CMB Polarization Lecture

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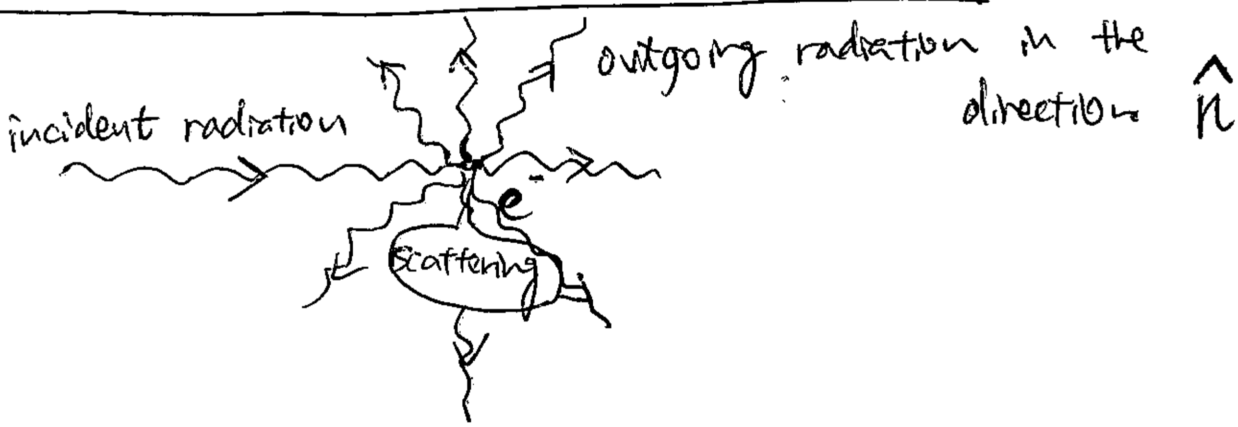
Q. What's required for generating CMB polarization ?

A. Scattering & Anisotropic (quadrupolar) radiation around an electron.

$\boxed{\text{Thomson scattering is anisotropic !}}$



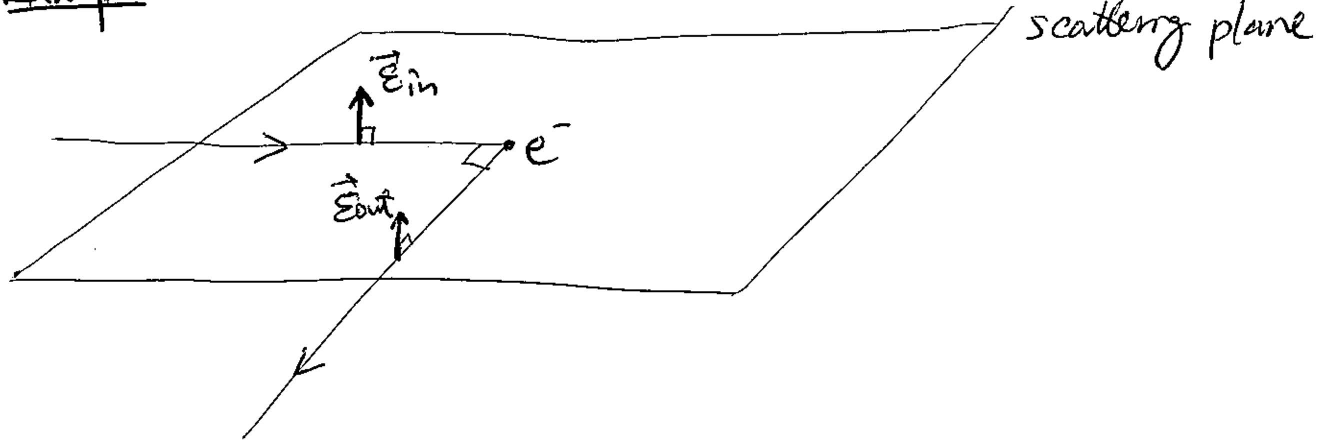
Probability of the scattering into a direction $\hat{n}$ is proportional to :

$$\boxed{\frac{d\sigma}{d\Omega_{\hat{n}}} = \frac{3}{8\pi}\sigma_T\left(\vec{\varepsilon}_{in}\cdot\vec{\varepsilon}_{out}\right)^2}$$

where

- $\vec{\varepsilon}_{in}$: polarization vector of the incident photon
- $\vec{\varepsilon}_{out}$: polarization vector of the outgoing photon

Example 1 : 90° scattering with polarization vector is perpendicular to the scattering plane.

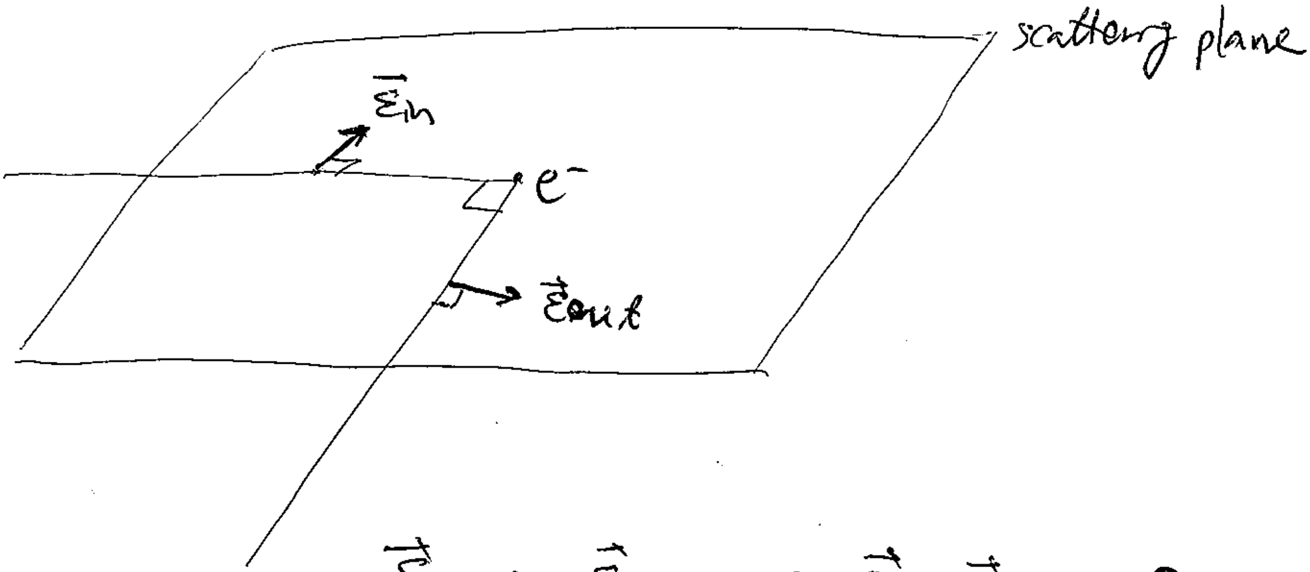


$$\vec{\varepsilon}_{in} \parallel \vec{\varepsilon}_{out} \quad \therefore \vec{\varepsilon}_{in} \cdot \vec{\varepsilon}_{out} = 1.$$

