arXiv: 1101.3636

Non-Gaussianity Consistency Relation for Multi-Field Inflation

$$\tau_{\rm NL} > \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$$ *for the local form*

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Cosmological Non-Gaussianity at Univ. of Michigan, May 15, 2011

Nao Sugiyama (Tohoku University) in the audience did most of the work!

Theme

- How to falsify inflation?

or

- Why bother measuring the trispectrum?

Motivation

- I will be focused on the local-form non-Gaussianity.
- The local-form bispectrum is particularly important because its detection would rule out all single-field inflation models (Creminelli & Zaldarriaga 2004).
  - $f_{NL}^{local} >> 1$ (like 30, as suggested by the current data)

    ➡ ALL single-field inflation models would be ruled out.

But, what about multi-field models?

Motivation

- **Can we rule out multi-field models also?**
  - If we rule out single-field AND multi-field, then...

Falsifying “inflation”

- We still need inflation to explain the flatness problem!
  - (Homogeneity problem can be explained by a bubble nucleation.)
- However, the observed fluctuations may come from different sources.
- So, what I ask is, “can we rule out inflation as a mechanism for generating the observed fluctuations?”

Conclusion

- It is almost possible.

Sugiyama, Komatsu & Futamase, arXiv: 1101.3636

Strategy

- We look at the local-form four-point function (trispectrum).
- Specifically, we look for a consistency relation between the local-form bispectrum and trispectrum that is respected by (almost) all models of multi-field inflation.
- We found one: $\tau_{\rm NL} > \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$

provided that 2-loop and higher-order terms are ignored.

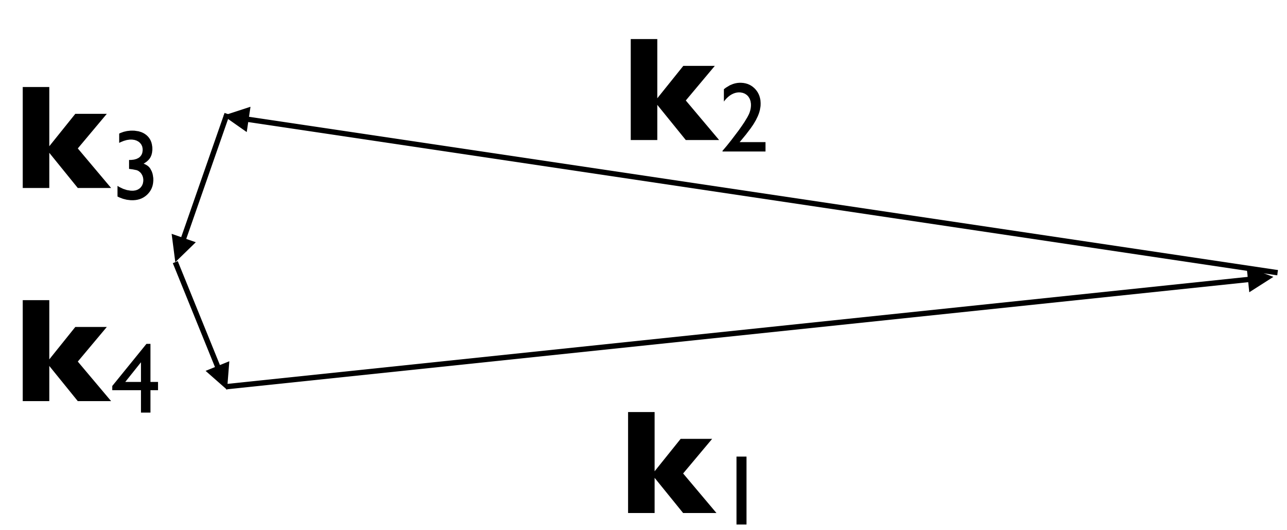
Which Local-form Trispectrum?

- The local-form bispectrum:
  - $B_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)f_{NL}[(6/5)P_\zeta(k_1)P_\zeta(k_2)+\text{cyc.}]$
- can be produced by a curvature perturbation in position space in the form of:
  - $\zeta(\mathbf{x})=\zeta_g(\mathbf{x}) + (3/5)f_{NL}[\zeta_g(\mathbf{x})]^2$
- This can be extended to higher-order:
  - $\zeta(\mathbf{x})=\zeta_g(\mathbf{x}) + (3/5)f_{NL}[\zeta_g(\mathbf{x})]^2 + (9/25)g_{NL}[\zeta_g(\mathbf{x})]^3$

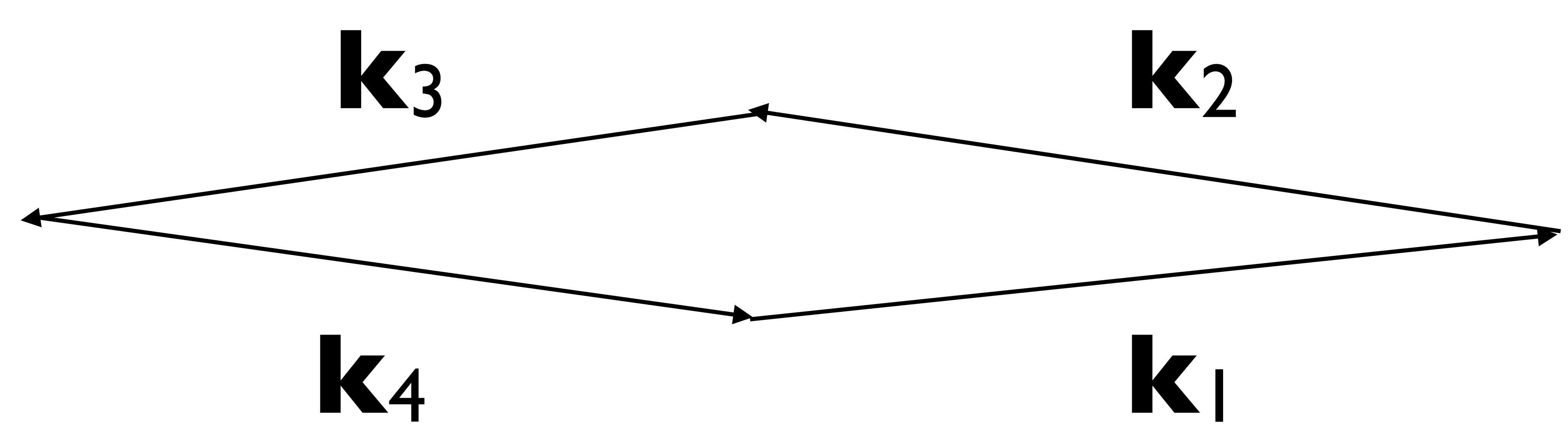
This term (ζ^3) is too small to see, so I will ignore this in this talk.

