introduced by Kovner (1987) in the frame of a multiple deflection gravitational lens system with at most one non-linear deflector; see also SEF, Sect. 9.3. The explicit algebraic proof for the symmetry of the 'telescope matrix' has been given in Seitz & Schneider (1994). We also note in passing that the magnification theorem (Schneider 1984; Seitz & Schneider 1992) is still satisfied: at least one of the images formed by the gravitational lens of any source is brighter than the source would appear in the same direction (i.e., with the same optical tidal matrix) in the absence of the lens, due to the non-negativity of $\Sigma(\boldsymbol{\xi})$. However, it should be noted that the source can appear fainter than the same source at the same redshift would appear in the corresponding homogeneous Universe, since $\delta\rho$ can be negative; indeed, $\langle\delta\rho\rangle = 0$ by definition.

We write the lens equation (2.11) in terms of the angle $\boldsymbol{\theta}$, by multiplying (2.11) from the left by $\mathcal{D}_{\rm d}^{\rm T}$, where the 'T' denotes the transpose of the matrix, and by defining $\boldsymbol{\eta}' = \mathcal{D}_{\rm d}^{\rm T}\boldsymbol{\beta}$:

$$\boldsymbol{\eta}' = \mathcal{D}_{\rm d}^{\rm T}\mathcal{D}_{\rm ds}^{-1}\mathcal{D}_{\rm s}\boldsymbol{\theta} - \mathcal{D}_{\rm d}^{\rm T}\boldsymbol{\alpha}\left(\mathcal{D}_{\rm d}\boldsymbol{\theta}\right) =: C_\theta\boldsymbol{\theta} - \mathcal{D}_{\rm d}^{\rm T}\boldsymbol{\alpha}\left(\mathcal{D}_{\rm d}\boldsymbol{\theta}\right) \quad . \tag{2.12}$$

Since $C_\theta = \mathcal{D}_{\rm d}^{\rm T} C_\xi \mathcal{D}_{\rm d}$, the symmetry of C_ξ implies that C_θ is also symmetric.

3 Relation to observables

The 'source position' $\boldsymbol{\eta}'$ is obtained from the 'true' source position $\boldsymbol{\eta}$ through a linear transformation. However, this transformation does not affect the observables. Let $\boldsymbol{\theta}^i$ be the observed image positions, $1 \le i \le N$, then the lens equation predicts that they have to satisfy

$$C_\theta \left(\boldsymbol{\theta}^i - \boldsymbol{\theta}^j\right) = \mathcal{D}_{\mathrm{d}}^{\mathrm{T}} \left[\boldsymbol{\alpha}\left(\mathcal{D}_{\mathrm{d}}\boldsymbol{\theta}^i\right) - \boldsymbol{\alpha}\left(\mathcal{D}_{\mathrm{d}}\boldsymbol{\theta}^j\right)\right] \quad , \tag{3.1}$$

for $i \neq j$, a relation not containing $\boldsymbol{\eta}'$. Secondly, whereas the magnification of the images depends on the linear transformation in the source plane, the observable magnification ratios are unaffected by it. What can be observed are the relative magnification matrices $\mathcal{A}_{ij}$ between images,

$$\mathcal{A}_{ij} = \left(\frac{\partial \boldsymbol{\eta}'(\boldsymbol{\theta}^i)}{\partial \boldsymbol{\theta}}\right) \left(\frac{\partial \boldsymbol{\eta}'(\boldsymbol{\theta}^j)}{\partial \boldsymbol{\theta}}\right)^{-1} \quad , \tag{3.2}$$

which are unchanged by a linear transformation in the source plane. Note that

$$\frac{\partial \boldsymbol{\eta}'}{\partial \boldsymbol{\theta}} = C_\theta - \mathcal{D}_{\mathrm{d}}^{\mathrm{T}} U\left(\mathcal{D}_{\mathrm{d}}\boldsymbol{\theta}\right)\mathcal{D}_{\mathrm{d}} \quad , \tag{3.3}$$

where

$$U(\boldsymbol{\xi}) = \frac{\partial \boldsymbol{\alpha}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \quad . \tag{3.4}$$

Since U is symmetric, so is $\partial\boldsymbol{\eta}'/\partial\boldsymbol{\theta}$.