therefore,

$$\boxed{\frac{d\sigma}{d\Omega_{\hat{n}}} = \frac{3}{8\pi}\sigma_T.}$$

Example 2 : 90° scattering with polarization vector is on the scattering plane

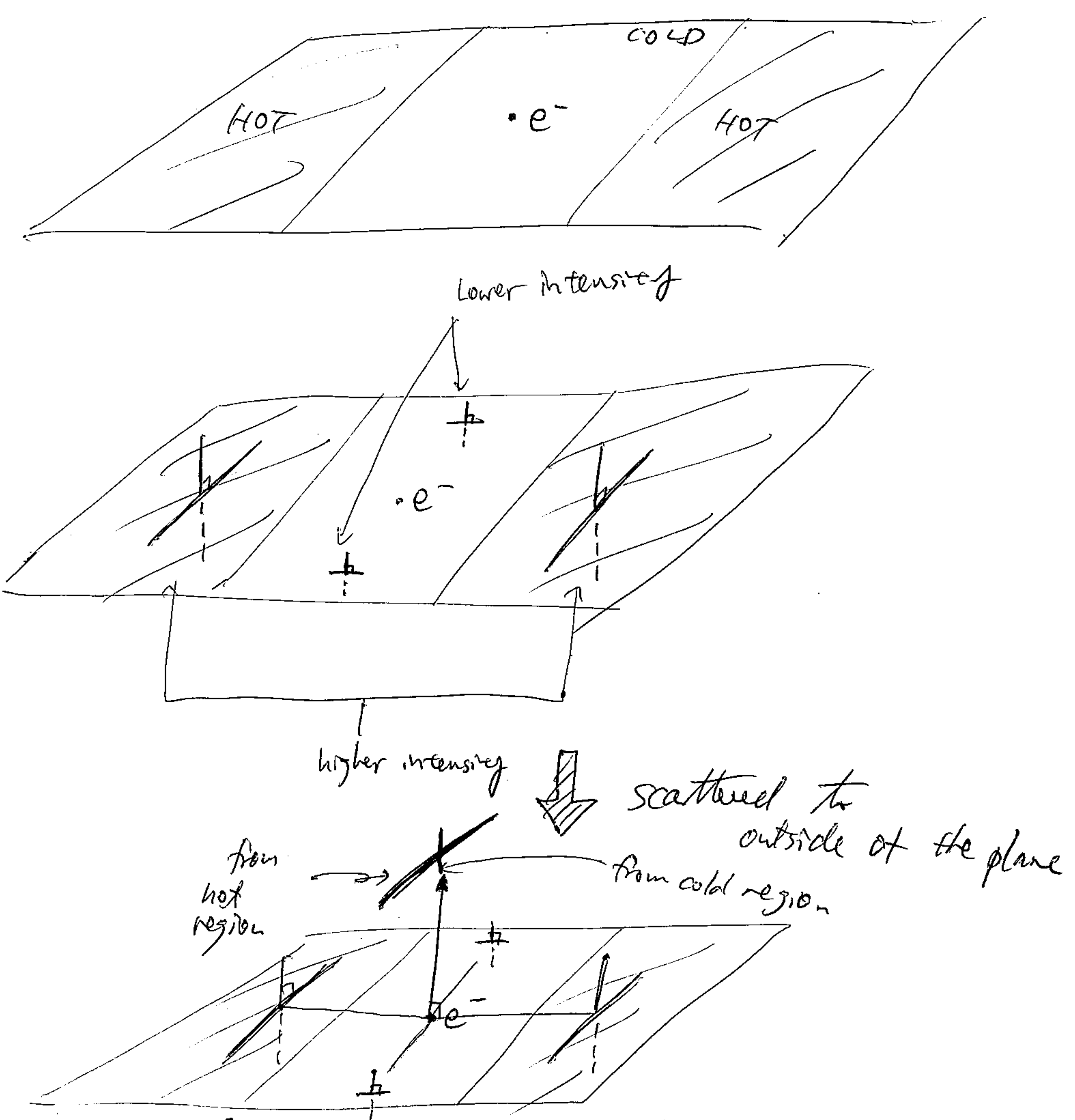


$$\vec{\varepsilon}_{in} \perp \vec{\varepsilon}_{out} \quad \therefore \vec{\varepsilon}_{in} \cdot \vec{\varepsilon}_{out} = 0.$$

therefore,

$$\boxed{\frac{d\sigma}{d\Omega_{\hat{n}}} = 0\ !}$$

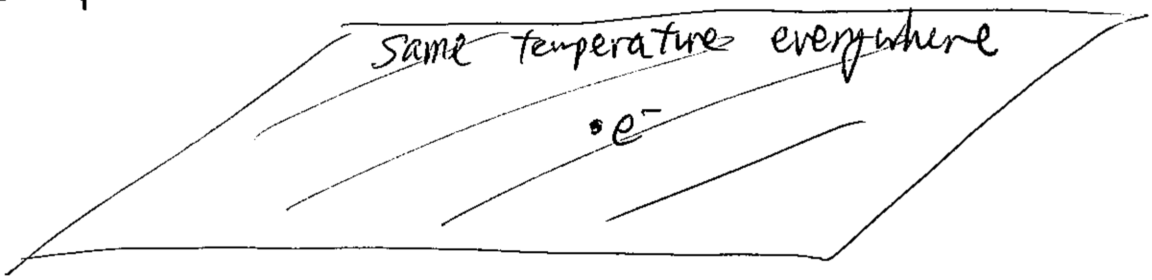
So, this gives the following picture :

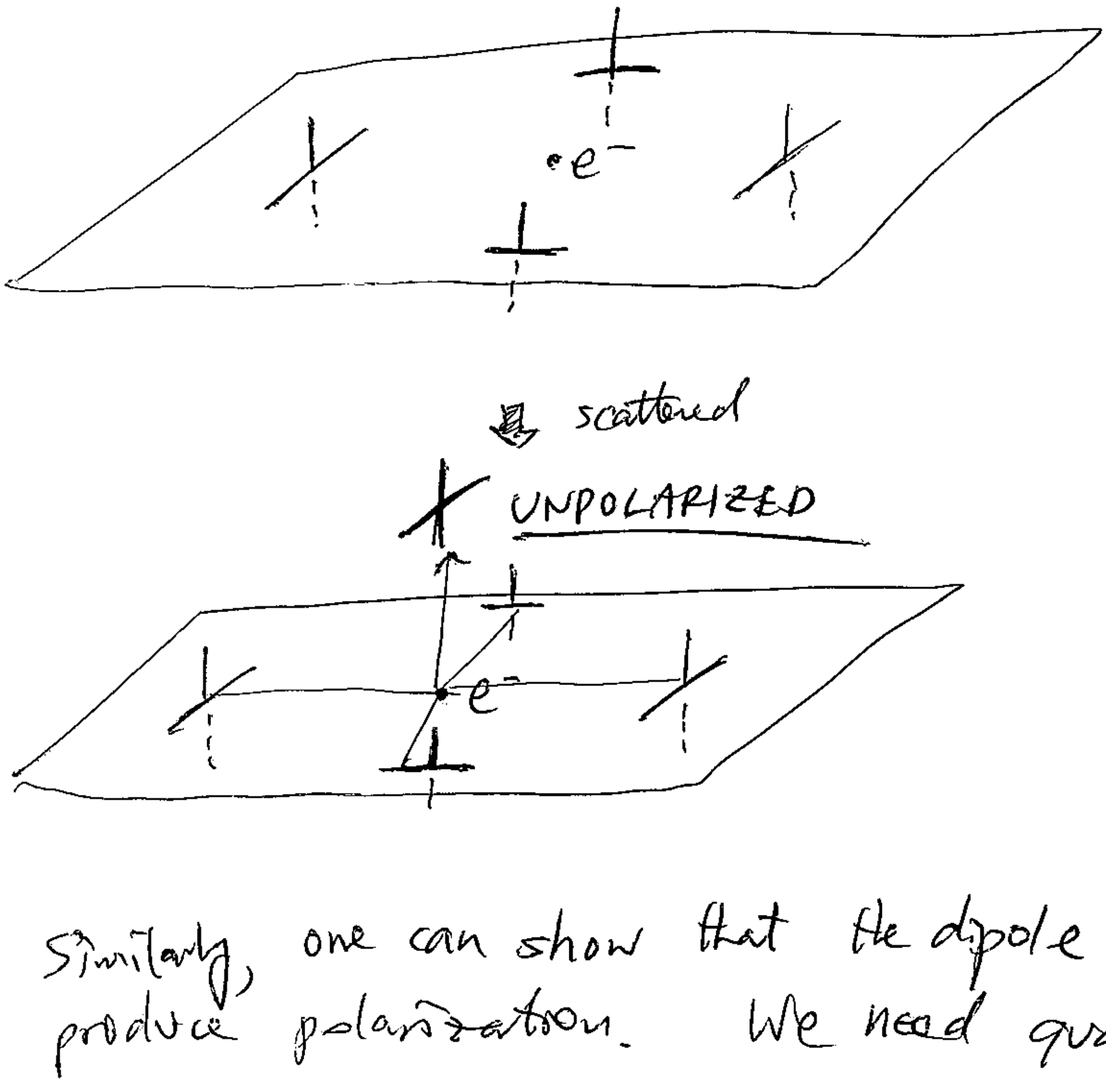


∴ $(\vec{\epsilon}_{in}\cdot\vec{\epsilon}_{out})^2$ dependence, coupled with the existence of the quadrupole, will produce polarization !!

What if there is **no** quadrupole?