Two Local-form Shapes

- For $\zeta(\mathbf{x})=\zeta_g(\mathbf{x}) + (3/5)f_{NL}[\zeta_g(\mathbf{x})]^2 + (9/25)g_{NL}[\zeta_g(\mathbf{x})]^3$, we obtain the trispectrum:
  - $T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$ $\{\mathbf{g_{NL}}[(54/25)P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3)+\text{cyc.}] +\mathbf{(f_{NL})^2}[(18/25)P_\zeta(k_1)P_\zeta(k_2)(P_\zeta(|\mathbf{k}_1+\mathbf{k}_3|)+P_\zeta(|\mathbf{k}_1+\mathbf{k}_4|))+\text{cyc.}]\}$



g_{NL}

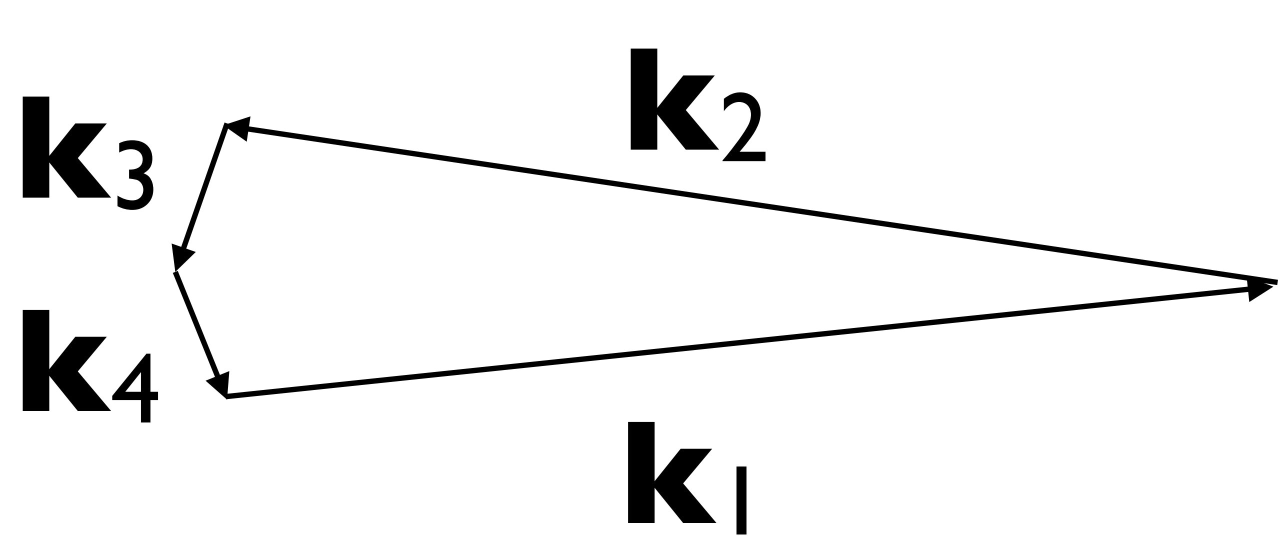


$f_{NL}{}^2$

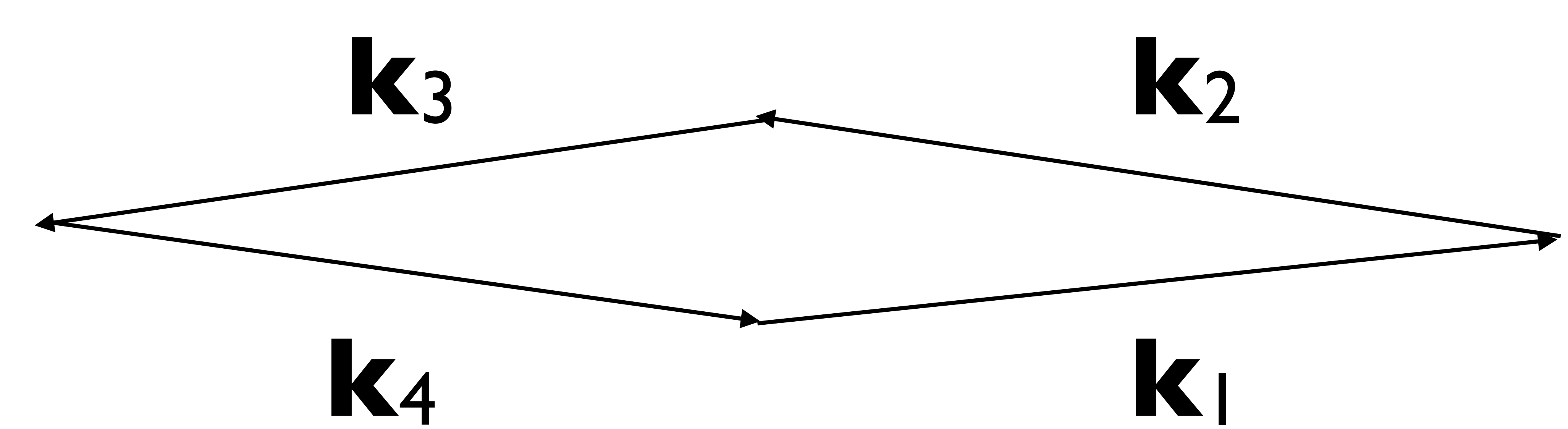
Generalized Trispectrum

- $T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$ $\{g_{NL}[(54/25) P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3)+cyc.] +\tau_{NL}[P_\zeta(k_1)P_\zeta(k_2)(P_\zeta(|\mathbf{k}_1+\mathbf{k}_3|)+P_\zeta(|\mathbf{k}_1+\mathbf{k}_4|))+cyc.]\}$

The single-source local form consistency relation, $\tau_{NL}=(6/5)(f_{NL})^2$, may not be respected – additional test of multi-field inflation!