We next consider the time delay function T. For that we first note that the potential part of the time delay is (Cooke & Kantowski 1975)

$$cT_{\mathrm{pot}} = (1+z_{\mathrm{d}})\Psi(\boldsymbol{\xi}) \quad , \tag{3.5}$$

where

$$\Psi(\boldsymbol{\xi}) = \frac{4G}{c^2}\int_{\mathbb{R}^2} \mathrm{d}^2\xi'\ \Sigma(\boldsymbol{\xi}')\ \ln\left(\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}'|}{\xi_*}\right) \quad , \tag{3.6}$$

is the deflection potential, and ξ_* is an arbitrary length scale; changing ξ_* changes Ψ by an additive constant which does therefore not affect the measureable time delays. From Fermat's principle (Schneider 1985, Kovner 1990) one knows that the lens equation is equivalent to $\nabla T = \mathbf{0}$, which determines T up to an affine transformation. The multiplicative constant is determined by that of T_{pot}, so that

$$cT = (1+z_{\mathrm{d}})\left[\boldsymbol{\theta}\cdot(C_\theta\boldsymbol{\theta})/2 - \boldsymbol{\eta}'\cdot\boldsymbol{\theta} - \Psi(\mathcal{D}_{\mathrm{d}}\boldsymbol{\theta})\right] \quad , \tag{3.7}$$

and noting that $\partial\Psi/\partial\boldsymbol{\theta} = \mathcal{D}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\alpha}(\mathcal{D}_{\mathrm{d}}\boldsymbol{\theta})$, one sees that $\partial T/\partial\boldsymbol{\theta} = \mathbf{0}$ is equivalent to (2.12). The time delay between any pair of images is then $\Delta t = T\left(\boldsymbol{\theta}^i\right) - T\left(\boldsymbol{\theta}^j\right)$. Whereas T as written in the form (3.7) contains the source position, one notes that $\boldsymbol{\eta}'$ is substituted in terms of the observed image positions using the lens equation (2.12). Therefore, the linear transformation in the source plane does not affect the calculation of the time delay.

4 The equivalent single-plane lens

We shall now show that the lens equation (2.12) is equivalent to a single-plane lens situation without 'cosmic shear'. The single-plane lens equation reads

$$\tilde{\boldsymbol{\eta}} = D_{\rm s}\boldsymbol{\theta} - D_{\rm ds}\tilde{\boldsymbol{\alpha}}(D_{\rm d}\boldsymbol{\theta}) \quad , \tag{4.1}$$

where $D_{\rm d}$, $D_{\rm ds}$, and $D_{\rm s}$ are the angular diameter distances in a homogeneous Universe, as described in Sect. 2. We characterize quantities in the single-plane lens with a tilde, to distinguish them from those of the cosmological lens equation. As before, $\tilde{\boldsymbol{\alpha}}(\tilde{\boldsymbol{\xi}})$ can be obtained from a deflection potential

$$\tilde{\boldsymbol{\alpha}}(\tilde{\boldsymbol{\xi}}) = \frac{{\rm d}\tilde{\Psi}(\tilde{\boldsymbol{\xi}})}{{\rm d}\tilde{\boldsymbol{\xi}}} \quad . \tag{4.2}$$

Theorem: For every mass distribution $\Sigma(\boldsymbol{\xi})$ and matrices $\mathcal{D}_{\rm d}$, $\mathcal{D}_{\rm s}$ and $\mathcal{D}_{\rm ds}$ giving rise to the cosmological lens equation (2.12), there exists a mass distribution $\tilde{\Sigma}(\tilde{\boldsymbol{\xi}})$ for which the single-plane lens equation (4.1) yields the same observables.