Example (Monopole)

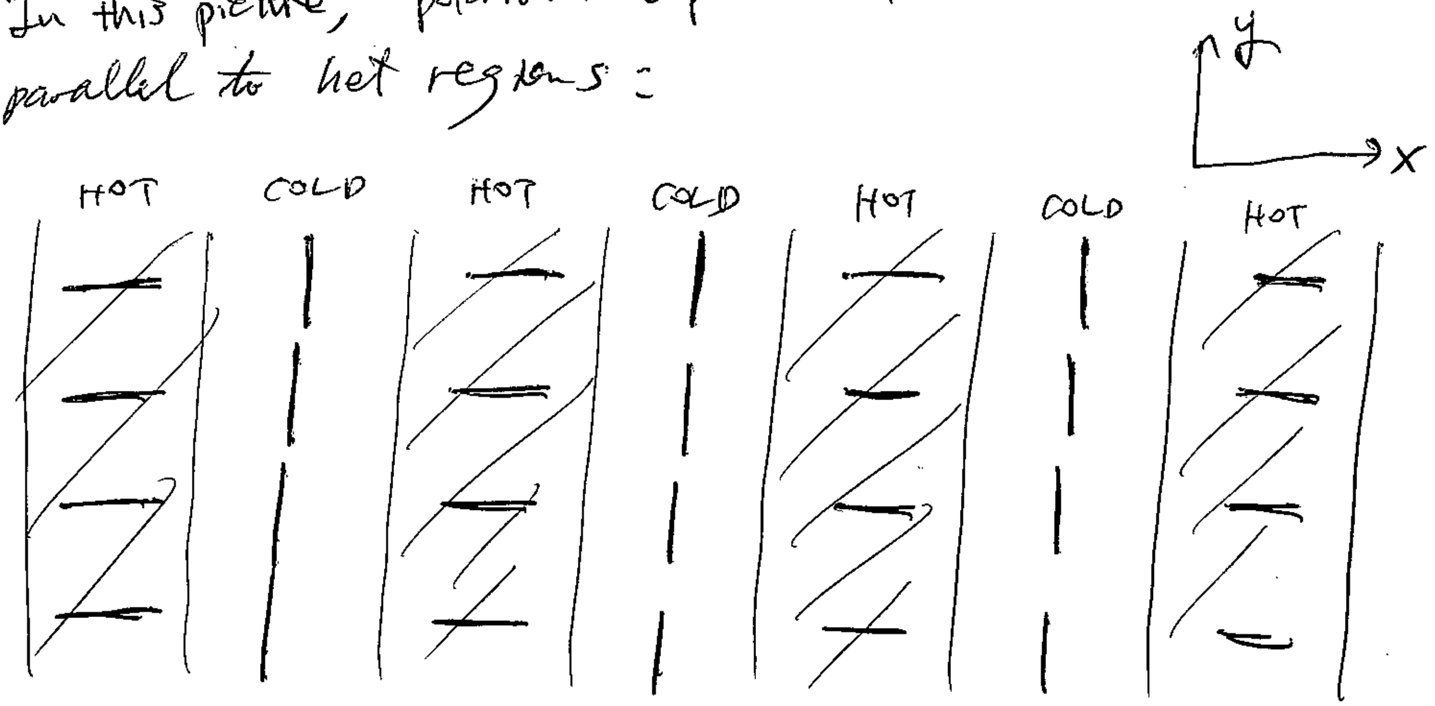




Similarly, one can show that the dipole does not produce polarization. We need quadrupole!!

Polarization Pattern

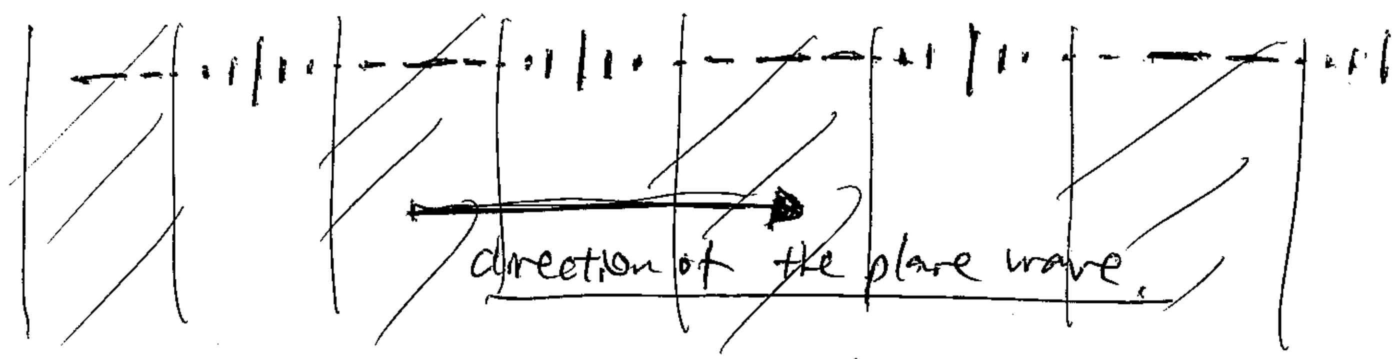
In this picture, polarization pattern tends to be parallel to hot regions:



$\vec{k}$ ⇨ direction of a plane-wave temperature fluctuation,

$$\frac{\delta T}{T} \propto e^{i\vec{k}\cdot\vec{x}}$$

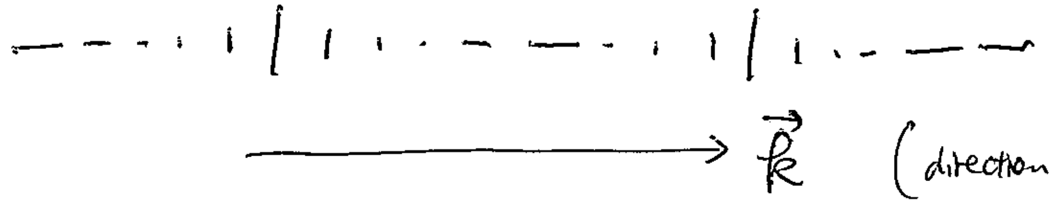
Looking at this more closely, one finds:



Therefore, in this picture, the polarization direction is either parallel or perpendicular to $\vec{k}$ (of temperature fluctuation.)

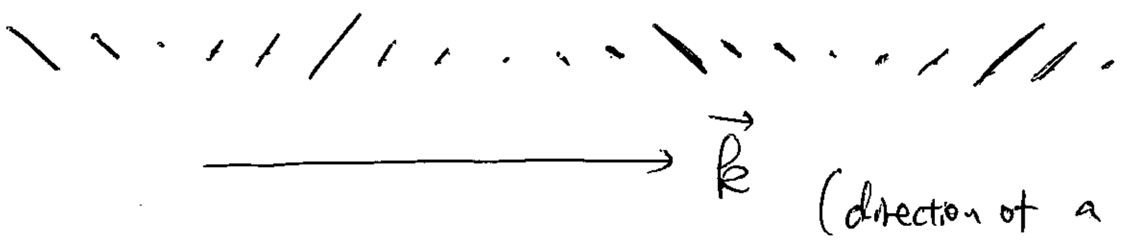
E-MODE polarization

This particular pattern, in which the polarization directions are either parallel or perpendicular to $\vec{k}$, is called E-mode (or grad-mode) polarization.



$\vec{k}$ (direction of a plane wave perturbation)

B-MODE ?



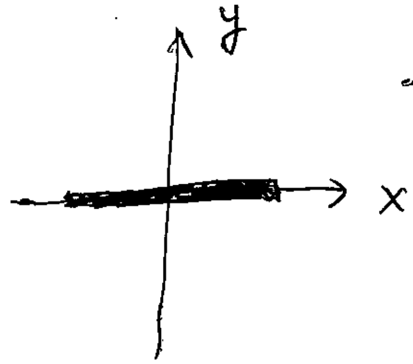
$\vec{k}$ (direction of a plane wave perturbation)

When the polarization directions are 45° tilted against $\vec{k}$, we have a pure "B-mode" polarization.

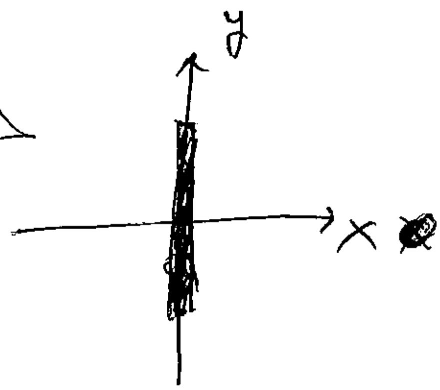
Precise, mathematical definition of E/B decomposition (in the flat-sky approximation).

Define "Stokes parameters" to characterize the linear polarization.

Stokes Q



$Q > 0$
$U = 0$
[purely positive Q]



$Q < 0$
$U = 0$
[purely negative Q]

$$\boxed{Q \equiv I_x - I_y}$$

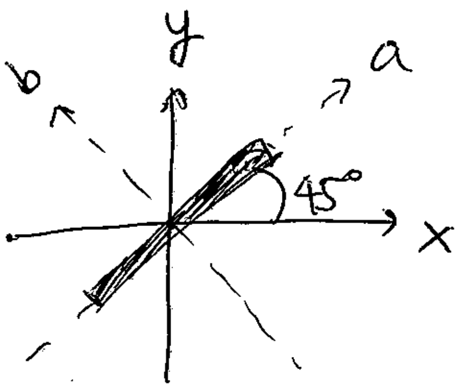
※ coordinate dependent !!