g_{NL}

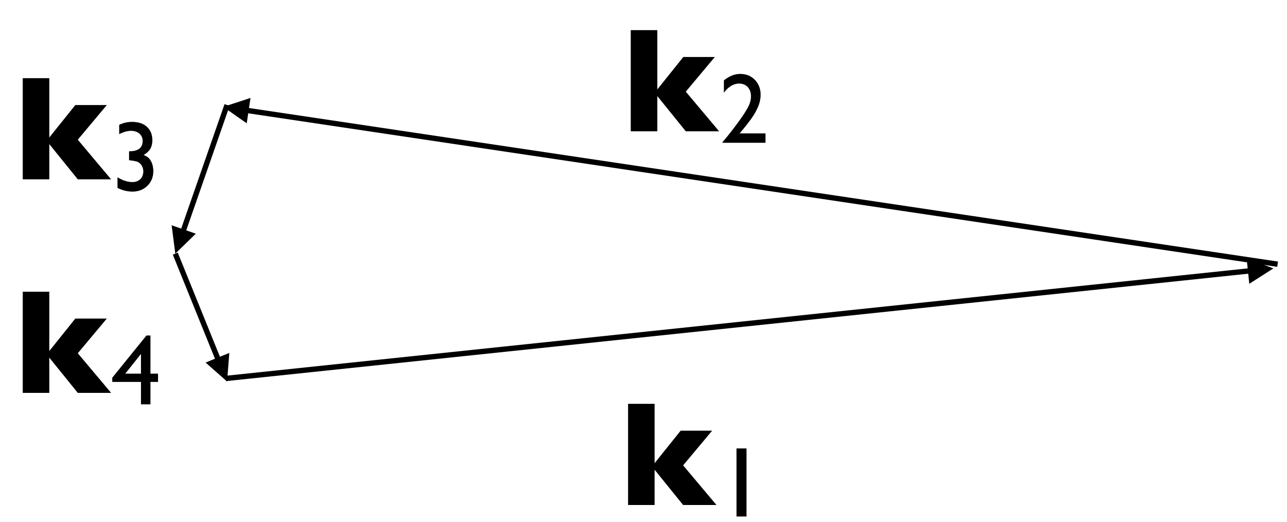


τ_{NL}

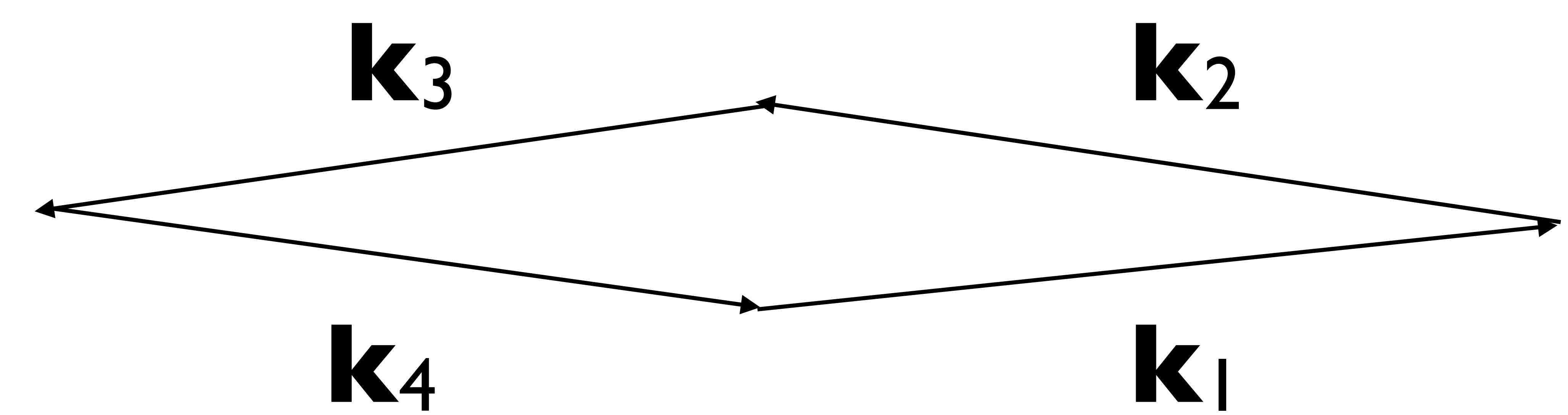
(Slightly) Generalized Trispectrum

- $T_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\ \{\mathbf{g_{NL}}[(54/25)P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3)+\text{cyc.}]\ +\boldsymbol{\tau_{NL}}[P_\zeta(k_1)P_\zeta(k_2)(P_\zeta(|\mathbf{k}_1+\mathbf{k}_3|)+P_\zeta(|\mathbf{k}_1+\mathbf{k}_4|))+\text{cyc.}]\}$

The single-source local form consistency relation, $\tau_{NL}=(6/5)(f_{NL})^2$, may not be respected – additional test of multi-field inflation!



g_{NL}



τ_{NL}

Tree-level Result (Suyama & Yamaguchi)

- Usual δN expansion to the second order

$$\zeta = \sum_I \frac{\partial N}{\partial \phi_I}\delta\phi_I + \frac{1}{2}\sum_{IJ}\frac{\partial^2 N}{\partial\phi_I\partial\phi_J}\delta\phi_I\delta\phi_J + \ldots$$

gives:

$$\frac{6}{5}f_{\rm NL}^{\rm local} = \frac{\sum_{IJ} N_{,IJ}N_{,I}N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\rm NL} = \frac{\sum_{IJK} N_{,IJ}N_{,J}N_{,IK}N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I(\sum_J N_{,IJ}N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

Now, stare at these.

$$\frac{6}{5} f_{\rm NL}^{\rm local} = \frac{\sum_{IJ} N_{,IJ} N_{,I} N_{,J}}{\left[\sum_I (N_{,I})^2\right]^2},$$

$$\tau_{\rm NL} = \frac{\sum_{IJK} N_{,IJ} N_{,J} N_{,IK} N_{,K}}{\left[\sum_I (N_{,I})^2\right]^3} = \frac{\sum_I \left(\sum_J N_{,IJ} N_{,J}\right)^2}{\left[\sum_I (N_{,I})^2\right]^3}$$

Change the variable...