This will be shown by construction. To simplify notation we define the matrices

$$\mathcal{R}_{\rm d} = \frac{\mathcal{D}_{\rm d}}{D_{\rm d}} \quad ; \quad \mathcal{R}_{\rm s} = \frac{\mathcal{D}_{\rm s}}{D_{\rm s}} \quad ; \quad \mathcal{R}_{\rm ds} = \frac{\mathcal{D}_{\rm ds}}{D_{\rm ds}} \quad , \tag{4.3}$$

which describe the deviation of the light propagation in an inhomogeneous Universe from that in the homogeneous one; i.e., in a homogeneous Universe these three matrices would become the unit matrix. Define the deflection potential

$$\tilde{\Psi}(\tilde{\boldsymbol{\xi}}) := \Psi\left(\mathcal{R}_{\rm d}\tilde{\boldsymbol{\xi}}\right) + \frac{1}{2}\tilde{\boldsymbol{\xi}} \cdot \left(B\tilde{\boldsymbol{\xi}}\right) \quad , \tag{4.4}$$

where B is the matrix

$$B = \frac{D_{\rm s}}{D_{\rm d}}\left(\mathcal{R}_{\rm d}^{\rm T}\mathcal{R}_{\rm ds}^{-1}\mathcal{R}_{\rm s} - \mathcal{I}\right) \quad ; \tag{4.5}$$

then the deflection angle becomes, according to (4.2),

$$\tilde{\boldsymbol{\alpha}}(\tilde{\boldsymbol{\xi}}) = \mathcal{R}_{\rm d}^{\rm T}\boldsymbol{\alpha}(\mathcal{R}_{\rm d}\tilde{\boldsymbol{\xi}}) + B\tilde{\boldsymbol{\xi}} \quad . \tag{4.6}$$

This is now inserted into the lens equation (4.1) to yield

$$\tilde{\boldsymbol{\eta}} = D_{\rm s}\boldsymbol{\theta} - D_{\rm ds}\mathcal{R}_{\rm d}^{\rm T}\boldsymbol{\alpha}(\mathcal{D}_{\rm d}\boldsymbol{\theta}) + D_{\rm d}B\boldsymbol{\theta} \quad . \tag{4.7}$$

If we now define $\tilde{\boldsymbol{\eta}}' \equiv (D_{\rm d}/D_{\rm ds})\tilde{\boldsymbol{\eta}}$, then the lens equation (4.7) becomes after multiplication with $(D_{\rm d}/D_{\rm ds})$

$$\tilde{\boldsymbol{\eta}}' = C_\theta\boldsymbol{\theta} - \mathcal{D}_{\rm d}^{\rm T}\boldsymbol{\alpha}(\mathcal{D}_{\rm d}\boldsymbol{\theta}) \quad , \tag{4.8}$$

which is obviously the same equation as (2.12), except for a linear transformation of the source coordinates. In particular, this single-plane lens yields the same relations (3.1), (3.2) and (3.7) between the image positions, the relative magnification matrices and the time delay function as the cosmological lens mapping. The surface mass density corresponding to the equivalent single-plane lens is obtained from (4.4) via Poisson's equation. The first term in (4.4) then yields a contribution to $\tilde{\Sigma}(\tilde{\boldsymbol{\xi}})$ which is obtained from $\Sigma(\boldsymbol{\xi})$ in a non-trivial way, although the corresponding potential contributions are related by a simple linear transformation. The second term in (4.4) yields a uniform surface mass density. Note that both contributions are not guaranteed to be non-negative.

If we use the angular source position $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\eta}}/D_\mathrm{s}$, the lens equation takes the more familiar form

$$\tilde{\boldsymbol{\beta}} = \mathcal{R}_\mathrm{d}^\mathrm{T}\mathcal{R}_\mathrm{ds}^{-1}\mathcal{R}_\mathrm{s}\boldsymbol{\theta} - \frac{D_\mathrm{ds}}{D_\mathrm{s}}\mathcal{R}_\mathrm{d}^\mathrm{T}\boldsymbol{\alpha}(\mathcal{D}_\mathrm{d}\boldsymbol{\theta}) =: \tilde{\Gamma}\boldsymbol{\theta} - \frac{D_\mathrm{ds}}{D_\mathrm{s}}\mathcal{R}_\mathrm{d}^\mathrm{T}\boldsymbol{\alpha}(\mathcal{D}_\mathrm{d}\boldsymbol{\theta}) \quad , \tag{4.9}$$

so that in fact the equivalent single-plane lens consists of the shear matrix $\tilde{\Gamma}$ plus a term corresponding to the 'main lens'. Since $\tilde{\Gamma}$ is proportional to C_θ it is also symmetric.