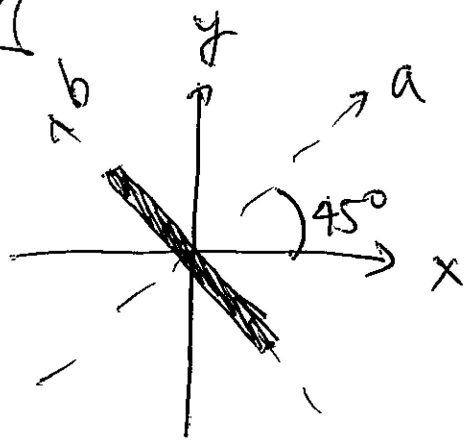
By rotating the coordinate by 90°, one can transform Q to $-Q$.

In terms of the electric field,

$$Q = |E_x|^2 - |E_y|^2$$

Stokes U





$Q = 0$

$U > 0$

[purely positive U]

$Q = 0$

$U < 0$

[purely negative U]

$$\boxed{U \equiv I_a - I_b}$$

in a clockwise rotation

By rotating the coordinate by 45° at a time, we have the following sequence of the transformation

$$\cdots \to +U \to -Q \to -U \to +Q \to +U \to \cdots$$

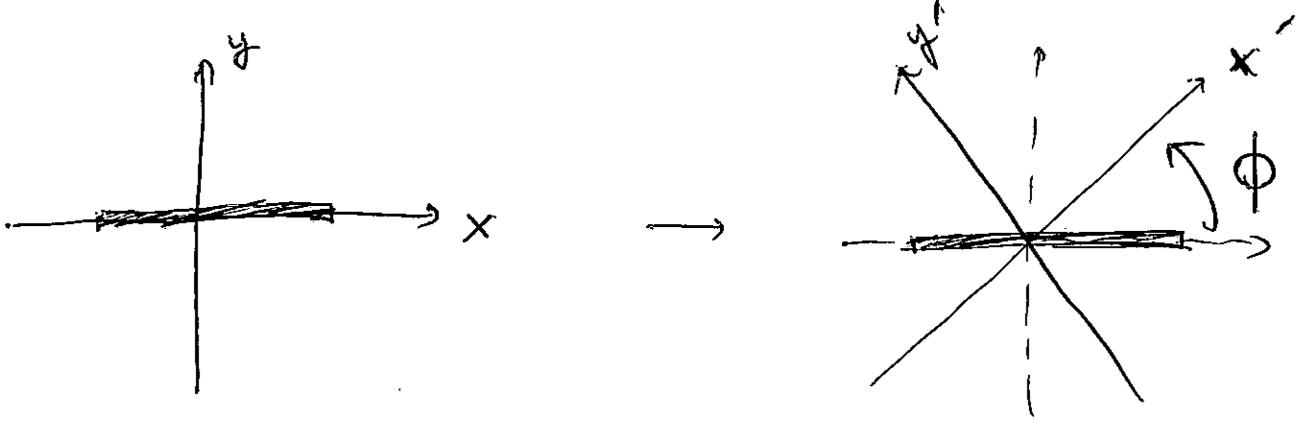
In terms of the electric field,

$$U = |E_a|^2 - |E_b|^2$$
$$= 2\mathrm{Re}(E_x E_y^*)$$

where $E_x = \frac{1}{\sqrt{2}}(E_a + iE_b)$

$E_y = \frac{1}{\sqrt{2}}(E_a - iE_b)$

For counter-clockwise rotations :



In this example, $\phi = 45^\circ$, and
pure Q (>0) transformed into pure U (<0).

$$\begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ -\sin 2\phi & \cos 2\phi \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix}$$

For the coordinate trans. of

$$\begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

It is often more convenient to work with a complex linear combination :

$$\boxed{Q + iU}$$

For counter-clockwise rotations,

$$Q' + iU' = e^{-2i\phi}(Q+iU)$$
$$Q' - iU' = e^{+2i\phi}(Q+iU)$$

This property shows that $Q \pm iU$ has the spin of ± 2 (or ∓ 2, depending on one's definition)

⇒ This is important : this means that we cannot decompose $Q \pm iU$ into the ordinary plane-wave decomposition, a.k.a. Fourier transform.

We need the spin-2 decomposition!!

Spherical / Spin-2 harmonics decomposition

As we measure temperature & polarization on the sky, which is the surface of a sphere, we need to use the spherical harmonics & spin 2 tensor harmonics to decompose T & $Q \pm iU$.

I.e.,

$$\underbrace{T(\hat{n})}_{\text{temperature at a given pixel } \hat{n}} = \sum_{\ell m} a_{\ell m} \underbrace{Y_{\ell m}(\hat{n})}_{\text{spherical harmonics}}$$

$$[Q \pm iU](\hat{n}) = \sum_{\ell m} {}_{\mp 2}a_{\ell m} \underbrace{{}_{\mp 2}Y_{\ell m}(\hat{n})}_{\text{spin 2 harmonics}}$$

Then, E- / B-modes are defined as:

$$\boxed{\begin{aligned} E_{\ell m} &\equiv -\frac{1}{2}\left({}_{2}a_{\ell m} + {}_{-2}a_{\ell m}\right) \\ B_{\ell m} &= -\frac{i}{2}\left({}_{2}a_{\ell m} - {}_{-2}a_{\ell m}\right) \end{aligned}}$$

※ physical meanings will be given later!!

Flat-sky Approximation

Zaldarriaga & Seljak 1997. PRD 55, 1830 [astro-ph/9609170]

The harmonics decomposition must be used for the actual data analysis, but it is more convenient to use the "flat-sky approximation" for obtaining the physical intuition.

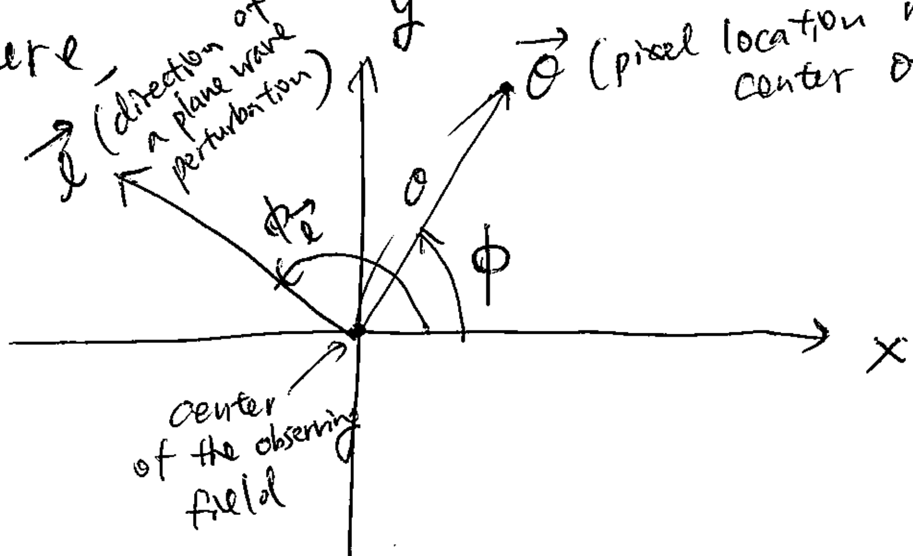
In this app., we can continue to use the ordinary (2d) Fourier transform for $T(\hat{n})$:

$$T(\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2}\, a_{\vec{\ell}}\, e^{i\vec{\ell}\cdot\vec{\theta}}$$

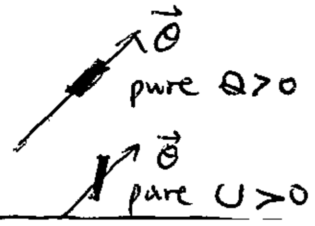
the corresponding expansion for spin-2 fields is

$$[Q \pm iU](\vec{\theta}) = -\int \frac{d^2\vec{\ell}}{(2\pi)^2}\, {}_{\mp 2}a_{\vec{\ell}}\, e^{\pm 2i(\phi - \phi_{\vec{\ell}})}\, e^{i\vec{\ell}\cdot\vec{\theta}}$$

Here,



In this setup, Q & U are defined with respect to the polar coordinate:



With E/B decomposition,

$$\boxed{[Q \pm iU](\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2}\,[E_{\vec{\ell}} \mp i B_{\vec{\ell}}]\, e^{\pm 2i(\phi-\phi_{\vec{\ell}})} e^{i\vec{\ell}\cdot\vec{\theta}}}$$

which gives :

$$Q(\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2} \left\{ E_{\vec{\ell}} \cos[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}} \sin[2(\phi-\phi_{\vec{\ell}})] \right\} e^{i\vec{\ell}\cdot\vec{\theta}}$$

$$U(\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2} \left\{ -E_{\vec{\ell}} \sin[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}} \cos[2(\phi-\phi_{\vec{\ell}})] \right\} e^{i\vec{\ell}\cdot\vec{\theta}}$$

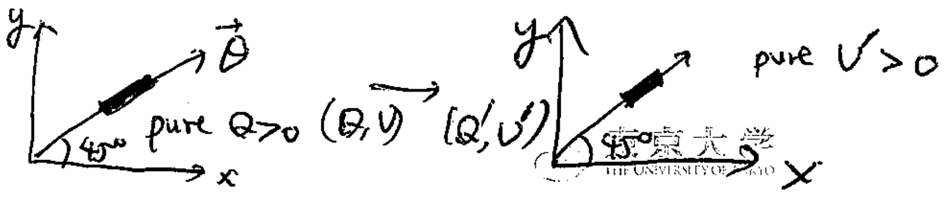
This formula is very useful for picturing E&B modes.