$$\frac{6}{5} f_{\rm NL}^{\rm local} = \frac{\sum_{IJ} N_{,IJ} N_{,I} N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\rm NL} = \frac{\sum_{IJK} N_{,IJ} N_{,J} N_{,IK} N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I (\sum_J N_{,IJ} N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

$$a_I = \frac{\sum_J N_{,IJ} N_{,J}}{[\sum_J (N_{,J})^2]^{3/2}}$$

$$b_I = \frac{N_{,I}}{[\sum_J (N_{,J})^2]^{1/2}}$$

$(6/5)f_{NL}=\sum_I a_I b_I$

$\tau_{NL}=(\sum_I a_I^2)(\sum_I b_I^2)$

Then apply the Cauchy-Schwarz Inequality

$$\left(\sum_I a_I^2\right)\left(\sum_J b_J^2\right) \geq \left(\sum_I a_I b_I\right)^2$$

- Implies (Suyama & Yamaguchi 2008)

$$\tau_{\mathrm{NL}} \geq \left(\frac{6 f_{\mathrm{NL}}^{\mathrm{local}}}{5}\right)^2$$

But, this is valid only at the tree level!

Harmless models can violate the tree-level result

- The Suyama-Yamaguchi inequality does not always hold because the Cauchy-Schwarz inequality can be 0=0. For example:

$$\zeta = \frac{\partial N}{\partial \phi_1}\delta\phi_1 + \frac{1}{2}\frac{\partial^2 N}{\partial \phi_2^2}\delta\phi_2^2$$

In this harmless two-field case, the Cauchy-Schwarz inequality becomes 0=0 (both f_{NL} and τ_{NL} result from the second term). In this case,

$$\tau_{\rm NL} \sim 10^3 (f_{\rm NL}^{\rm local})^{4/3}$$

(Suyama & Takahashi 2008)

"1 Loop"

$$\zeta = \frac{\partial N}{\partial \phi_1}\delta\phi_1 + \frac{1}{2}\frac{\partial^2 N}{\partial \phi_2^2}\delta\phi_2^2$$

Fourier transform this,
and multiply 3 times

$$\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3s}{(2\pi)^3}\langle \delta\tilde{\phi}_2(\mathbf{k}_1-\mathbf{p})\delta\tilde{\phi}_2(\mathbf{p})\delta\tilde{\phi}_2(\mathbf{k}_2-\mathbf{q})\delta\tilde{\phi}_2(\mathbf{q})\delta\tilde{\phi}_2(\mathbf{k}_3-\mathbf{s})\delta\tilde{\phi}_2(\mathbf{s})\rangle$$

$$= \left(\frac{H^2}{2}\right)^3 (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\int \frac{d^3p}{(2\pi)^3}\frac{1}{p^3|\mathbf{k}_1-\mathbf{p}|^3|\mathbf{k}_3+\mathbf{p}|^3} + \text{(permutations)}$$

$p_{min}=1/L$

$$\approx \left(\frac{H^2}{2}\right)^3 (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{8\ln(k_bL)}{2\pi^2}\left[\frac{1}{k_1^3k_3^2}+\frac{1}{k_2^3k_3^2}+\frac{1}{k_1^3k_2^2}\right]$$

- $k_b=\min(k_1,k_2,k_3)$

Assumptions

- Scalar fields are responsible for generating fluctuations.
- Fluctuations are Gaussian and scale-invariant at the horizon crossing.
    - All (local-form) non-Gaussianity was generated outside the horizon by δN

Starting point

$$\zeta(\mathbf{x},t) = N_a(t,t_*)\delta\varphi_*^a(\mathbf{x}) + \frac{1}{2}N_{ab}(t,t_*)\delta\varphi_*^a(\mathbf{x})\delta\varphi_*^b(\mathbf{x})$$
$$+ \frac{1}{3!}N_{abc}\delta\varphi_*^a\delta\varphi_*^b\delta\varphi_*^c + \frac{1}{4!}N_{abcd}\delta\varphi_*^a\delta\varphi_*^b\delta\varphi_*^c\delta\varphi_*^d$$

- We need the fourth-order expansion for the complete calculation at the 1-loop level.
- Then, Fourier transform this and calculate the bispectrum and trispectrum...

- $$\frac{6}{5}f_{\rm NL} \simeq \left[\tilde{N}_a\tilde{N}_a + {\rm Tr}(\tilde{N}^2)\mathcal{P}_* \ln(k_0 L)\right]^{-2} \times \left[\tilde{N}_a\tilde{N}_b\tilde{N}_{ab} + \left({\rm Tr}(\tilde{N}^3) + 2\tilde{N}_a\tilde{N}_{bc}\tilde{N}_{abc}\right)\mathcal{P}_* \ln(k_0 L)\right]$$

where
$$\tilde{N}_a \equiv N_a + \frac{1}{2}N_{abb}\mathcal{P}_* \ln(k_{\rm max}L),$$
$$\tilde{N}_{ab} \equiv N_{ab} + \frac{1}{2}N_{abcc}\mathcal{P}_* \ln(k_{\rm max}L).$$
[Byrnes et al. (2007)]

- $$\tau_{\rm NL} \simeq \left[\tilde{N}_a\tilde{N}_a + {\rm Tr}(\tilde{N}^2)\mathcal{P}_* \ln(k_0 L)\right]^{-3} \times \left[\tilde{N}_a\tilde{N}_{ab}\tilde{N}_{bc}\tilde{N}_c + \left(2\tilde{N}_a\tilde{N}_{ab}\tilde{N}_{cd}\tilde{N}_{bcd} + {\rm Tr}(\tilde{N}^4) + 2\tilde{N}_a\tilde{N}_{bc}\tilde{N}_{bd}\tilde{N}_{acd} + \tilde{N}_a\tilde{N}_b\tilde{N}_{acd}\tilde{N}_{bcd}\right)\mathcal{P}_* \ln(k_0 L)\right]$$

where $\tilde{N}_{abc} \equiv N_{abc} + \frac{1}{2}N_{abcdd}\mathcal{P}_* \ln(k_{\rm max}L)$

$$\mathcal{P}_{\rm loop} \equiv \frac{{\rm Tr}(\tilde{N}^2)}{\tilde{N}_a\tilde{N}_a}\mathcal{P}_* \ln(kL)$$

