f_{NL}

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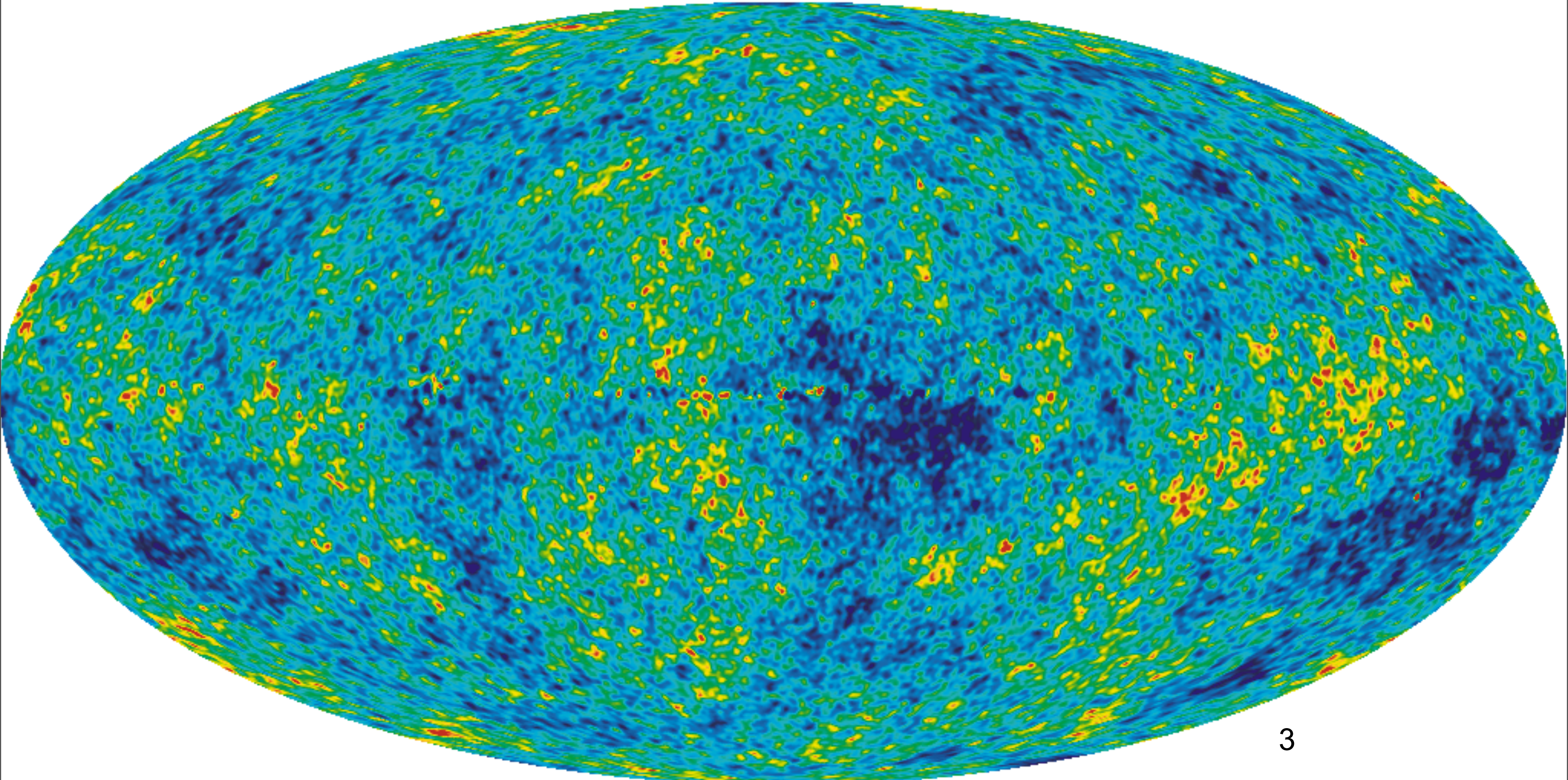
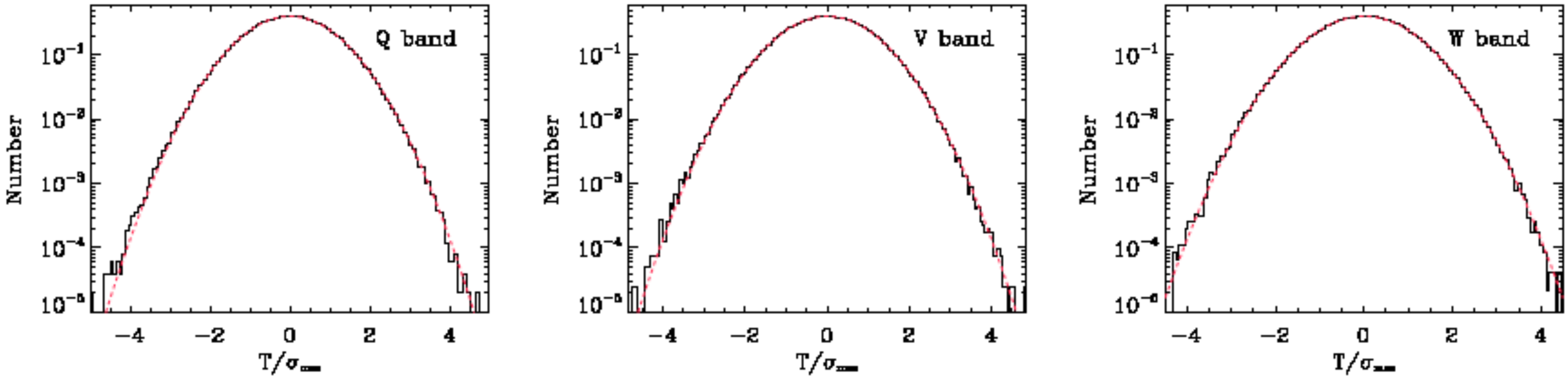
Why Study Non-Gaussianity?

- **Who said that CMB must be Gaussian?**
  - **Don't let people take it for granted.**
  - It is rather remarkable that the distribution of the observed temperatures is so close to a Gaussian distribution.
  - The WMAP map, when smoothed to 1 degree, is entirely dominated by the CMB signal.
    - If it were still noise dominated, no one would be surprised that the map is Gaussian.
  - The WMAP data are telling us that primordial fluctuations are pretty close to a Gaussian distribution.
    - How common is it to have something so close to a Gaussian distribution in astronomy?
  - It is not so easy to explain why CMB is Gaussian, unless we have a compelling early universe model that predicts Gaussian primordial fluctuations: e.g., ***Inflation***.

How Do We Test *Gaussianity of CMB*?



One-point PDF from WMAP



- The one-point distribution of CMB temperature anisotropy looks pretty Gaussian.
  - Left to right: Q (41GHz), V (61GHz), W (94GHz).
- We are therefore talking about quite a subtle effect.

Finding NG.

- **Two approaches to**
- **I. Null (Blind) Tests / "Discovery" Mode**
  - This approach has been most widely used in the literature.
  - One may apply one's favorite statistical tools (higher-order correlations, topology, isotropy, etc) to the data, and show that the data are (*in*)*consistent* with Gaussianity at xx% CL.
  - PROS: This approach is model-independent. Very generic.
  - CONS: We don't know how to interpret the results.
    - "**The data are consistent with Gaussianity**" --- what physics do we learn from that? It is not clear what could be ruled out on the basis of this kind of test.
- **II. "Model-testing," or "Strong Prior" Mode**
  - Somewhat more recent approaches.
  - Try to constrain "Non-gaussian parameter(s)" (e.g., f_{NL})
  - PROS: We know what we are testing, we can quantify our constraints, and we can compare different data sets.
  - CONS: Highly model-dependent. We may well be missing other important non-Gaussian signatures.

Cosmology and Strings: 6 Numbers

- Successful early-universe models **must** satisfy the following observational constraints:
  - The observable universe is nearly flat, **$|\Omega_K| < O(0.02)$**
  - The primordial fluctuations are
    - Nearly Gaussian, **$|f_{NL}| < O(100)$**
    - Nearly scale invariant, **$|n_s - 1| < O(0.05)$**, **$|dn_s/d\ln k| < O(0.05)$**
    - Nearly adiabatic, **$|S/R| < O(0.2)$**

Cosmology and Strings: 6 Numbers

- A “generous” theory would make cosmologists very happy by producing detectable primordial gravity waves ($r>0.01$)…
  - But, this is not a requirement yet.
  - Currently, **$r<O(0.5)$**

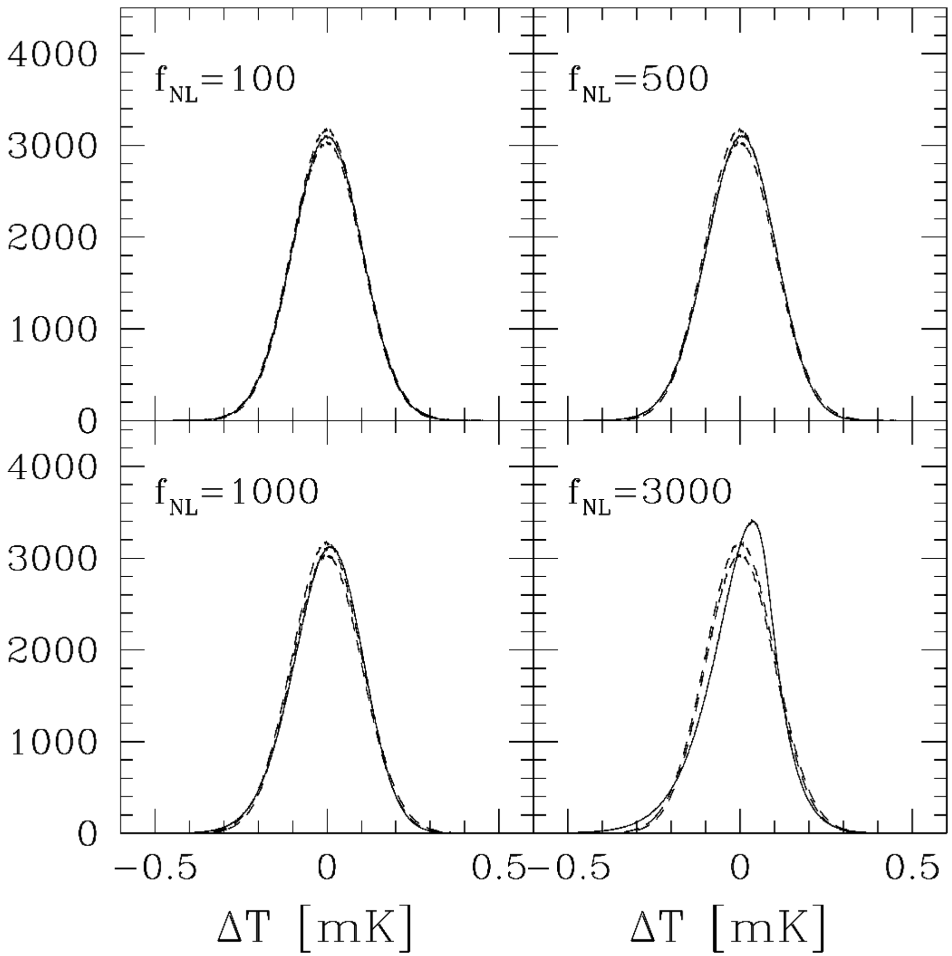
Gaussianity vs Flatness

- We are generally happy that geometry of our observable Universe is flat.
  - Geometry of our Universe is consistent with a flat geometry to **~2%** accuracy at 95% CL. (Spergel et al., WMAP 3yr)