Let's take a single plane wave, $\vec{\ell}$. Then this wave generates $Q(\vec{\theta})$ & $U(\vec{\theta})$ as

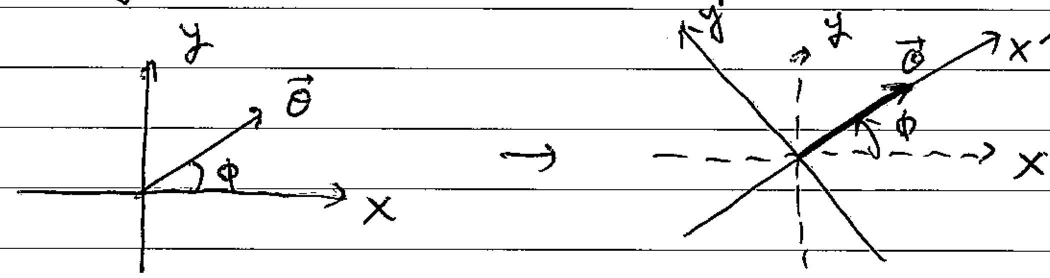
$$Q(\vec{\theta}) = \left\{ E_{\vec{\ell}} \cos[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}} \sin[2(\phi-\phi_{\vec{\ell}})] \right\} e^{i\ell\theta\cos\varphi}$$

$$U(\vec{\theta}) = \left\{ -E_{\vec{\ell}} \sin[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}} \cos[2(\phi-\phi_{\vec{\ell}})] \right\} e^{i\ell\theta\cos\varphi}$$

Now, we need to convert these results to Q & U defined in the Cartesian coordinate :



We can do the coordinate transformation by rotating the coordinate (counter-clock wise) by ϕ:



In other words, we choose the coordinates such that $\vec{\theta}$ is always on the x-axis.

$$Q' \pm iU' = e^{\mp 2i\phi}(Q \pm iU)$$

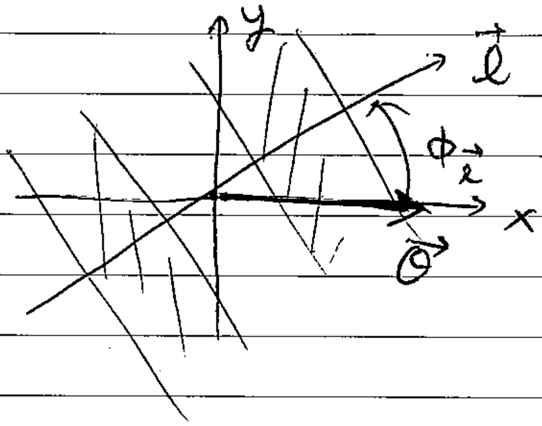
Therefore (Zaldarriaga & Seljak 1997)

$$\begin{cases} Q'(\vec{\theta}) = +\int \frac{d^2\vec{l}}{(2\pi)^2}\left[E_{\vec{l}}\cos(2\phi_{\vec{l}}) - B_{\vec{l}}\sin(2\phi_{\vec{l}})\right]e^{i\vec{l}\cdot\vec{\theta}} \\ U'(\vec{\theta}) = +\int \frac{d^2\vec{l}}{(2\pi)^2}\left[E_{\vec{l}}\sin(2\phi_{\vec{l}}) + B_{\vec{l}}\cos(2\phi_{\vec{l}})\right]e^{i\vec{l}\cdot\vec{\theta}} \end{cases}$$

From now on, we use Q & U for Q' & U'.

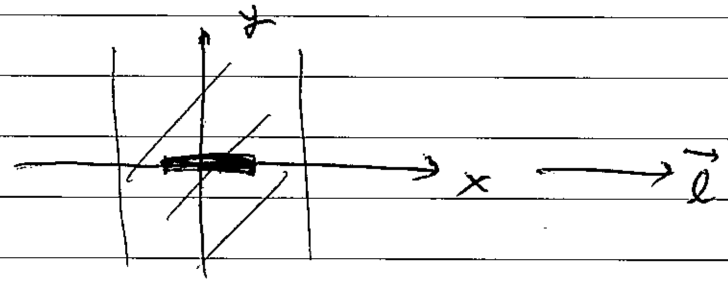
pure E-modes

Consider a pure E-mode from a single plane wave.

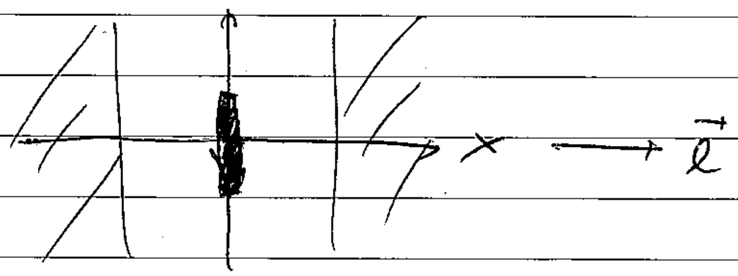


$$\begin{cases} Q(\vec{\theta}) = E_{\vec{l}} \cos(2\phi_{\vec{l}}) e^{il\theta\cos\phi_{\vec{l}}} \\ U(\vec{\theta}) = E_{\vec{l}} \sin(2\phi_{\vec{l}}) e^{il\theta\cos\phi_{\vec{l}}} \end{cases}$$

$\boxed{\phi_{\vec{l}} = 0}$ $\quad E_{\vec{l}} = +1 \rightarrow \begin{cases} Q = +1 \\ U = 0 \end{cases}$ (we take $l\theta \ll 1$)

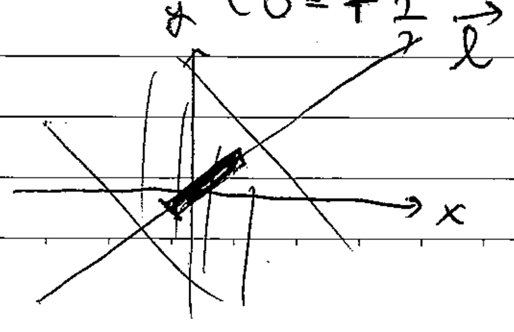


$E_{\vec{l}} = -1 \rightarrow \begin{cases} Q = -1 \\ U = 0 \end{cases}$

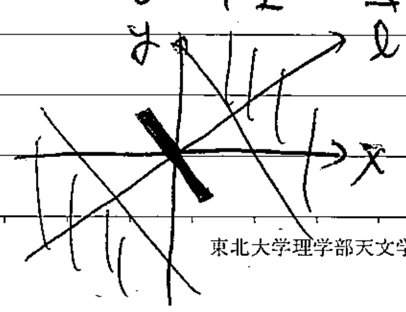


$\boxed{\phi_{\vec{l}} = 45^\circ}$

$E_{\vec{l}} = +1$ $\quad \begin{cases} Q = 0 \\ U = +1 \end{cases}$



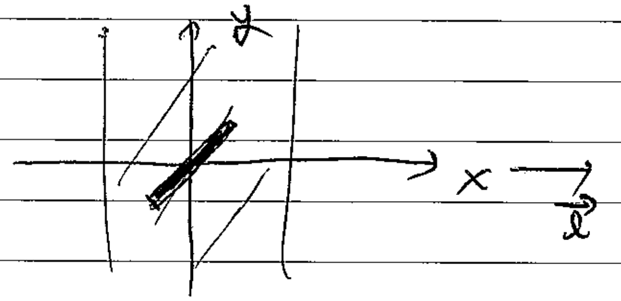
$E_{\vec{l}} = -1$ $\quad \begin{cases} Q = 0 \\ U = +1 \end{cases}$