- $$\frac{6}{5}f_{\rm NL} \simeq \left[\tilde{N}_a\tilde{N}_a + {\rm Tr}(\tilde{N}^2)\mathcal{P}_* \ln(k_0 L)\right]^{-2} \times \left[\tilde{N}_a\tilde{N}_b\tilde{N}_{ab} + \left({\rm Tr}(\tilde{N}^3) + 2\tilde{N}_a\tilde{N}_{bc}\tilde{N}_{abc}\right)\mathcal{P}_* \ln(k_0 L)\right]$$

$$\alpha \equiv [N_a N_a(1+\mathcal{P}_{\rm loop})]^{-2}\,[N_a N_b N_{ab} + N_a N_{bc} N_{abc}\mathcal{P}_* \ln(k_0 L)]$$

$$\beta \equiv [N_a N_a(1+\mathcal{P}_{\rm loop})]^{-2}\,[{\rm Tr}(N^3) + N_a N_{bc} N_{abc}]\,\mathcal{P}_* \ln(k_0 L),$$

$$\alpha^2 + \beta^2 \geq \frac{1}{2}(\alpha+\beta)^2$$

$$[N_a N^a(1+\mathcal{P}_{\rm loop})]^{-4} \times \left[\left(N_a N_b N_{ab} + N_a N_{bc} N_{abc}\mathcal{P}_* \ln(k_0 L)\right)^2 + \left({\rm Tr}(N^3) + N_a N_{bc} N_{abc}\right)^2 \mathcal{P}_*^2 \ln^2(k_0 L)\right] \geq \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$$

$$\mathcal{P}_{\rm loop} \equiv \frac{{\rm Tr}(\tilde{N}^2)}{\tilde{N}_a\tilde{N}_a}\mathcal{P}_* \ln(kL)$$

- $$\frac{6}{5}f_{\rm NL} \simeq \left[\tilde{N}_a\tilde{N}_a + {\rm Tr}(\tilde{N}^2)\mathcal{P}_* \ln(k_0 L)\right]^{-2} \times \left[\tilde{N}_a\tilde{N}_b\tilde{N}_{ab} + \left({\rm Tr}(\tilde{N}^3) + 2\tilde{N}_a\tilde{N}_{bc}\tilde{N}_{abc}\right)\mathcal{P}_* \ln(k_0 L)\right]$$

$$\alpha \equiv \left[N_a N_a (1+\mathcal{P}_{\rm loop})\right]^{-2}\left[N_a N_b N_{ab} + N_a N_{bc} N_{abc}\mathcal{P}_* \ln(k_0 L)\right]$$

$$\beta \equiv \left[N_a N_a (1+\mathcal{P}_{\rm loop})\right]^{-2}\left[{\rm Tr}(N^3) + N_a N_{bc} N_{abc}\right]\mathcal{P}_* \ln(k_0 L),$$

$$\alpha^2 + \beta^2 \geq \frac{1}{2}(\alpha+\beta)^2$$

$$\left[N_a N^a (1+\mathcal{P}_{\rm loop})\right]^{-4} \times \left[\left(N_a N_b N_{ab} + N_a N_{bc} N_{abc}\mathcal{P}_* \ln(k_0 L)\right)^2 + \left({\rm Tr}(N^3) + N_a N_{bc} N_{abc}\right)^2 \mathcal{P}_*^2 \ln^2(k_0 L)\right] \geq \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$$

1st term

$$\frac{\left(N_a N_b N_{ab} + N_a N_{bc} N_{abc} \mathcal{P}_* \ln(k_0 L)\right)^2}{(N_a N_a)^4 (1+\mathcal{P}_{\rm loop})^4}$$

$$< \frac{N_b N_{ba} N_{ad} N_d + 2 N_d N_{da} N_{abc} N_{bc} \mathcal{P}_* \ln(k_0 L)}{(N_a N_a)^3 \left(1+\mathcal{P}_{\rm loop}\right)^3}$$

$$+ \frac{N_{ab} N_{abc} N_{cde} N_{de} \mathcal{P}_*^2 \ln^2(k_0 L)}{(N_a N_a)^3 (1+\mathcal{P}_{\rm loop})^3},$$

- where we have used the Cauchy-Schwarz inequality:

$$(\textstyle\sum_a u_a v_a)^2 \leq (\sum_a {u_a}^2)(\sum_a {v_a}^2)$$

$$v_a \equiv N_a \quad u_a \equiv N_b N_{ba} + N_{bc} N_{abc} \mathcal{P}_* \ln(k_0 L)$$

- $$\frac{6}{5}f_{\rm NL} \simeq \left[\tilde{N}_a\tilde{N}_a + {\rm Tr}(\tilde{N}^2)\mathcal{P}_* \ln(k_0L)\right]^{-2} \times \left[\tilde{N}_a\tilde{N}_b\tilde{N}_{ab} + \left({\rm Tr}(\tilde{N}^3) + 2\tilde{N}_a\tilde{N}_{bc}\tilde{N}_{abc}\right)\mathcal{P}_*\ln(k_0L)\right]$$

$$\alpha \equiv [N_aN_a(1+\mathcal{P}_{\rm loop})]^{-2}\left[N_aN_bN_{ab} + N_aN_{bc}N_{abc}\mathcal{P}_*\ln(k_0L)\right]$$

$$\beta \equiv [N_aN_a(1+\mathcal{P}_{\rm loop})]^{-2}\left[{\rm Tr}(N^3) + N_aN_{bc}N_{abc}\right]\mathcal{P}_*\ln(k_0L),$$

↓

$$\alpha^2 + \beta^2 \geq \frac{1}{2}(\alpha+\beta)^2$$

↓

$$[N_aN^a(1+\mathcal{P}_{\rm loop})]^{-4} \times \left[\left(N_aN_bN_{ab} + N_aN_{bc}N_{abc}\mathcal{P}_*\ln(k_0L)\right)^2 + \left({\rm Tr}(N^3) + N_aN_{bc}N_{abc}\right)^2\mathcal{P}_*^2\ln^2(k_0L)\right] \geq \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$$