- What do we know about Gaussianity?
  - Parameterize non-Gaussianity: $\Phi=\Phi_L+f_{NL}\Phi_L^2$
    - $\Phi_L \sim 10^{-5}$ is a Gaussian, linear curvature perturbation in the matter era
  - Therefore, $f_{NL}<100$ means that the distribution of Φ is consistent with a Gaussian distribution to $\sim 100\times(10^{-5})^2/(10^{-5})=$**0.1%** accuracy at 95% CL.

- **Remember this fact: "Inflation is supported more by Gaussianity than by flatness."**

How Would f_{NL} Modify PDF?



One-point PDF is not useful for measuring primordial NG. We need something better:

- Three-point Function
  - Bispectrum
- Four-point Function
  - Trispectrum
- Morphological Test
  - Minkowski Functionals

Bispectrum of Primordial Perturbations

- Bispectrum is the Fourier transform of three-point correlation function.
  - Cf. Power spectrum is the Fourier transform of two-point correlation function.
- Bispectrum(k_1,k_2,k_3)=$\langle\Phi(k_1)\Phi(k_2)\Phi(k_3)\rangle$

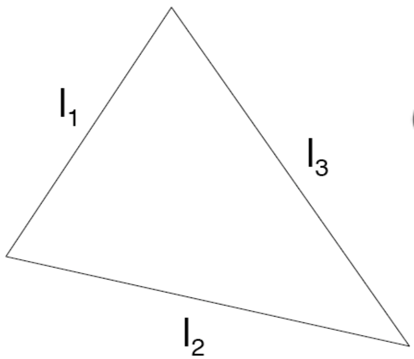
$$= 2(2\pi)^3\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) f_{NL} P_\Phi(k_1) P_\Phi(k_2)$$

where

$$\langle\Phi_L(\mathbf{k}_1)\Phi_L(\mathbf{k}_2)\rangle = (2\pi)^3 P_\Phi(k_1)\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2).$$

Komatsu & Spergel (2001)

Bispectrum of CMB



$$a_{lm} \equiv \int d^2\hat{\mathbf{n}} \frac{\Delta T(\hat{\mathbf{n}})}{T} Y_{lm}^*(\hat{\mathbf{n}})$$

$$= 4\pi(-i)^l \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Phi(\mathbf{k}) g_{Tl}(k) Y_{lm}^*(\hat{\mathbf{k}})$$

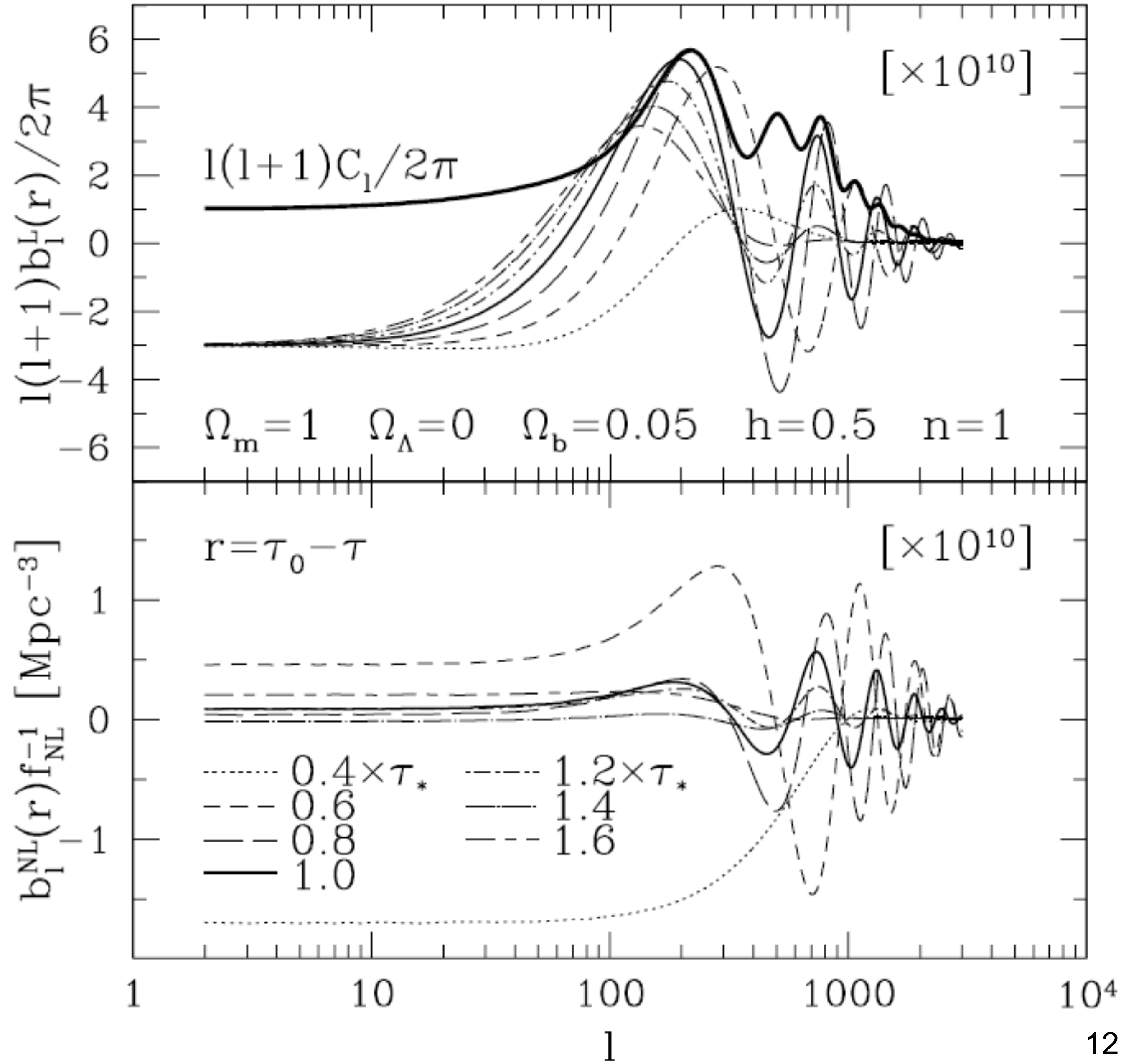
$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} \equiv \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = \mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3} b_{l_1 l_2 l_3}$$

$$b_{l_1 l_2 l_3}^{primary} = 2\int_0^\infty r^2 dr \left[b_{l_1}^L(r) b_{l_2}^L(r) b_{l_3}^{NL}(r) + \right.$$ (cyclic)

$$b_l^L(r) \equiv \frac{2}{\pi}\int_0^\infty k^2 dk P_\Phi(k) g_{Tl}(k) j_l(kr),$$

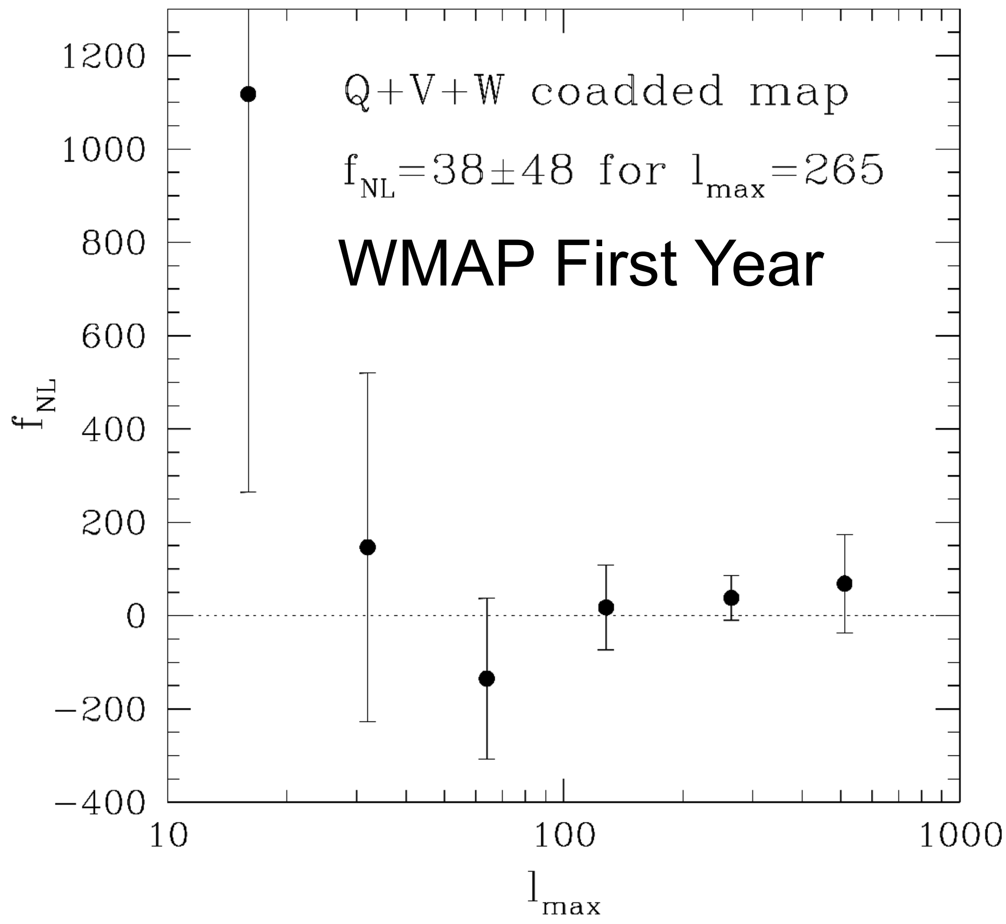
$$b_l^{NL}(r) \equiv \frac{2}{\pi}\int_0^\infty k^2 dk f_{NL} g_{Tl}(k) j_l(kr).$$