In general, the trace of $\tilde{\Gamma}$ is different from 2, which means that the shear matrix contains an equivalent surface mass density term (which is not necessarily positive). As was pointed out by Gorenstein, Falco & Shapiro (1988), adding a homogeneous surface mass density to a lens and at the same time rescaling the lens strength leaves all observables invariant except the time delay. This mass-sheet degeneracy is therefore also present in the single-plane lens and is not a novel feature of the cosmological lens equation.

5 Special case: Elliptical isopotentials

We now consider the case that the lens is described by elliptical isopotential curves, so that

$$\Psi(\boldsymbol{\xi}) = F(u) \quad , \tag{5.1}$$

with

$$u = \xi^2\left(1 - \epsilon\cos\left[2\left(\varphi - \vartheta\right)\right]\right) \quad ; \tag{5.2}$$

here, $\boldsymbol{\xi}$ has been expressed in polar coordinates $\boldsymbol{\xi} = \xi(\cos\varphi, \sin\varphi)$, ϵ describes the ellipticity of the potential, and ϑ is the direction of the major axis of the potential. This kind of lens model has been introduced by Blandford & Kochanek (1987) and investigated in detail by several authors (e.g., Kassiola & Kovner 1993). The corresponding mass distribution has no elliptical isodensity curves, and depending on the shape of the radial profile $F(u)$ and the ellipticity ϵ, dumb bell shaped isodensity curves or even negative surface mass densities are obtained from Ψ. Leaving aside these potential difficulties, models of the form (5.1) have often been used for modeling gravitational lens systems. Whereas the surface mass density is not elliptical, one can still define a major and minor axis, which coincide with the major and minor axes of Ψ.

Expressed in terms of the observable $\boldsymbol{\theta}$, $\Psi(\mathcal{D}_\mathrm{d}\boldsymbol{\theta})$, the observed major axis of the lens does not coincide with ϑ, but is rotated by the matrix $\mathcal{D}_\mathrm{d}$. But the same is true for the equivalent single-plane lens, see (4.4). Note that the first term in (4.4) is again an elliptical potential in $\tilde{\boldsymbol{\xi}}$, however with different ellipticity and orientation, and a modified radial profile. However, the matrix which rotates $\tilde{\Psi}(D_\mathrm{d}\boldsymbol{\theta})$ with respect to Ψ is the same as that rotating the intrinsic lens direction ϑ into the observed one. Hence, the observer in the inhomogeneous Universe sees the same orientation of the lens as the corresponding observer in the homogeneous Universe sees of the equivalent single-plane lens. Since the image positions $\boldsymbol{\theta}^i$ are the same in both cases, this implies that the positions of the images relative to the observed orientation of the lens are the same in both cases, in contrast to the conjecture made in Blandford & Kundić (1996). In fact, not only is the observed orientation the same in both situations, but the potentials do agree in both cases when expressed in terms of $\boldsymbol{\theta}$! Note that this is not the case for the surface mass densities.

Finally, it can be easily seen that if one considers a family of models where the primary lens has elliptical potential curves and in addition a shear matrix is included, then the cosmological contributions C_θ and $\mathcal{R}_{\rm d}$ yield an equivalent single-plane lens which is included within this family.

6 Summary and discussion

In this paper the effects of large-scale matter inhomogeneities on gravitational lensing by galaxies were considered. The lens equation in an inhomogeneous Universe was derived for the case that there is only a single strong lens between the source and observer. The only approximation entering (2.9) is that the length scale of the matter inhomogeneities is much larger than the linear size of a light bundle which encloses all light rays from the source to the observer (i.e., for a typical multiply imaged QSO, this size is of order 20 kpc). In that case, light propagating from the source to the lens and from the lens to the observer can be described by the Jacobi differential equation. Note that for the validity of (2.9) one does not need to assume that the inhomogeneities are weak – or in other words, that the matrices $\mathcal{R}_{\rm d}$, $\mathcal{R}_{\rm s}$ and $\mathcal{R}_{\rm ds}$ are close to the unit matrix. However, it was assumed that the matter inhomogeneities of the large-scale structure – outside the main lens – are not so strong as to produce multiple images on its own. The analogy of the 'telescope matrix' which was defined by Kovner (1987) has been shown to be symmetric in general. This symmetry property is essential for showing that for arbitrary propagation matrices $\mathcal{R}_{\rm d}$, $\mathcal{R}_{\rm s}$ and $\mathcal{R}_{\rm ds}$ caused by the large-scale structure, there exists an equivalent single-plane gravitational lens such that all observables are the same – image positions, relative magnification matrices and time-delays. However, in general this equivalent single-plane lens contains a homogeneous matter sheet, not necessarily with positive surface mass density. Therefore, the mass-sheet degeneracy as discussed in Gorenstein, Falco & Shapiro (1988) does apply.