2nd term

$$\frac{\left(\mathrm{Tr}(N^3) + N_a N_{bc} N_{abc}\right)^2 \mathcal{P}_*^2 \ln^2(k_0 L)}{(N_a N_a)^4 (1 + \mathcal{P}_{\rm loop})^4}$$

$$< \frac{\left(\mathrm{Tr}(N^4) + 2N_{ac}N_{cb}N_{dab}N_d + N_c N_{cab} N_{abd} N_d\right) \mathcal{P}_* \ln(k_0 L)}{(N_a N_a)^3 (1 + \mathcal{P}_{\rm loop})^3}$$

- where we have used the Cauchy-Schwarz inequality:

$$\mathrm{Tr}^2(LM) \leq \mathrm{Tr}(M^2)\mathrm{Tr}(L^2)$$

with

$$L_{ab} \equiv N_{ab} \qquad M_{ab} \equiv N_{ac}N_{cb} + N_c N_{cab}$$

- and $\mathcal{P}_{\rm loop}/(1 + \mathcal{P}_{\rm loop}) < 1$

Sugiyama, Komatsu & Futamase, arXiv:1101.3636

Collecting terms, here comes a simple result

$$\tau_{\rm NL} + (2\ \text{loop}) > \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$$

- where (2 loop) denotes the following particular term:

$$\mathcal{P}_* \equiv (H_*/2\pi)^2$$

$$(2\ \text{loop}) = \frac{N_{ab}N_{abc}N_{cde}N_{de}\mathcal{P}_*^2 \ln^2(k_0 L)}{(N_a N_a)^3(1+\mathcal{P}_{\rm loop})^3}$$

$$\mathcal{P}_{\rm loop} \equiv \frac{{\rm Tr}(\tilde{N}^2)}{\tilde{N}_a\tilde{N}_a}\mathcal{P}_* \ln(kL)$$

Now, ignore this 2-loop term:

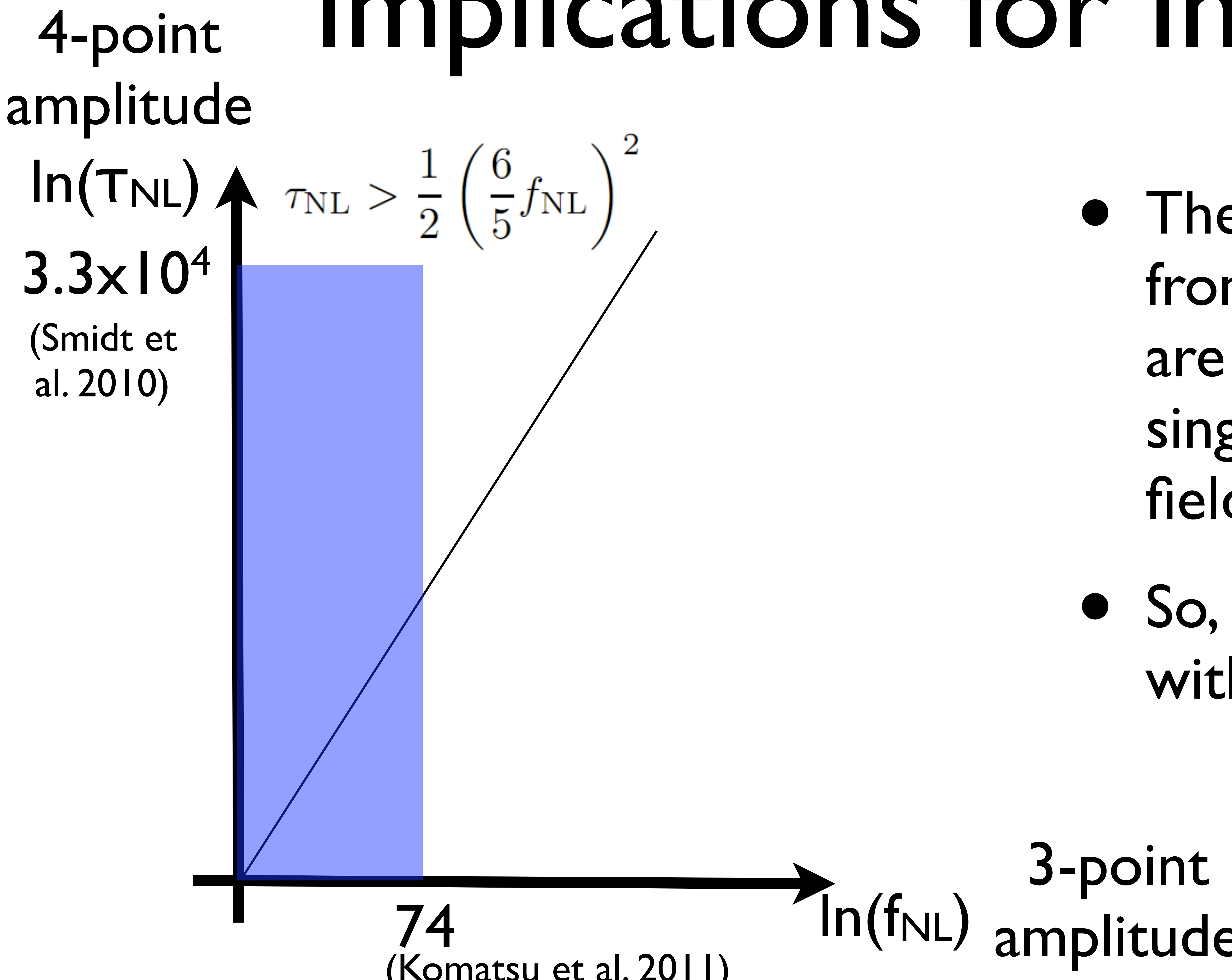
$$\tau_{\rm NL} > \frac{1}{2}\left(\frac{6}{5}f_{\rm NL}\right)^2$$

- The effect of including all 1-loop terms is to change the coefficient of Suyama-Yamaguchi inequality, $\tau_{NL} \geq (6f_{NL}/5)^2$
- This relation *can* have a logarithmic scale dependence.