Komatsu & Spergel (2001)



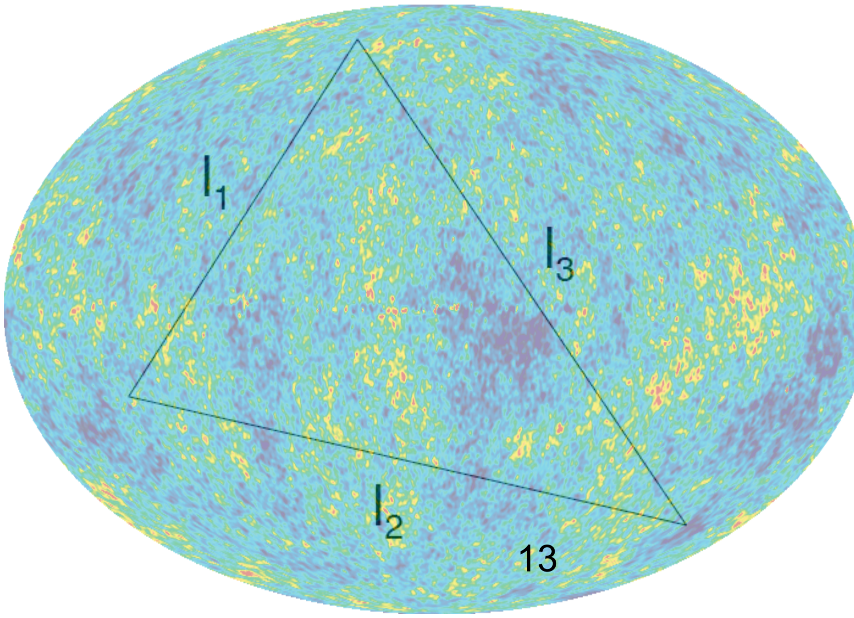
Komatsu et al. (2003); Spergel et al. (2007)

Bispectrum Constraints



-58 < f_{NL} < +134 (95% CL) (1yr)

-54 < f_{NL} < +114 (95% CL) (3yr)



Trispectrum of Primordial Perturbations

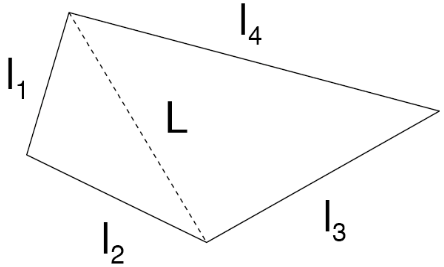
- Trispectrum is the Fourier transform of four-point correlation function.
- Trispectrum(k_1,k_2,k_3,k_4)

$$=\langle\Phi(k_1)\Phi(k_2)\Phi(k_3)\Phi(k_4)\rangle$$

  which can be sensitive to the higher-order terms:

$$\Phi(\boldsymbol{x}) = \Phi_{\rm L}(\boldsymbol{x}) + f_{\rm NL}\left[\Phi_{\rm L}^2(\boldsymbol{x}) - \langle\Phi_{\rm L}^2(\boldsymbol{x})\rangle\right] + f_2\Phi_{\rm L}^3(\boldsymbol{x})$$

Okamoto & Hu (2002); Kogo & Komatsu (2006)



Trispectrum of CMB

$$\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{l_4m_4}\rangle = \sum_{LM}(-1)^M \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix}\begin{pmatrix} l_3 & l_4 & L \\ m_3 & m_4 & M \end{pmatrix} T^{l_1l_2}_{l_3l_4}(L)$$

$$T_c{}^{l_1l_2}_{l_3l_4}(L) = P^{l_1l_2}_{l_3l_4}(L) + (2L+1)\sum_{L'}\left[(-1)^{l_2+l_3}\begin{Bmatrix} l_1 & l_2 & L \\ l_4 & l_3 & L' \end{Bmatrix} P^{l_1l_3}_{l_2l_4}(L') + (-1)^{L+L'}\begin{Bmatrix} l_1 & l_2 & L \\ l_3 & l_4 & L' \end{Bmatrix} P^{l_1l_4}_{l_3l_2}(L')\right],$$

where

$$P^{l_1l_2}_{l_3l_4}(L) = t^{l_1l_2}_{l_3l_4}(L) + (-1)^{2L+l_1+l_2+l_3+l_4} t^{l_2l_1}_{l_4l_3}(L) + (-1)^{L+l_3+l_4} t^{l_1l_2}_{l_4l_3}(L) + (-1)^{L+l_1+l_2} t^{l_2l_1}_{l_3l_4}(L).$$

$$t^{l_1l_2}_{l_3l_4}(L) = \int r_1^2dr_1r_2^2dr_2\, F_L(r_1,r_2)\alpha_{l_1}(r_1)\beta_{l_2}(r_1)\alpha_{l_3}(r_2)\beta_{l_4}(r_2)h_{l_1Ll_2}h_{l_3Ll_4}$$
$$+\int r^2dr\, \beta_{l_2}(r)\beta_{l_4}(r)\left[\mu_{l_1}(r)\beta_{l_3}(r)+\beta_{l_1}(r)\mu_{l_3}(r)\right]h_{l_1Ll_2}h_{l_3Ll_4},$$

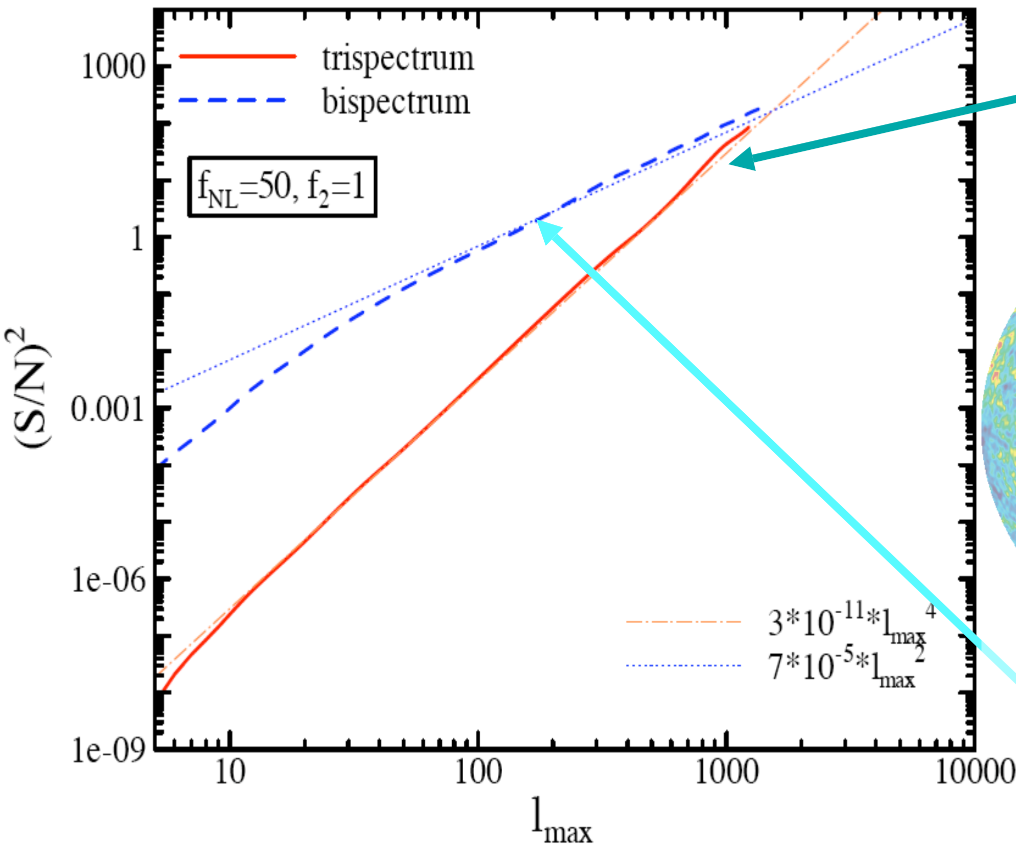
alpha$_l$(r)=2b$_l^{NL}$(r); beta$_l$(r)=b$_l^L$(r); $\mu_l(r) \equiv \frac{2}{\pi}\int k^2dk\, f_2 g_{Tl}(k) j_l(kr)$

Measuring Trispectrum

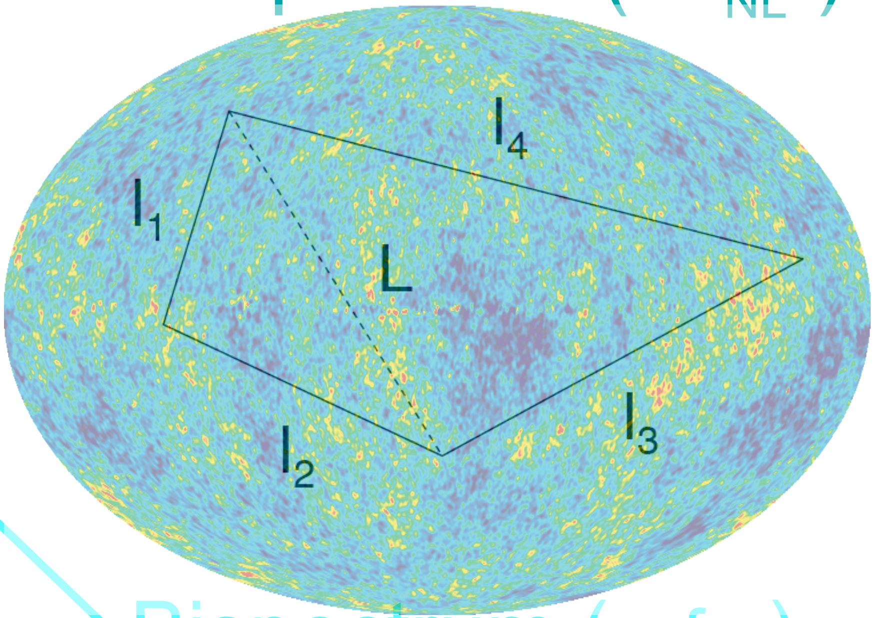
- It's pretty painful to measure all the quadrilateral configurations.
  - Measurements from the COBE 4-year data (Komatsu 2001; Kunz et al. 2001)
- Only limited configurations measured from the WMAP 3-year data
  - Spergel et al. (2007)
- No evidence for non-Gaussianity, but f_{NL} has not been constrained by the trispectrum yet. (Work to do.)