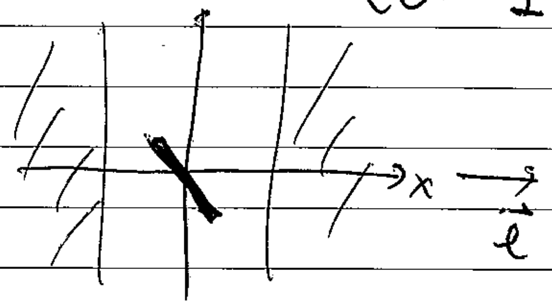


$\boxed{\text{pure B-modes}}$ Single plane wave

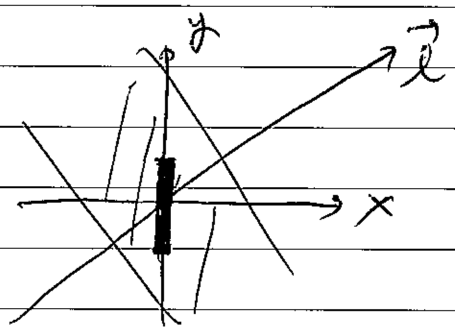
$$\begin{cases} Q(\vec{\theta}) = -B_{\vec{\ell}} \sin(2\phi_{\vec{\ell}})\, e^{i\ell\theta\cos\phi_{\vec{\ell}}} \\ U(\vec{\theta}) = +B_{\vec{\ell}} \cos(2\phi_{\vec{\ell}})\, e^{i\ell\theta\cos\phi_{\vec{\ell}}} \end{cases}$$

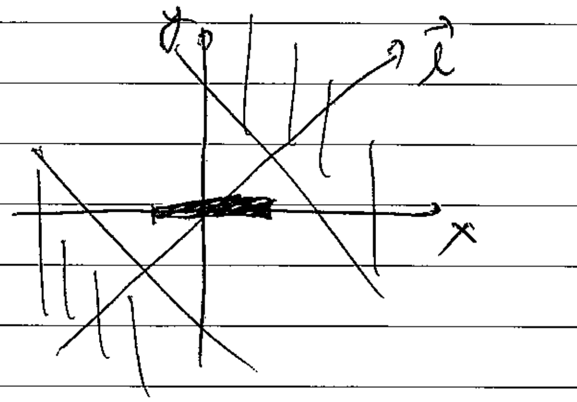
$\boxed{\phi_{\vec{\ell}} = 0}$ $B_{\vec{\ell}} = +1 \begin{cases} Q=0 \\ U=+1 \end{cases}$ $B_{\vec{\ell}} = -1 \begin{cases} Q=0 \\ U=-1 \end{cases}$





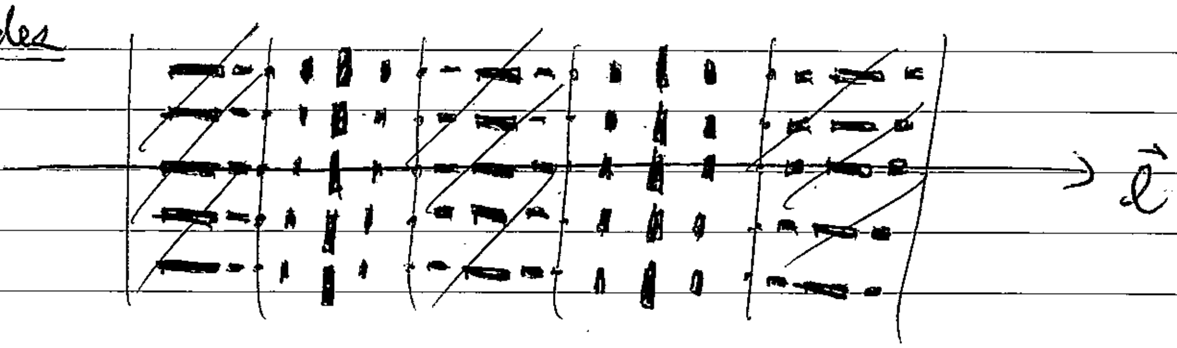
$\boxed{\phi_{\vec{\ell}} = 45^\circ}$ $B_{\vec{\ell}} = +1 \begin{cases} Q=-1 \\ U=0 \end{cases}$ $B_{\vec{\ell}} = -1 \begin{cases} Q=+1 \\ U=0 \end{cases}$



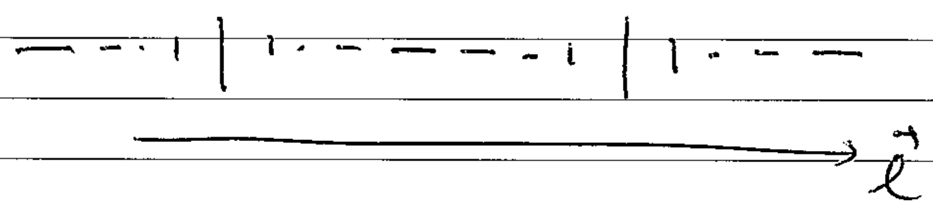


Therefore, we obtain the following picture :

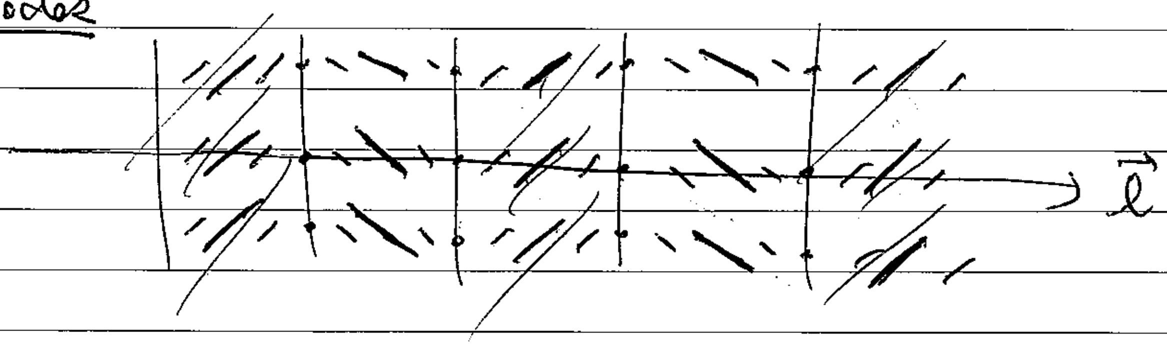
E-modes



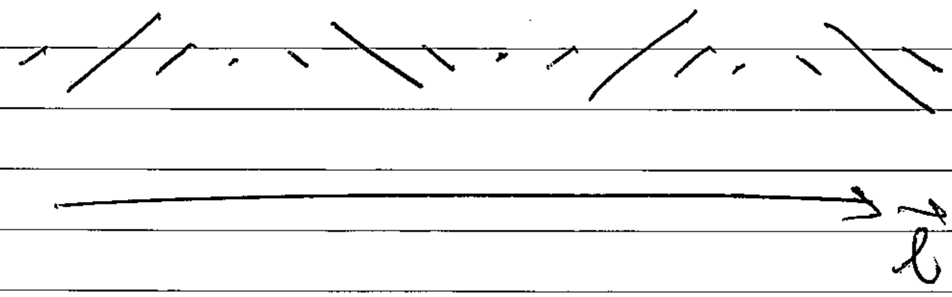
i.e.,



B-modes



i.e.,

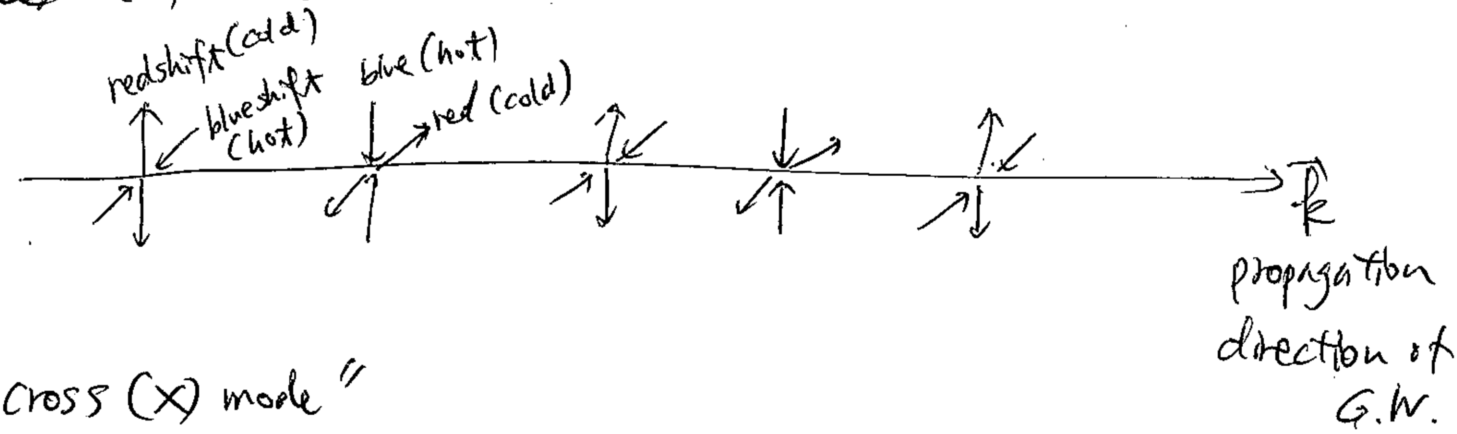


How do we generate B-modes?