What we have learned

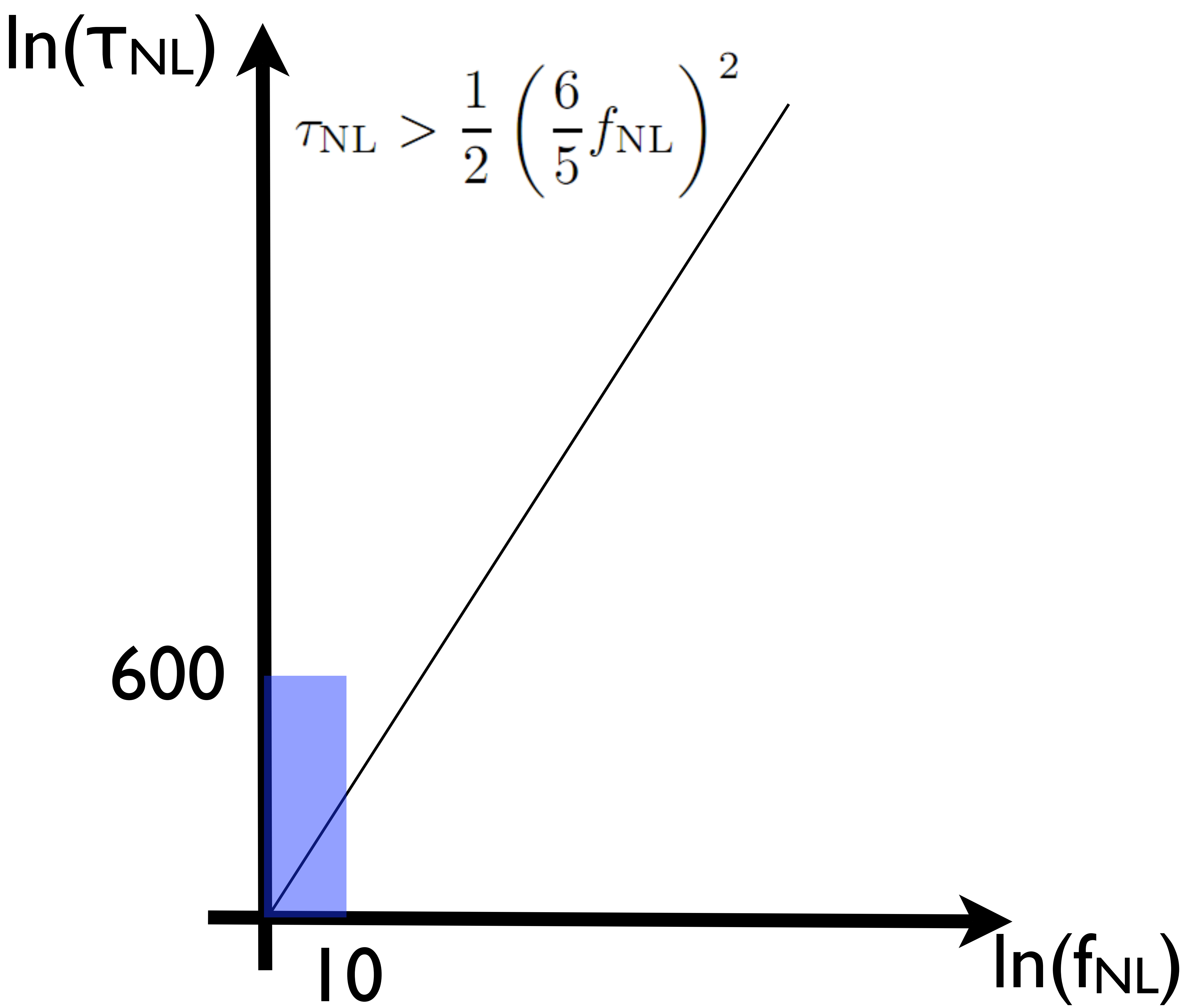
- The tree-level inequality cannot be taken at the face value.
- 1-loop corrections do not destroy the inequality completely (it just modifies the coefficient), so it can still be used to falsify inflation as a mechanism for generating the observed fluctuations.

Implications for Inflation



- The current limits from WMAP 7-year are consistent with single-field or multi-field models.
- So, let's play around with the future.