Kogo & Komatsu (2006)

Trispectrum: Not useful for WMAP, but maybe useful for Planck, if f_{NL} is greater than ~50

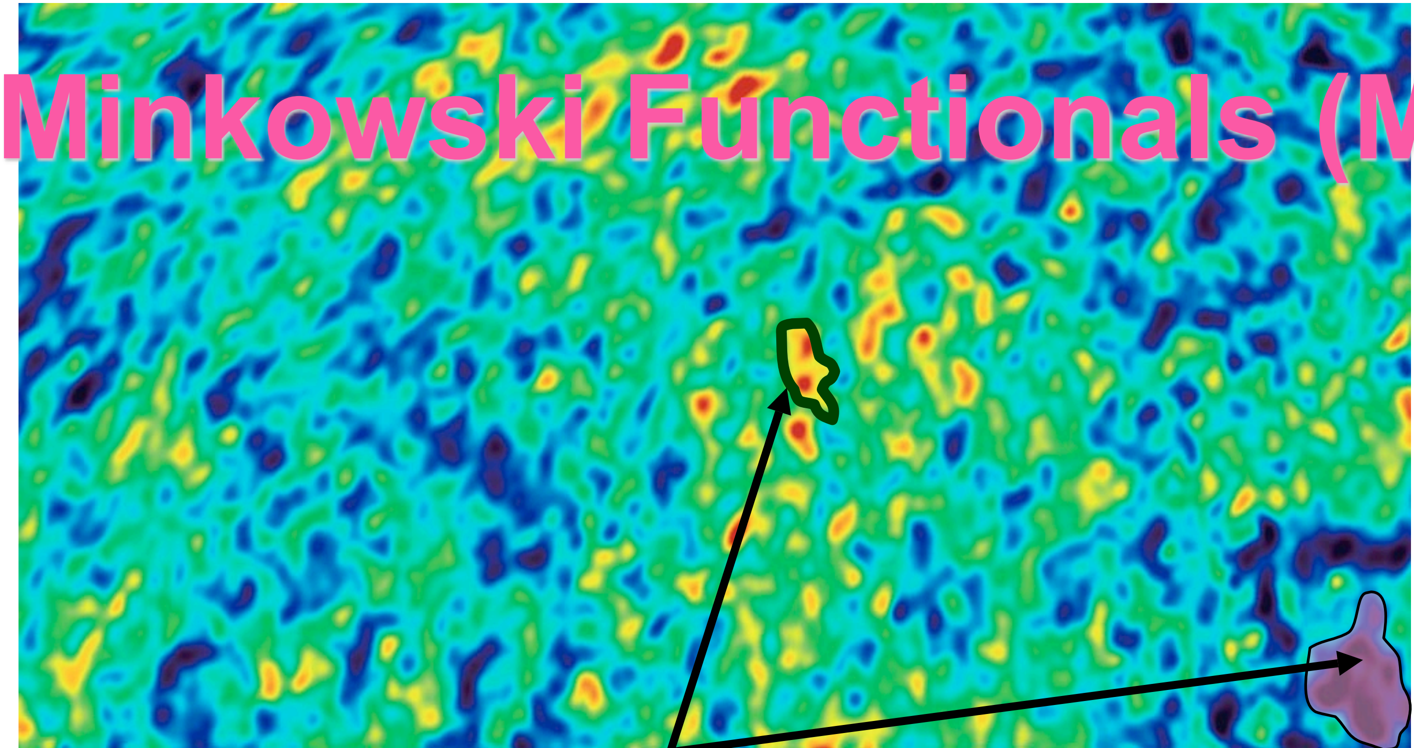


- Trispectrum (~ f_{NL}^2)



- Bispectrum (~ f_{NL})

Minkowski Functionals (MFs)

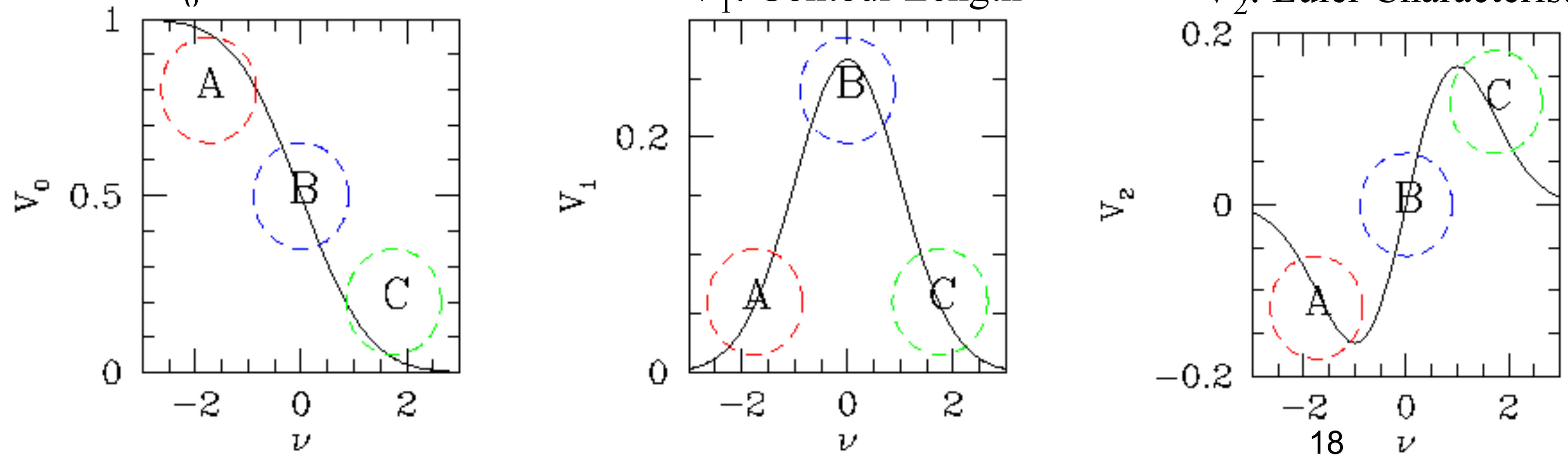


The number of hot spots minus cold spots.

V_0:surface area

V_1: Contour Length

V_2: Euler Characteristic



Hikage, Komatsu & Matsubara (2006)

Analytical formulae of MFs

Perturbative formulae of MFs (Matsubara 2003)

Gaussian term

$$V_k(\nu) = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_2}{\omega_{2-k}\omega_k} \left(\frac{\sigma_1}{\sqrt{2}\sigma_0} \right)^k e^{-\nu^2/2} \{ H_{k-1}(\nu) + \left[\frac{1}{6} S^{(0)} H_{k+2}(\nu) + \frac{k}{3} S^{(1)} H_k(\nu) + \frac{k(k-1)}{6} S^{(2)} H_{k-2}(\nu) \right] \sigma_0 + O(\sigma_0^2) \}$$

$(k = 0,1,2)$

leading order of Non-Gaussian term

$$\sigma_j^2 = \frac{1}{4} \sum_l (2l+1)[l(l+1)]^j C_l W_l^2$$

W_l: smoothing kernel

$$\omega_0 = 1, \omega_1 = 1, \omega_2 = \pi, \omega_3 = 4\pi/3$$

H_k : k- th Hermite polynomial

$S^{(a)}$: skewness parameters (a = 0,1,2)

In weakly non-Gaussian fields (σ_0<<1) , the non-Gaussianity in MFs is characterized by three skewness parameters $S^{(a)}$.

Matsubara (2003)

3 "Skewness Parameters"

- Ordinary skewness

$$S^{(0)} \equiv \frac{\langle f^3 \rangle}{\sigma_0^4},$$

- Second derivative

$$S^{(1)} \equiv -\frac{3}{4} \frac{\langle f^2 (\nabla^2 f) \rangle}{\sigma_0^2 \sigma_1^2},$$

- (First derivative)^2 x Second derivative

$$S^{(2)} \equiv -\frac{3d}{2(d-1)} \frac{\langle (\nabla f) \cdot (\nabla f)(\nabla^2 f) \rangle}{\sigma_1^4},$$

$$S^{(0)} = \frac{3}{2\pi\sigma_0^4} \sum_{2\le l_1 \le l_2 \le l_3} I_{l_1 l_2 l_3}^2 b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3},$$

$$S^{(1)} = \frac{3}{8\pi\sigma_0^2\sigma_1^2} \sum_{2\le l_1 \le l_2 \le l_3} [l_1(l_1+1) + l_2(l_2+1) + l_3(l_3+1)] \times I_{l_1 l_2 l_3}^2 b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3},$$