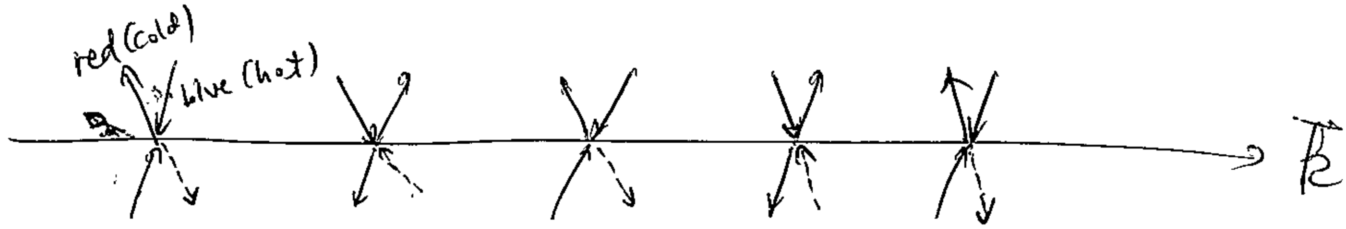
One way: gravitational waves!!

Gravitational waves can produce polarization, as it can create the quadrupolar temperature anisotropy.

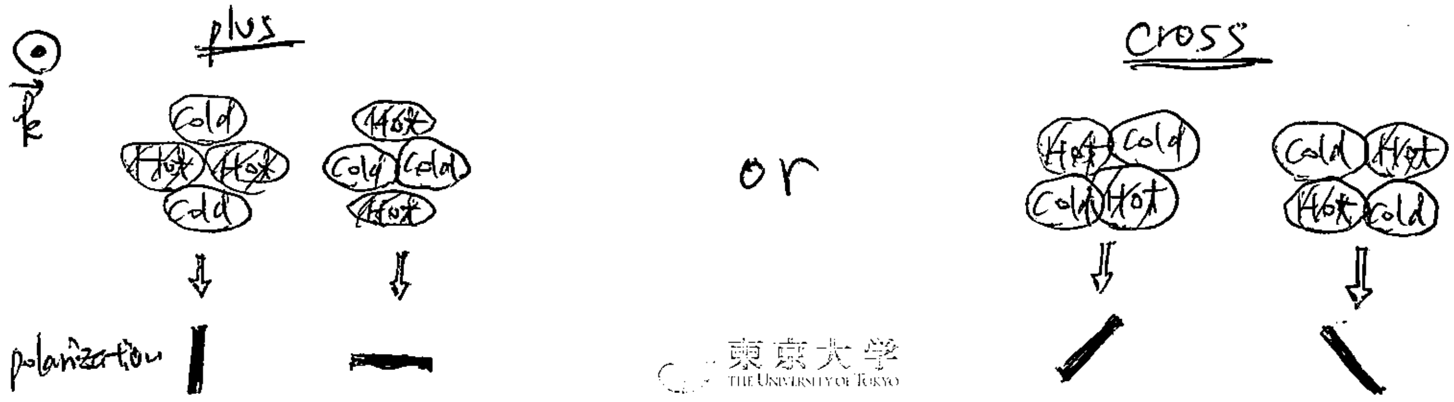
"~~cross~~ plus (+) mode"



"cross (×) mode"



So, when GW is coming toward us:

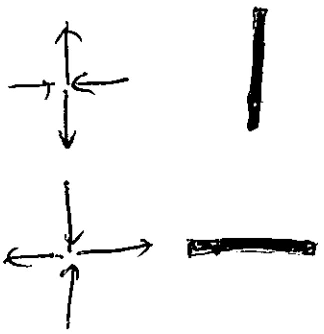


Gravitational waves can generate BOTH E & B modes.

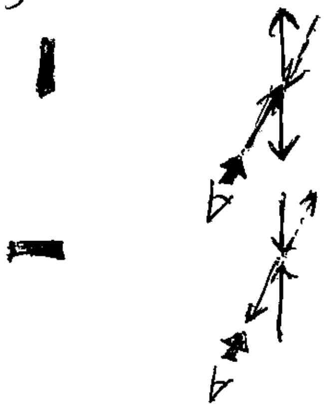
because they have two modes: + & X.

"+" mode

Face-on



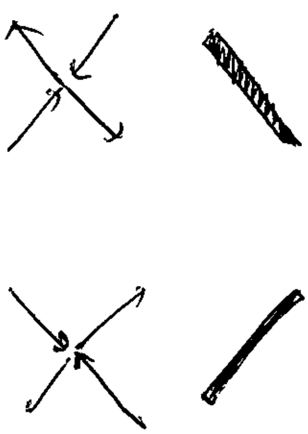
Edge-on



[Calculations show that
(the edge-on amplitude) = 1/2 (face-on amplitude)]

"X" mode

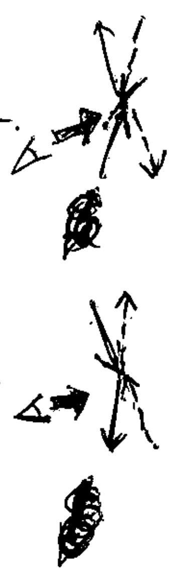
Face-on



Edge-on

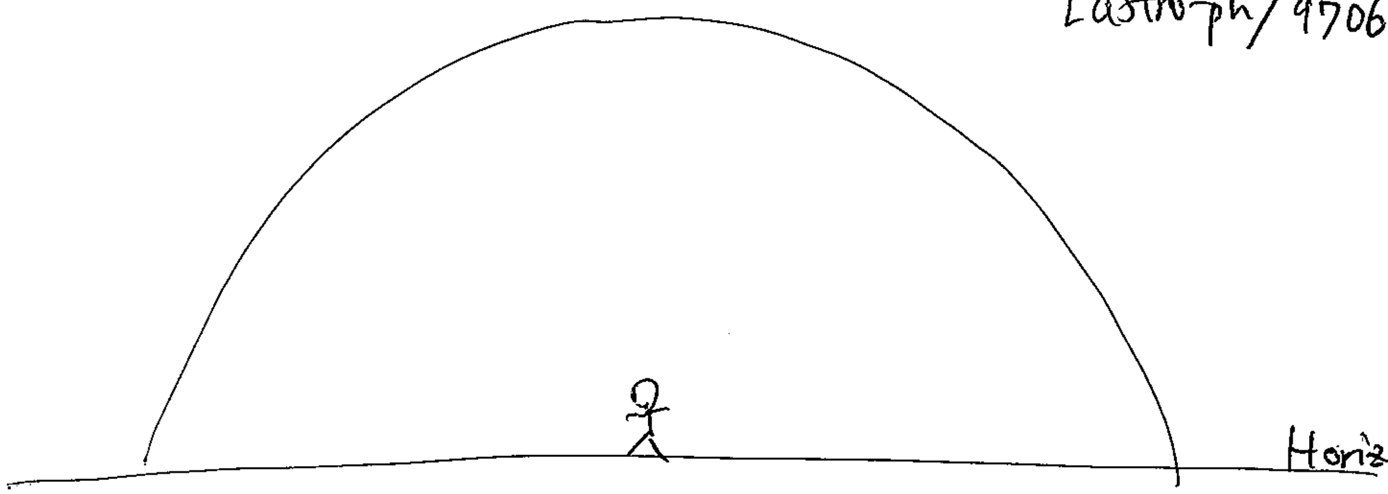
NO POL.

NO POL.