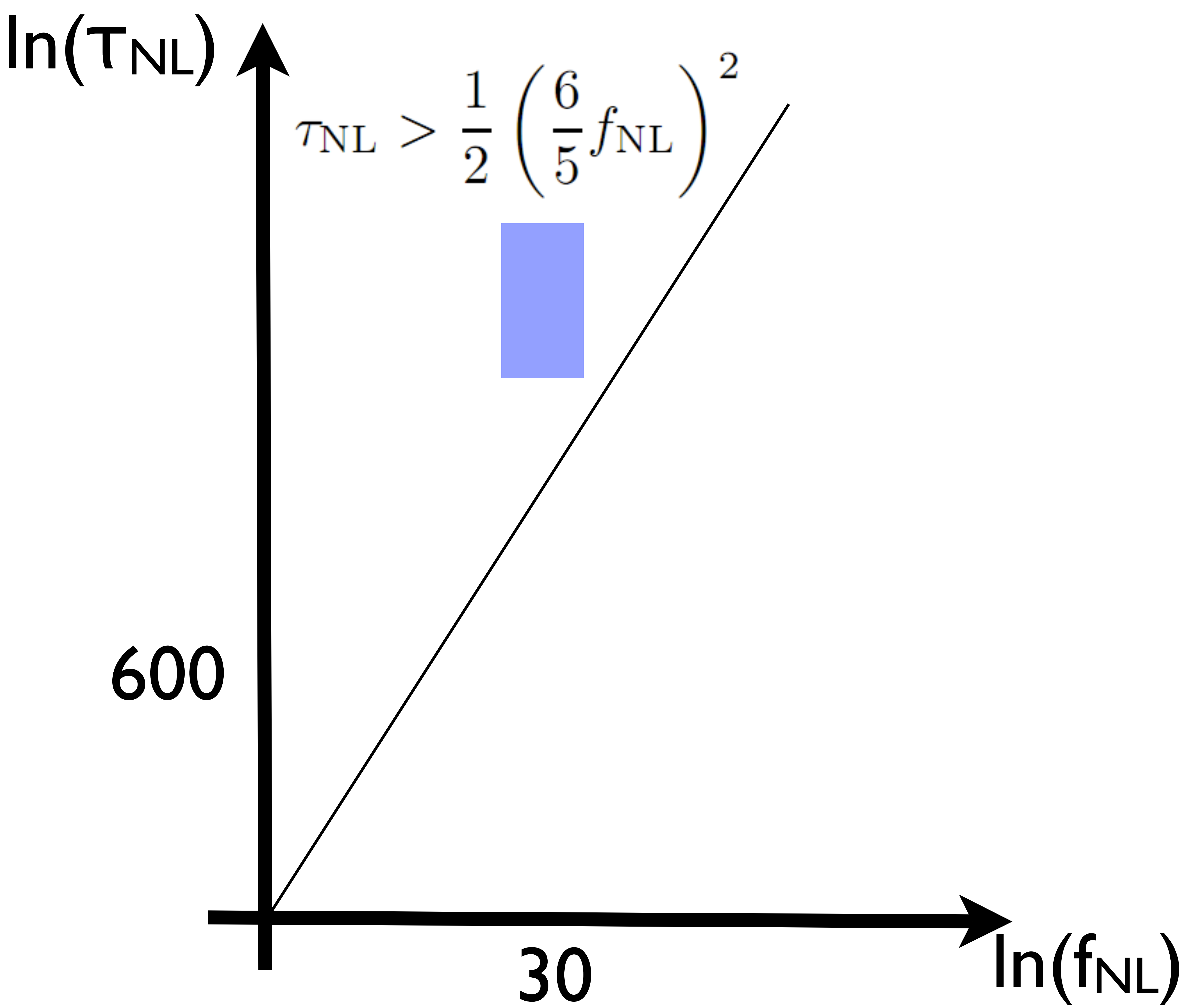
Case A: Single-field Happiness



- No detection of anything (f_{NL} or τ_{NL}) after Planck. Single-field survived the test (for the moment: the future galaxy surveys can improve the limits by a factor of ten).

(Suyama & Yamaguchi 2008; Komatsu 2010; Sugiyama, Komatsu & Futamase 2011)

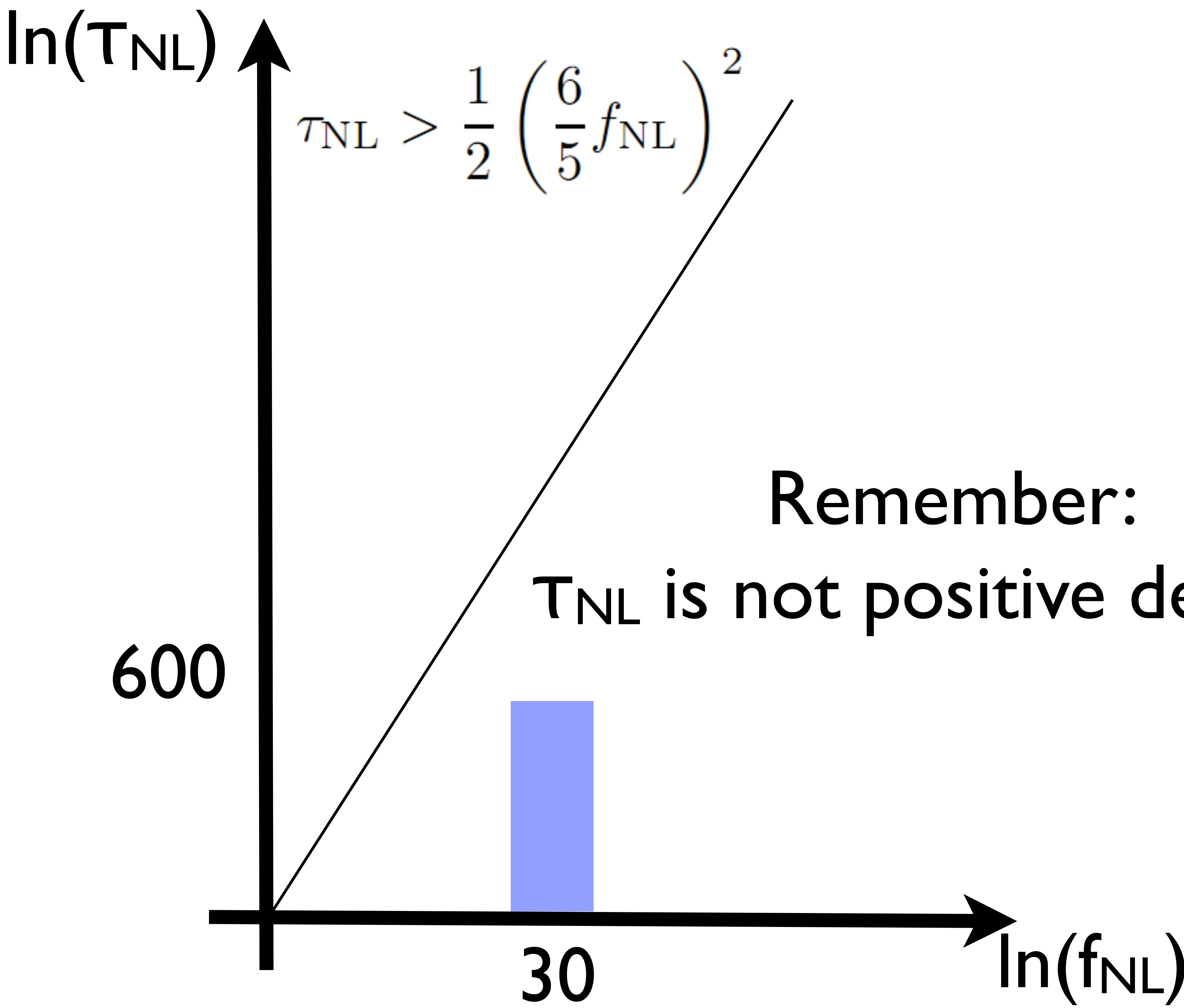
Case B: Multi-field Happiness(?)



- **f_{NL} is detected. Single-field is gone.**
- But, τ_{NL} is also detected, in accordance with **$\tau_{NL}>0.5(6f_{NL}/5)^2$** expected from most multi-field models.

(Suyama & Yamaguchi 2008; Komatsu 2010; Sugiyama, Komatsu & Futamase 2011)

Case C: Madness



Remember:
τ_{NL} is not positive definite

- f_{NL} is detected. Single-field is gone.
- But, τ_{NL} is not detected, or **found to be negative**, inconsistent with $\tau_{NL}>0.5(6f_{NL}/5)^2$.
- **Single-field <u>AND</u> most of multi-field models are gone.**