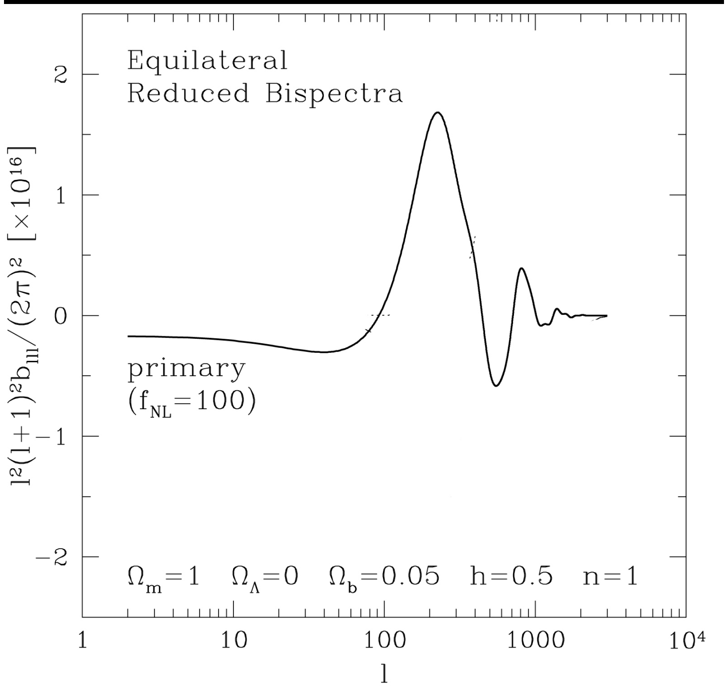
$$S^{(2)} = \frac{3}{4\pi\sigma_1^4} \sum_{2\le l_1 \le l_2 \le l_3} \{[l_1(l_1+1) + l_2(l_2+1) - l_3(l_3+1)] \times l_3(l_3+1) + (\text{cyc.})\} I_{l_1 l_2 l_3}^2 b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3},$$

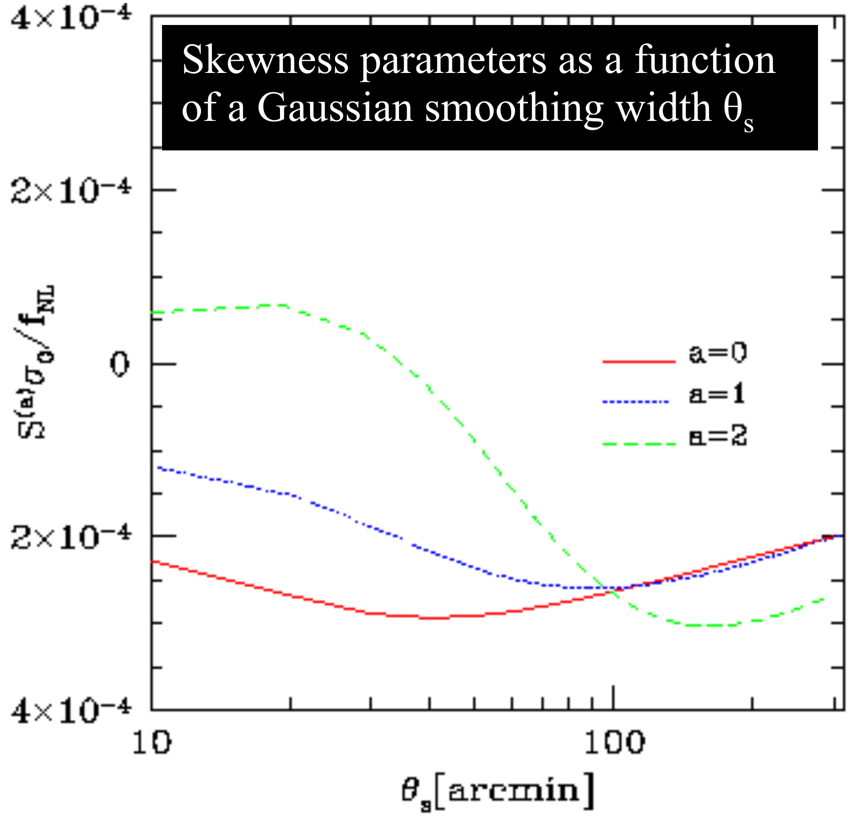
$S^{(0)}$: Simple average of b_{l1l2l3}

$S^{(1)}$: l^2 weighted average

$S^{(2)}$: l^4 weighted average

Analytical predictions of bispectrum at f_{NL}=100 (Komatsu & Spergel 2001)





Skewness parameters as a function of a Gaussian smoothing width θ_s

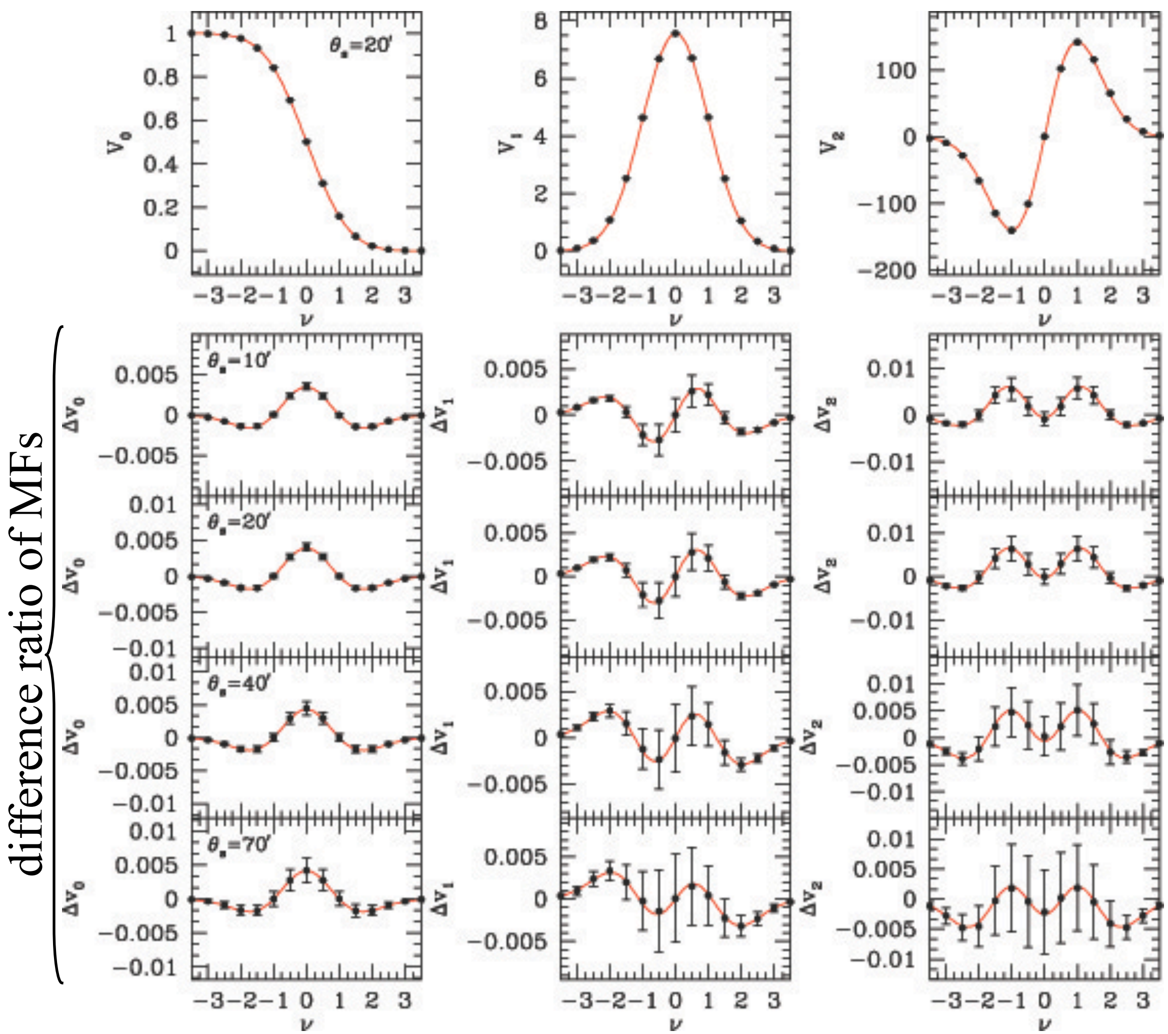
Note: This is Generic.

- The skewness parameters are the direct observables from the Minkowski functionals.
- The skewness parameters can be calculated directly from the bispectrum.
- It can be applied to *any* form of the bispectrum!
  - Statistical power is weaker than the full bispectrum, but the application can be broader than a bispectrum estimator that is tailored for a specific form of non-Gaussianity, like f_{NL}.

Hikage et al. (2007)

Comparison of analytical formulae with Non-Gaussian simulations

Surface area   Contour Length   Euler Characteristic



Comparison of MFs between analytical predictions and non-Gaussian simulations with f_{NL}=100 at different Gaussian smoothing scales, $\boldsymbol{\theta_s}$

Simulations are done for WMAP.

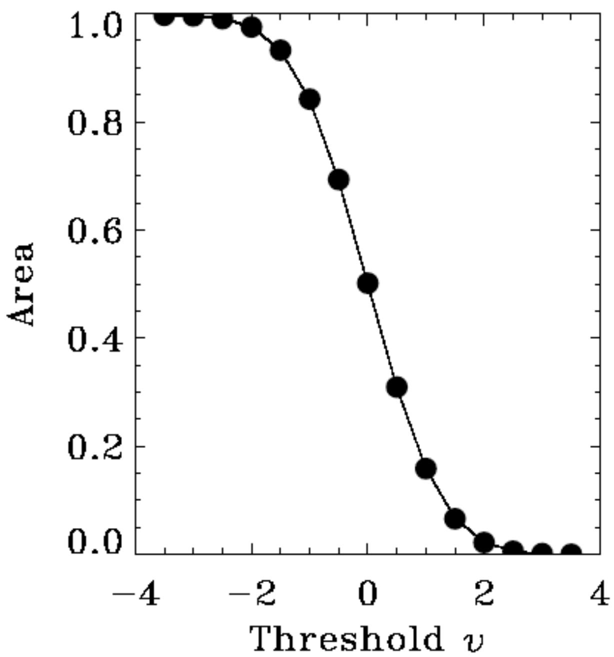
Analytical formulae agree with non-Gaussian simulations very well.