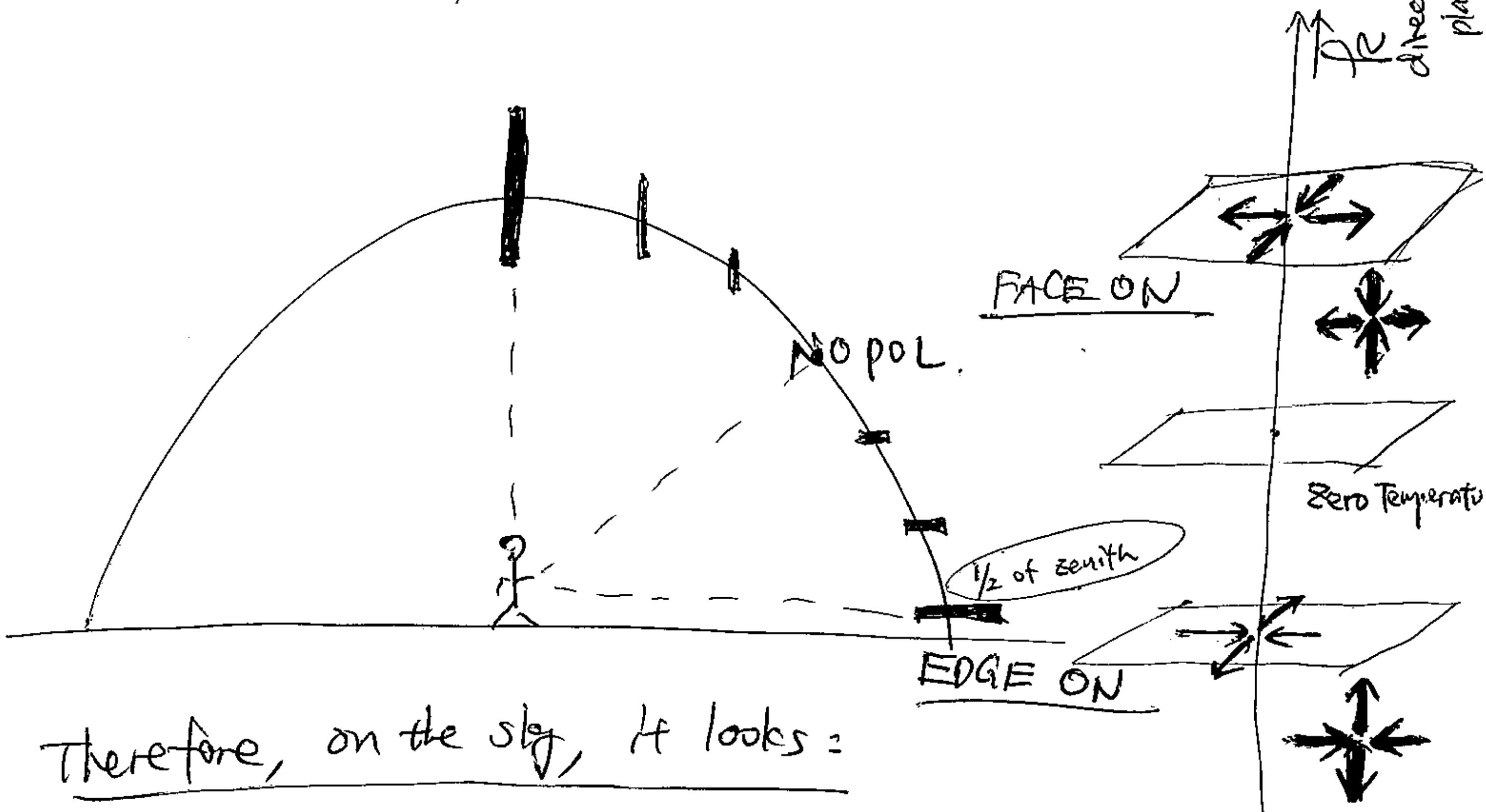


Now, let's project this on the sky.

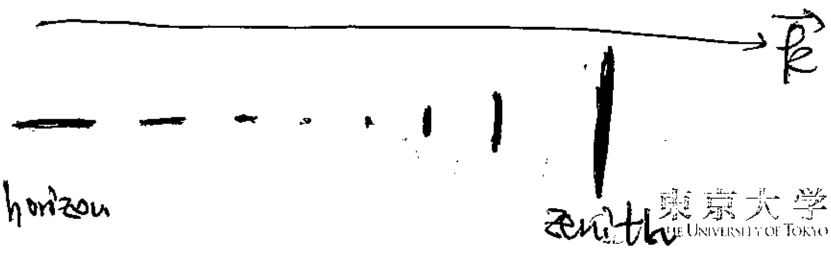
Hu & White [astroph/9706147]



For "+" mode (1/2 wavelength from Horizon to zenith)

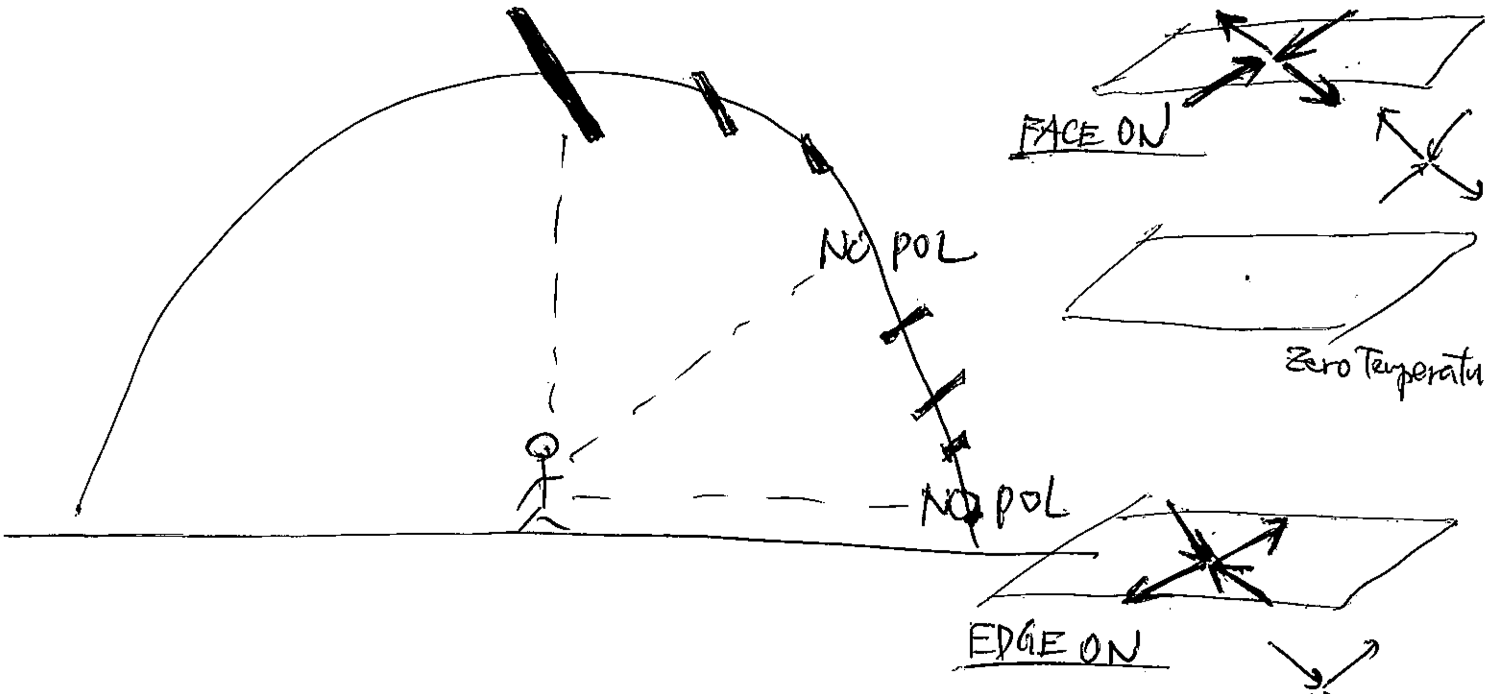


Therefore, on the sky, it looks :

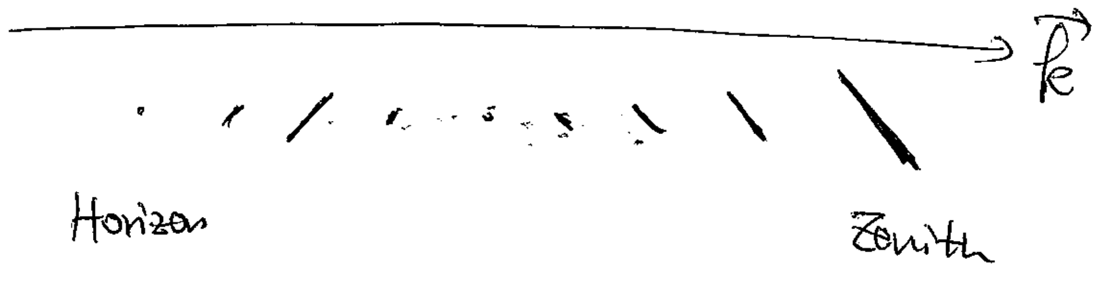


This is E mode.

For "X" mode (1/2 wavelength from Horizon to zenith)



Therefore, on the sky, it looks :



This is B mode.

Because there is no polarization on the horizon for B modes, the amplitude of B mode power spectrum is smaller than that of E modes!

$\therefore C_\ell^{BB} < C_\ell^{EE}$.