For a single lens plane, this mass-sheet degeneracy can be broken if independent observational information about the lens can be obtained. For example, if the lens is well described by an isothermal mass distribution, the measurement of the velocity dispersion determines the lens strength and thus fixes the mass-sheet transformation. If this is done, the time delay then depends on the matrix $\mathcal{R}_{\rm d}$, and thus matter inhomogeneities between us and the lens affect the determination of the Hubble constant. This fact was noted by Narayan (1991) who showed that the measurement of the time delay determines the angular diameter distance to the lens, which is affected by $\mathcal{R}_{\rm d}$, in agreement with B-K96, where the size of this effect has been calculated using the non-linear evolution of the power spectrum. Thus, lensing by the large-scale structure does affect the determination of the Hubble constant, although the magnitude of the effect (i.e., the rms deviation of $\mathcal{R}_{\rm d}$ from the unit matrix) is of the order of a few percent; see Fig. 2 of B-K96. Therefore, in agreement with the conclusion of Surpi, Harari & Frieman (1996), cosmic shear does not seriously compromise the lensing method for the determination of H_0.

Whereas the mass distribution of the equivalent single-plane lens depends non-trivially on the lens mass distribution and the propagation matrices $\mathcal{R}$, it has been shown that if the lens is chosen to have elliptical isopotentials then the equivalent single-plane lens consists of a main component with elliptical isopotentials plus a shear matrix. Furthermore, if a family of lens models is considered, consisting of an elliptical isopotential component and a shear matrix, then the equivalent single-plane lens is also within this family. For this case it was shown that the image positions relative to the orientation of the mass distribution as seen by an observer is the same in the inhomogeneous Universe and in the single-plane lensing situation. Note that lens models consisting of an elliptical lens (either elliptical isodensity contour or elliptical isopotentials) plus external shear seems to be required for those lens systems in which there is a large number of observational constraints (e.g., Keeton, Kochanek & Seljak 1997, Witt & Mao 1997).

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Appendix

We shall show in this Appendix that the matrix C_ξ multiplying $\boldsymbol{\xi}$ in (2.11) is symmetric. To do this, we define the matrix

$$A(\lambda) := \mathcal{D}(\lambda)\mathcal{D}^{-1}(\lambda_{\rm d}) \tag{A1}$$

and the matrix

$$C(\lambda) = \hat{\mathcal{D}}^{-1}(\lambda)\mathcal{D}(\lambda)\mathcal{D}^{-1}(\lambda_{\rm d}) = \hat{\mathcal{D}}^{-1}(\lambda)A(\lambda) \quad , \tag{A2}$$

in both cases for $\lambda \geq \lambda_{\rm d}$. The proof proceeds in the following four steps:

(1) The matrix $A(\lambda)$ satisfies the differential equation (2.3), which implies that A and $\hat{D}$ satisfy the same differential equation. This can be trivially shown by inserting (A1) into (2.3).