Komatsu et al. (2003); Spergel et al. (2007); Hikage et al. (2007)

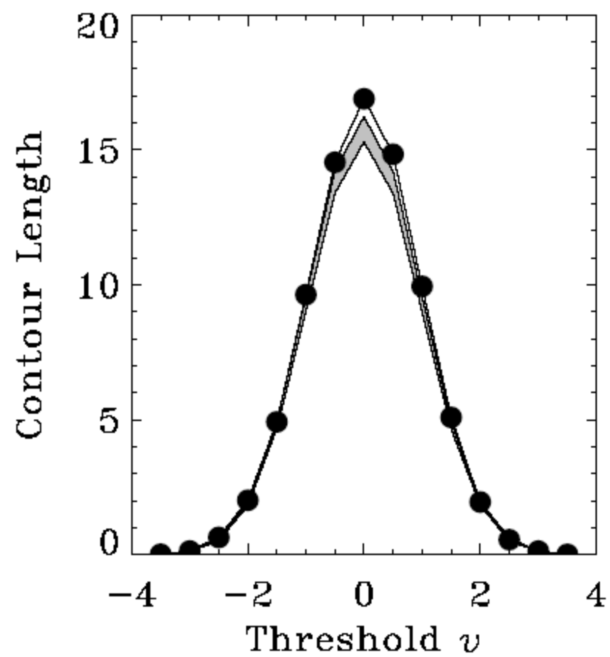
MFs from *WMAP*

(1yr) **f_{NL} < +117** (95% CL)

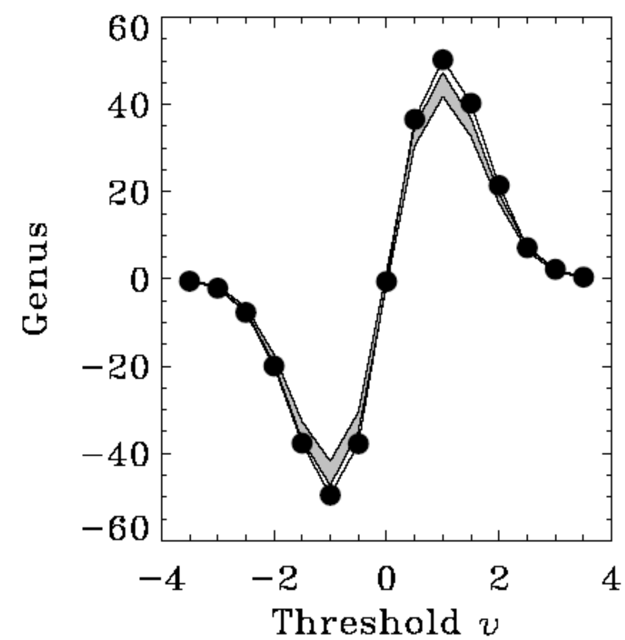
(3yr) **-70 < f_{NL} < +90** (95% CL)



Area



Contour Length



Euler Characteristic

Gaussianity vs Flatness: Future

- **Flatness will never beat Gaussianity.**
  - In 5-10 years, we will know **flatness** to **0.1%** level.
  - In 5-10 years, we will know **Gaussianity** to **0.01%** level (f_{NL}~10), or even to **0.005%** level (f_{NL}~5), at 95% CL.
- However, a real potential of Gaussianity test is that **we might detect something at this level** (multi-field, curvaton, DBI, ghost cond., new ekpyrotic…)
  - Or, we might detect curvature first?
  - Is 0.1% curvature interesting/motivated?

Confusion about f_{NL} (1): Sign

- What is f_{NL} that is actually measured by WMAP?
- When we expand Φ as $\Phi=\Phi_L+f_{NL}\Phi_L^2$, Φ is **Bardeen's curvature perturbation** (metric space-space), Φ_H, in the **matter dominated era**.
  - Let's get this stright: Φ is **not** Newtonian potential (which is metric time-time, not space-space)
  - Newtonian potential in this notation is $-\Phi$. (There is a minus sign!)
  - In the large-scale limit, temperature anisotropy is $\Delta T/T=-(1/3)\Phi$.
  - A positive f_{NL} results in a negative skewness of ΔT.
- **It is useful to remember the physical effects:**

  f_{NL} positive

  = Temperature skewed negative (more cold spots)

  = Matter density skewed positive (more objects)

Confusion about f_{NL} (2): Primordial vs Matter Era

- In terms of the **primordial** curvature perturbation in the comoving gauge, *R*, Bardeen's curvature perturbation in the matter era is given by $\Phi_L=+(3/5)R_L$ at the linear level (notice the plus sign).

- Therefore, $R=R_L+(3/5)f_{NL}R_L^2$

  x $R=R_L-(3/5)f_{NL}R_L^2$

  x $R=R_L+f_{NL}R_L^2$

- There is another popular quantity, $\zeta=+R$. (Bardeen, Steinhardt & Turner (1983); Notice the plus sign.)

  $\zeta=\zeta_L+(3/5)f_{NL}\zeta_L^2$

  x $\zeta=\zeta_L-(3/5)f_{NL}\zeta_L^2$

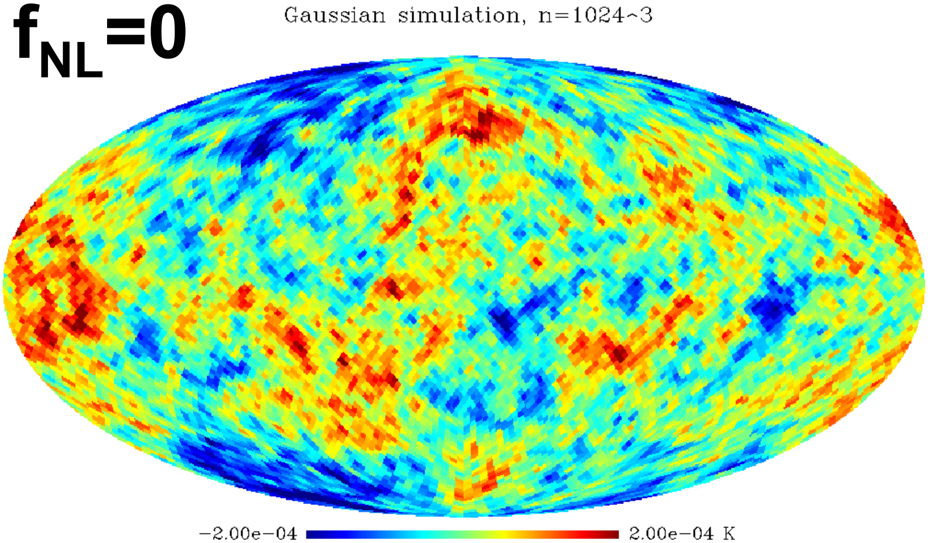
Confusion about f_{NL} (3): Maldacena Effect

- Juan Maldacena's celebrated non-Gaussianity paper (Maldacena 2003) uses the sign convention that is minus of that in Komatsu & Spergel (2001):
  - $+f_{NL}$(Maldacena) = $-f_{NL}$(Komatsu&Spergel)
- The result: cosmologists and high-energy physicists have often been using different sign conventions.
- It is always useful to ask ourselves, "*do we get more cold spots in CMB for $f_{NL}>0$?*"
  - If yes, it's Komatsu&Spergel convention.
  - If no, it's Maldacena convention.

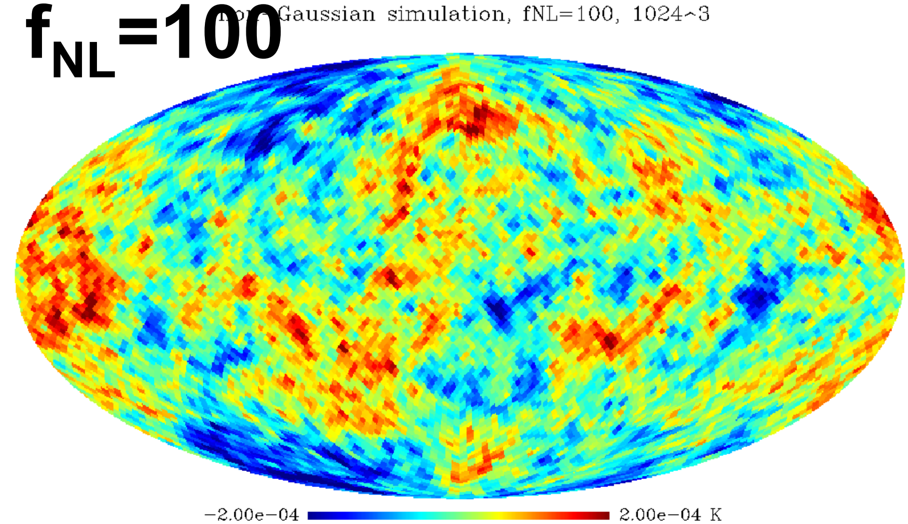
Positive f_{NL} = More Cold Spots

Simulated temperature maps from $\Phi(x) = \Phi_G(x) + f_{NL}\Phi_G^2(x)$

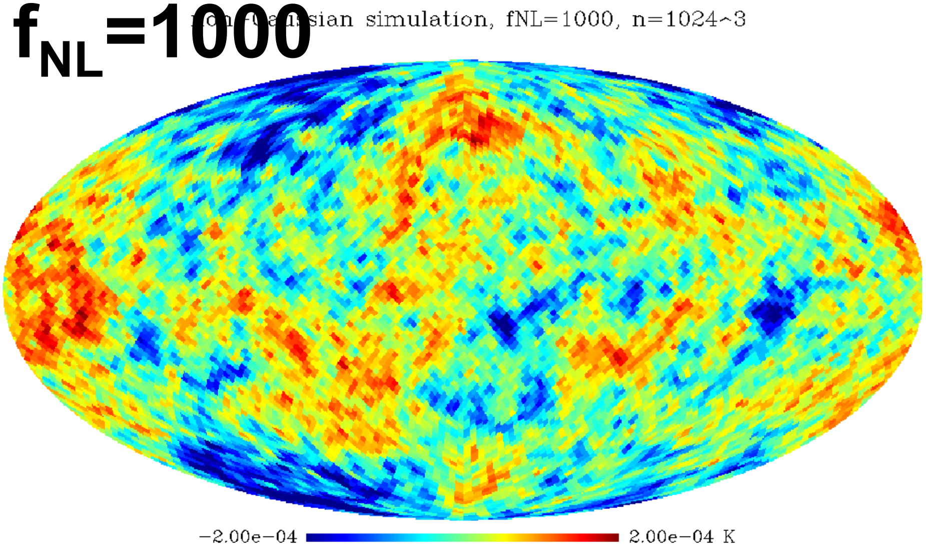
f_{NL}=0



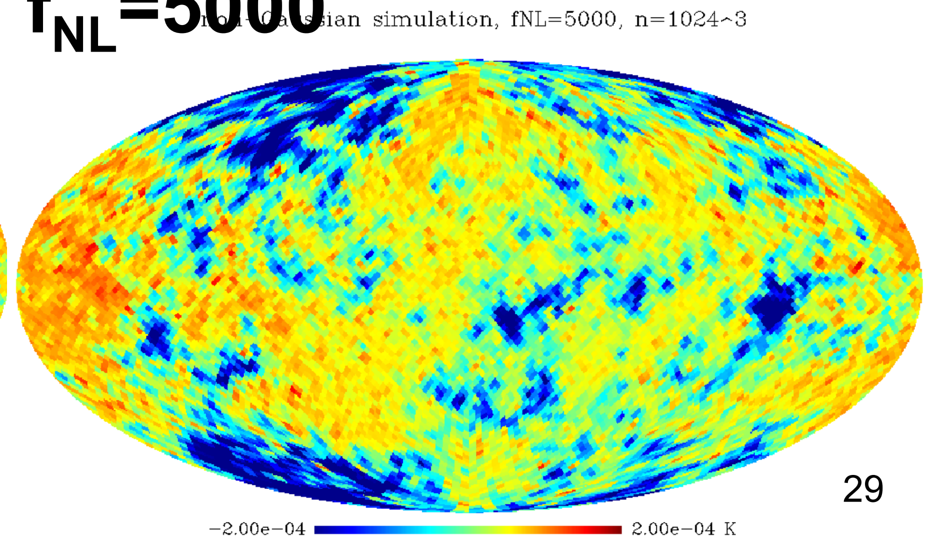
f_{NL}=100



f_{NL}=1000



f_{NL}=5000

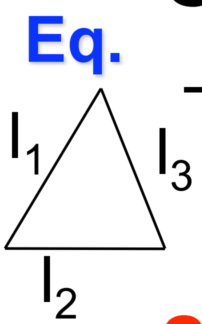


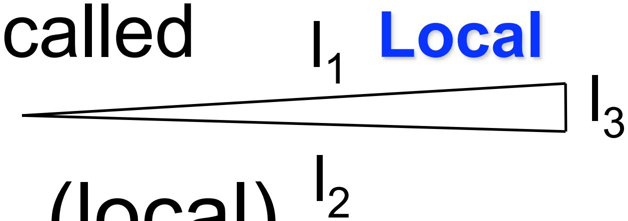
Journey For Measuring f_{NL}

- **2001**: Bispectrum method proposed and developed for f_{NL} (*Komatsu & Spergel*)
- **2002**: First observational constraint on f_{NL} from the COBE 4-yr data (*Komatsu, Wandelt, Spergel, Banday & Gorski*)
  - **$-3500 < f_{NL} < +2000$** (95%CL; **$l_{max}=20$**)
- **2003**: First numerical simulation of CMB with f_{NL} (*Komatsu*)
- **2003**: WMAP 1-year (*Komatsu, WMAP team*)
  - **$-58 < f_{NL} < +134$** (95% CL; **$l_{max}=265$**)

Journey For Measuring f_{NL}

- **2004**: Classification scheme of triangle dependence proposed (Babich, Creminelli & Zaldarriaga)
  - There are two "f_{NL}": the original f_{NL} is called "local," and the new one is called "equilateral."





- **2005**: Fast estimator for f_{NL}(local) developed ("KSW" estimator; *Komatsu, Spergel & Wandelt*)

Journey For Measuring f_{NL}

- **2006**: Improvement made to the KSW method, and applied to WMAP 1-year data by Harvard group (*Creminelli, et al.*)
  - **$-27 < f_{NL}$(local) $< +121$** (95% CL; **lmax=335**)
- **2006**: Fast estimator for f_{NL}(equilateral) developed, and applied to WMAP 1-year data by Harvard group (*Creminelli, et al.*)
  - **$-366 < f_{NL}$(equilateral) $< +238$** (95% CL; **lmax=405**)

Journey For Measuring f_{NL}

- **2007**: WMAP 3-year constraints
  - **$-54 < f_{NL}$(local) $< +114$** (95% CL; **lmax=350**) (*Spergel, WMAP team*)
  - **$-36 < f_{NL}$(local) $< +100$** (95% CL; **lmax=370**) (*Creminelli, et al.*)
  - **$-256 < f_{NL}$(equilateral) $< +332$** (95% CL; **lmax=475**) (*Creminelli, et al.*)
- **2007**: We've made further improvement to Harvard group's extension of the KSW method; **now, the estimator is very close to optimal** (*Yadav, Komatsu, Wandelt*)

Latest News on f_{NL}

- **2007**: Latest constraint from the WMAP 3-year data using the new YKW estimator
  - **+27 < f_{NL}(local) < +147** (95% CL; **lmax=750**) (Yadav & Wandelt, arXiv:0712.1148)
  - Note a significant jump in lmax.
  - A "hint" of f_{NL}(local)>0 at more than two σ?
- **Our independent analysis showed a similar level of f_{NL}(local), but no evidence for f_{NL}(equilateral).**

There have been many claims of non-Gaussianity at the 2-3 σ.
This is the best physically motivated one, and will be testable with more data.

WMAP: Future Prospects

- Could more years of data from WMAP yield a definitive answer?
  - 3-year latest [Y&W]: f_{NL}(local) = 87 +/- 60 (95%)
- Projected 95% uncertainty from WMAP
  - 5yr: Error[f_{NL}(local)] ~ 50
  - 8yr: Error[f_{NL}(local)] ~ 42
  - 12yr: Error[f_{NL}(local)] ~ 38

An unambiguous (>4σ) detection of f_{NL}(local) at this level with the future (e.g., 8yr) WMAP data could be a truly remarkable discovery.

More On Future Prospects

- CMB: Planck (temperature + polarization): **f_{NL}(local)<6 (95%)**
  - Yadav, Komatsu & Wandelt (2007)
- Large-scale Structure: e.g., ADEPT, CIP: **f_{NL}(local)<7 (95%); f_{NL}(equilateral)<90 (95%)**
  - Sefusatti & Komatsu (2007)
- CMB and LSS are independent. By combining these two constraints, we get **f_{NL}(local)<4.5**. This is currently the best constraint that we can possibly achieve in the foreseeable future (~10 years)

A Comment on Jeong&Smoot

- Jeong&Smoot (arXiv:0710.2371) claim significant detections of f_{NL} from the WMAP 3-yr data, $+23 < f_{NL}(\mathrm{local}) < +75$ (95% CL)
- Their analysis is based on one-point distribution of temperature, which is mostly measuring skewness.
- However, we know that it is not possible to see f_{NL} at this level from just skewness of the WMAP data (as proved by Komatsu&Spergel 2001). So, what is going on?

Here is the Reason…

- The biggest issue is that their simulations of CMB are not correct.
  - They completely ignored pixel-to-pixel correlation of the CMB signal.
  - In other words, they simulated "CMB" as a pure random, white noise (just like detector noise).
  - Their simulation therefore underestimated the uncertainty in their f_{NL} grossly; the 95% error should be more like 160 rather than 13, which is what they report.

If f_{NL} is large, what are the implications?

Three Sources of Non-Gaussianity

- It is important to remember that f_{NL} receives **<u>three contributions</u>**:

1. Non-linearity in inflaton fluctuations, $\delta\phi$
   - Falk, Rangarajan & Srendnicki (1993)
   - Maldacena (2003)
2. Non-linearity in Φ-$\delta\phi$ relation
   - Salopek & Bond (1990; 1991)
   - Matarrese et al. (2nd order PT papers)
   - δN papers; gradient-expansion papers
3. Non-linearity in $\Delta T/T$-Φ relation
   - Pyne & Carroll (1996)
   - Mollerach & Matarrese (1997)

Gaussian quantum fluctuation

↓ $\delta\phi \sim g_{\delta\phi}(\eta + m_{pl}^{-1} f_\eta \eta^2)$

Non-Gaussian inflaton fluctuation

↓ $\Phi \sim m_{pl}^{-1} g_\Phi(\delta\phi + m_{pl}^{-1} f_{\delta\phi} \delta\phi^2)$

Non-Gaussian curvature perturbation

↓ $\Delta T/T \sim g_T(\Phi + f_\Phi \Phi^2)$

Non-Gaussian CMB anisotropy

$$\Delta T/T \sim g_T[\Phi_L + (f_\Phi + g_\Phi^{-1} f_{\delta\phi} + g_\Phi^{-1} g_{\delta\phi}^{-1} f_\eta)\Phi_L^2]$$

- $g_{\delta\phi}=1$
- $f_\eta \sim O(\varepsilon^{1/2})$

in slow-roll

- $g_\Phi \sim O(1/\varepsilon^{1/2})$
- $f_{\delta\phi} \sim O(\varepsilon^{1/2})$

in slow-roll

- $g_T = -1/3$
- $f_\Phi \sim O(1)$

for Sachs-Wolfe

$f_{NL} \sim f_\Phi + g_\Phi^{-1} f_{\delta\phi} + g_\Phi^{-1} g_{\delta\phi}^{-1} f_\eta \sim O(1) + O(\varepsilon)$ in slow-roll