(2) For any two solutions X and Y of the differential equation (2.3), one finds

$$\frac{\rm d}{{\rm d}\lambda}\left[(X^{\rm T})'Y - X^{\rm T}Y'\right] = 0 \quad , \tag{A3}$$

as can be easily verified by straight insertion into (2.3) and using the fact that $\mathcal{T}$ is symmetric. This relation implies that

$$S(\lambda) := \hat{\mathcal{D}}'(\lambda)\hat{\mathcal{D}}^{-1}(\lambda) \tag{A4}$$

is symmetric. This can be shown as follows: Eq. (A3) implies, by setting $X = Y = \hat{\mathcal{D}}$, that $(\hat{\mathcal{D}}^{\rm T})'\hat{\mathcal{D}} - \hat{\mathcal{D}}^{\rm T}\hat{\mathcal{D}}'$ is constant. Due to the initial condition at $\lambda = \lambda_{\rm d}$, $\hat{\mathcal{D}}^{\rm T}(\lambda_{\rm d}) = 0 = \hat{\mathcal{D}}(\lambda_{\rm d})$, this constant is the zero matrix, so that $(\hat{\mathcal{D}}^{\rm T})'\hat{\mathcal{D}} = \hat{\mathcal{D}}^{\rm T}\hat{\mathcal{D}}'$ for all $\lambda \geq \lambda_{\rm d}$. In particular this implies

$$(\hat{\mathcal{D}}^{\rm T})' = \hat{\mathcal{D}}^{\rm T}\hat{\mathcal{D}}'\hat{\mathcal{D}}^{-1} \quad . \tag{A5}$$

To show the symmetry of S, we consider

$$S - S^{\rm T} = \hat{\mathcal{D}}'\hat{\mathcal{D}}^{-1} - \left(\hat{\mathcal{D}}^{-1}\right)^{\rm T}(\hat{\mathcal{D}}^{\rm T})' = \hat{\mathcal{D}}'\hat{\mathcal{D}}^{-1} - \left(\hat{\mathcal{D}}^{-1}\right)^{\rm T}\hat{\mathcal{D}}^{\rm T}\hat{\mathcal{D}}'\hat{\mathcal{D}}^{-1} = 0 \quad , \tag{A6}$$

where we have used (A5) in the second step, and in the final step the fact was employed that the inverse and transpose operations do commute. Hence, S is symmetric.

(3) If we set $X = A$ and $Y = \hat{\mathcal{D}}$, we find that the constant in the bracket of (A3) is

$$(A^{\mathrm{T}})'\hat{\mathcal{D}} - A^{\mathrm{T}}\hat{\mathcal{D}}' = -(1+z_{\mathrm{d}})\mathcal{I} \quad , \tag{A7}$$

due to the intial conditions $A(\lambda_{\mathrm{d}}) = \mathcal{I}$, $\hat{\mathcal{D}}(\lambda_{\mathrm{d}}) = 0$, $\hat{\mathcal{D}}'(\lambda_{\mathrm{d}}) = (1+z_{\mathrm{d}})\mathcal{I}$. Hence,

$$(1+z_{\mathrm{d}})\hat{\mathcal{D}}^{-1} = A^{\mathrm{T}}S - (A^{\mathrm{T}})' \quad . \tag{A8}$$

(4) The final step is taken by noting that $\lim_{\lambda\to\lambda_{\mathrm{d}}}[(\lambda-\lambda_{\mathrm{d}})C(\lambda)]$ is symmetric. We now show that C' is symmetric for all $\lambda \geq \lambda_{\mathrm{d}}$, which then proves that C is symmetric for all $\lambda - \lambda_{\mathrm{d}}$, and in particular this is true for $C_\xi = C(\lambda_{\mathrm{s}})$. Using

$$\hat{\mathcal{D}}\hat{\mathcal{D}}^{-1} = \mathcal{I} \to (\hat{\mathcal{D}}^{-1})' = -\hat{\mathcal{D}}^{-1}S \quad ,$$

as follows from differentiation, we find in turn:

$$\begin{aligned} C' &= \hat{\mathcal{D}}^{-1}A' + (\hat{\mathcal{D}}^{-1})'A = \hat{\mathcal{D}}^{-1}A' - \hat{\mathcal{D}}^{-1}SA \\ &= \frac{1}{(1+z_{\mathrm{d}})}\left(A^{\mathrm{T}}SA' - (A^{\mathrm{T}})'A' - A^{\mathrm{T}}SSA + (A^{\mathrm{T}})'SA\right) \quad . \end{aligned} \tag{A9}$$

However, this final expression is manifestly symmetric, which completes the proof.

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