Komatsu, astro-ph/0206039

1. Generating Non-Gaussian $\delta\phi$

- You need cubic interaction terms (or higher order) of fields.
  - $V(\phi) \sim \phi^3$: Falk, Rangarajan & Srendnicki (1993) [gravity not included yet]
  - Full expansion of the action, including gravity action, to cubic order was done a decade later by Maldacena (2003)

$$\phi = \phi(t) + \varphi(t,x)$$

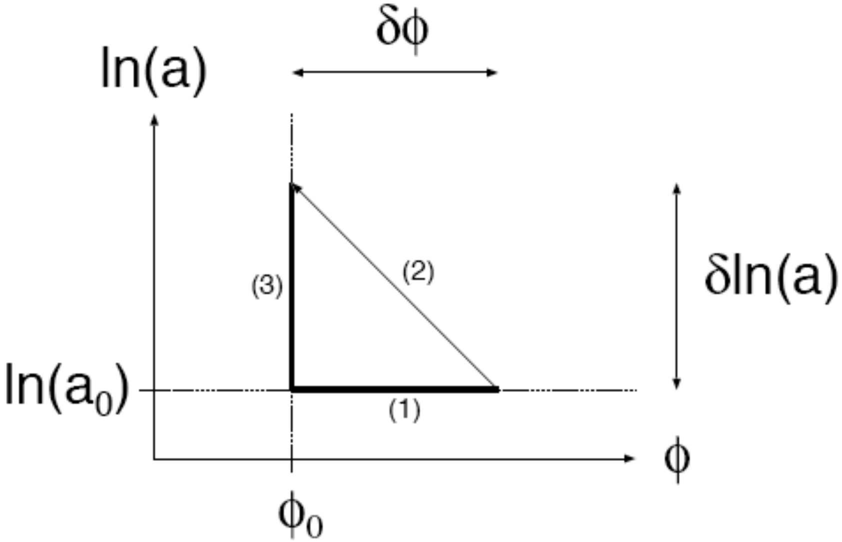
$$\partial^2\chi = \frac{\dot\phi^2}{2\dot\rho^2}\frac{d}{dt}\left(-\frac{\dot\rho}{\dot\phi}\varphi\right)$$

$$h_{ij} = e^{2\rho}\hat{h}_{ij}$$

$$S_3 = \int e^{3\rho}\left(-\frac{\dot\phi}{4\dot\rho}\varphi\dot\varphi^2 - e^{-2\rho}\frac{\dot\phi}{4\dot\rho}\varphi(\partial\varphi)^2 - \dot\varphi\partial_i\chi\partial_i\varphi + \right.$$
$$+ \frac{3\dot\phi^3}{8\dot\rho}\varphi^3 - \frac{\dot\phi^5}{16\dot\rho^3}\varphi^3 - \frac{\dot\phi V''}{4\dot\rho}\varphi^3 - \frac{V'''}{6}\varphi^3 + \frac{\dot\phi^3}{4\dot\rho^2}\varphi^2\dot\varphi + \frac{\dot\phi^2}{4\dot\rho}\varphi^2\partial^2\chi$$
$$\left. + \frac{\dot\phi}{4\dot\rho}(-\varphi\partial_i\partial_j\chi\partial_i\partial_j\chi + \varphi\partial^2\chi\partial^2\chi)\right)$$

2. Non-linear Mapping

- The observable is the curvature perturbation, R. How do we relate R to the scalar field perturbation $\delta\phi$?
- Hypersurface transformation (Salopek & Bond 1990); a.k.a. δN formalism.



(1)Scalar field perturbation
(2)Evolve the scale factor, a, until ϕ matches ϕ_0
(3)$R=\ln(a)-\ln(a_0)$

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Result of Non-linear Mapping

$$N = -\frac{4\pi G}{\partial H/\partial \phi}$$ [N is the Lapse function.]

$$\mathcal{R}_{\rm com} = -\int_{\phi_0}^{\phi_0+\delta\phi_{\rm flat}} d\phi\, \frac{N(\phi)H(\phi)}{\dot{\phi}} = 4\pi G \int_{\phi_0}^{\phi_0+\delta\phi_{\rm flat}} d\phi \left[\frac{\partial \ln H}{\partial \phi}\right]^{-1}$$

Expand R to the quadratic order in $\delta\phi$:

$$\mathcal{R}_{\rm com} = \mathcal{R}^{\rm L}_{\rm com} + \mathcal{R}^{\rm NL}_{\rm com} \Longleftrightarrow$$

$$\mathcal{R}^{\rm L}_{\rm com} = 4\pi G \left(\frac{\partial \ln H}{\partial \phi}\right)^{-1} \delta\phi_{\rm flat},$$

$$\mathcal{R}^{\rm NL}_{\rm com} = -\frac{1}{8\pi G}\left(\frac{\partial^2 \ln H}{\partial \phi^2}\right)\left(\mathcal{R}^{\rm L}_{\rm com}\right)^2.$$

$$f_{\rm NL} = -\frac{5}{24\pi G}\left(\frac{\partial^2 \ln H}{\partial \phi^2}\right) \approx -\frac{5}{48\pi G}\left(\frac{\partial^2 \ln V}{\partial \phi^2}\right).$$ [For Gaussian $\delta\phi$]

For standard slow-roll inflation models, this is of order the slow-roll parameters, O(0.01).

Multi-field Generalization

$$\mathcal{R}_{\rm com} = -\int_{\phi_{0A}}^{\phi_{0A}+\delta\phi_{A\,\rm flat}} d\phi_A \frac{N(\phi_A)H(\phi_A)}{\dot{\phi}_A}$$

A=1,..., # of fields in the system

Then, again by expanding R to the quadratic order in $\delta\phi_A$, one can find f_{NL} for the multi-field case.

Example: the curvaton scenario, in which the second derivative of the integrand with respect to ϕ_2, the "curvaton field," divided by the square of the first derivative is much larger than slow-roll param.

3. Curvature Perturbation to CMB

- The linear Sachs-Wolfe effect is given by $dT/T = -(1/3)\Phi_H = +(1/3)\Phi_A$
- The non-linear SW effect is

$$\frac{\Delta T}{T} = \frac{1}{3}\Phi_A + \frac{1}{18}\Phi_A^2 - \nabla^{-4}\partial_i\partial^j(\partial^i\Phi_H\partial_j\Phi_H) - \frac{1}{3}\nabla^{-2}(\partial^i\Phi_H\partial_i\Phi_H)$$

where time-dependent terms (called the integrated SW effect) are not shown. (Bartolo et al. 2004)

- These terms generate f_{NL} of order unity.

Implications of large f_{NL}

- f_{NL} never exceeds 10 in the conventional picture of inflation in which
  - All fields are **slowly rolling**, and
  - All fields have the **canonical kinetic term**.
- Therefore, an unambiguous detection of $f_{NL} > 10$ rules out most of the existing inflation models.
- Who would the “survivors” be?

3 Ways to Get Larger Non-Gaussianity from Early Universe

$$\mathbf{f_{NL} \sim f_{\Phi} + g_{\Phi}^{-1} f_{\delta\phi} + g_{\Phi}^{-1} g_{\delta\phi}^{-1} f_{\eta}}$$

1. **Break slow-roll:** $f_{\delta\phi}$, $f_{\eta} >> 1$
   - Features (steps, bumps…) in V(ϕ)
     - Kofman, Blumenthal, Hodges & Primack (1991); Wang & Kamionkowski (2000); Komatsu et al. (2003); Chen, Easther & Lim (2007)
   - Ekpyrotic model, old and new
     - Buchbinder, Khoury & Ovrut (2007); Koyama, Mizuno, Vernizzi & Wands (2007)

3 Ways to Get Larger Non-Gaussianity from Early Universe

$$f_{NL} \sim f_{\Phi} + g_{\Phi}^{-1} f_{\delta\phi} + g_{\Phi}^{-1} g_{\delta\phi}^{-1} f_{\eta}$$

2. **Amplify field interactions: $f_{\eta} >> 1$**
   - Often done by **non-canonical kinetic terms**
   - Ghost inflation $S = \int d^4x \, \frac{1}{2}\dot{\pi}^2 - \frac{\alpha^2}{2M^2}(\nabla^2\pi)^2 - \frac{\beta}{2M^2}\dot{\pi}(\nabla\pi)^2 + \cdots$
     - Arkani-Hamed, Creminelli, Mukohyama & Zaldarriaga (2004)
   - DBI Inflation $\mathcal{L}_{\rm eff} = -\frac{1}{g_s}\sqrt{-g}\left(f(\phi)^{-1}\sqrt{1 + f(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} + V(\phi)\right)$
     - Alishahiha, Silverstein & Tong (2004)
   - Any other models with a low effective sound speed of scalar field because $f_{\eta} \sim 1/(c_s)^2$
     - Chen, Huang, Kachru & Shiu (2004); Cheung, Creminelli, Fitzpatrick, Kaplan & Senatore (2007)

3 Ways to Get Larger Non-Gaussianity from Early Universe

$$f_{NL} \sim f_{\Phi} + g_{\Phi}^{-1} f_{\delta\phi} + g_{\Phi}^{-1} g_{\delta\phi}^{-1} f_{\eta}$$

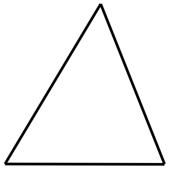
3. **Suppress the perturbation conversion factor, g_{Φ}, $g_{\delta\phi} << 1$**
   - Generate curvature perturbations from isocurvature (entropy) fluctuations with an efficiency given by g.
     - Linde & Mukhanov (1997); Lyth & Wands (2002)
   - Curvaton predicts $g_{\Phi} \sim \Omega_{curvaton}$ which can be arbitrarily small
     - Lyth, Ungarelli & Wands (2002)

Subtlety: Triangle Dependence

- Remember that there are two f_{NL}
  - "Local," which has the largest amplitude in the squeezed configuration

Local

Eq.



  - "Equilateral," which has the largest amplitude in the equilateral configuration

- So the question is, "which model gives f_{NL}(local), and which f_{NL}(equilateral)?"

Classifying Non-Gaussianities in the Literature

- Local Form
  - Ekpyrotic models
  - Curvaton models
- Equilateral Form
  - Ghost condensation, DBI, low speed of sound models
- Other Forms
  - Features in potential, which produce large non-Gaussianity within narrow region in l

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•Is any of these a winner?
•Non-Gaussianity may tell us soon. We will find out!

Summary

- Since the introduction of f_{NL}, the research on non-Gaussianity as a probe of the physics of early universe has evolved tremendously.
- I hope I convinced you that f_{NL} is as important a tool as Ω_K, n_s, $dn_s/d\ln k$, and r, for constraining inflation models.
- In fact, it has the best chance of ruling out the largest population of models...

Concluding Remarks

- Stay tuned: WMAP continues to observe, and Planck will soon be launched.
- Non-Gaussianity has provided cosmologists and string theorists with a unique opportunity to work together.
- For me, this is one of the most important contributions that f_{NL} has made to the community.