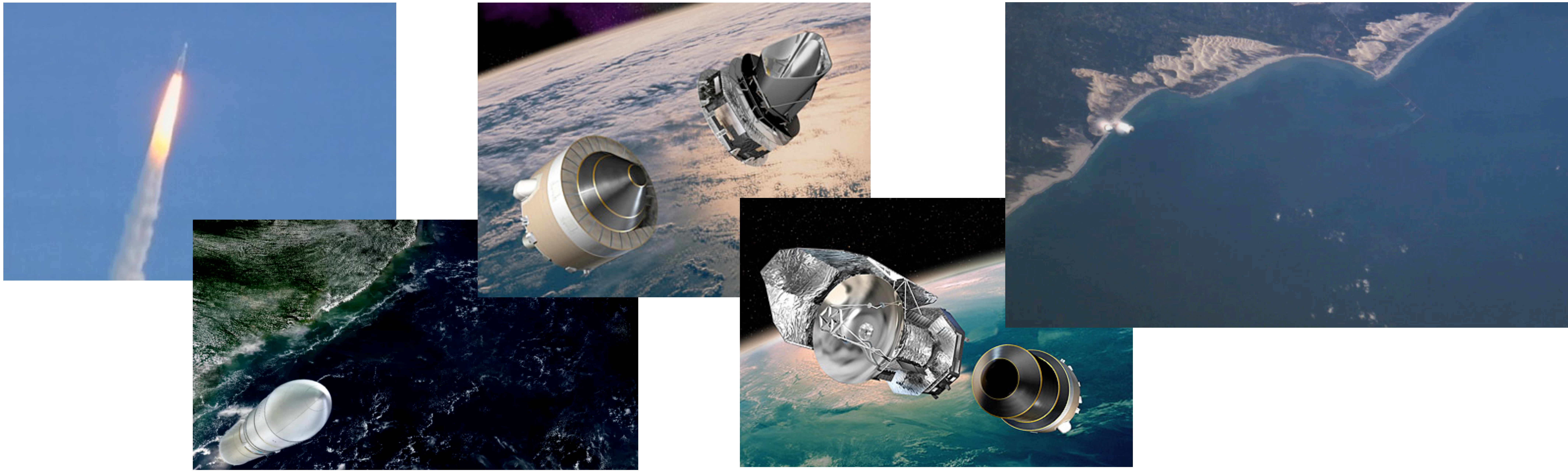
B-mode Polarization & Primordial non-Gaussianity

Eiichiro Komatsu (Texas Cosmology Center, Univ. of Texas Austin)

Lecture at IPMU, June 17, 2009

Planck Launched!

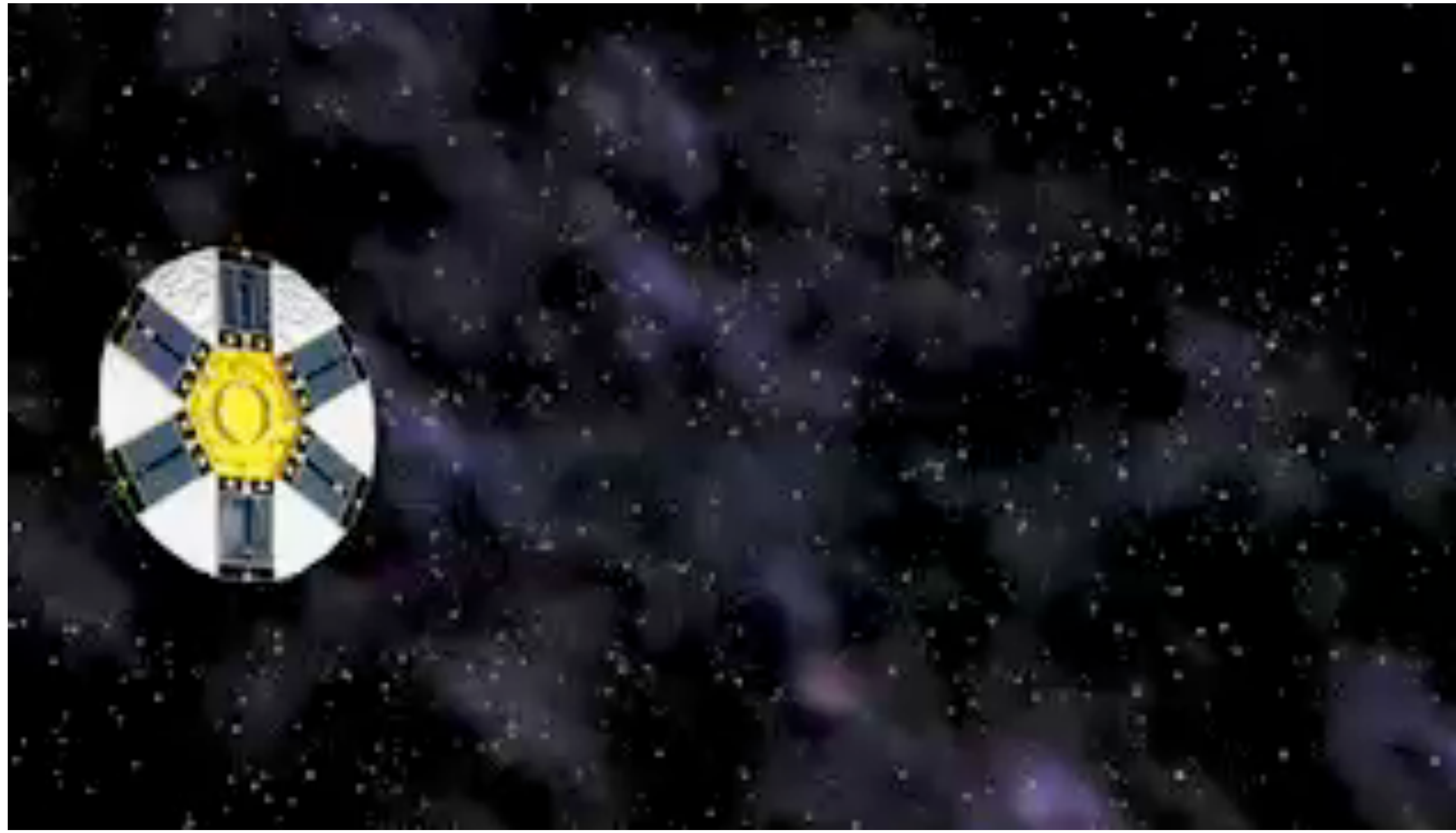


- The Planck satellite was successfully launched from French Guiana on May 14.
- Separation from the Herschell satellite was also successful.
- Both Planck and Hershell are on their ways to L2.

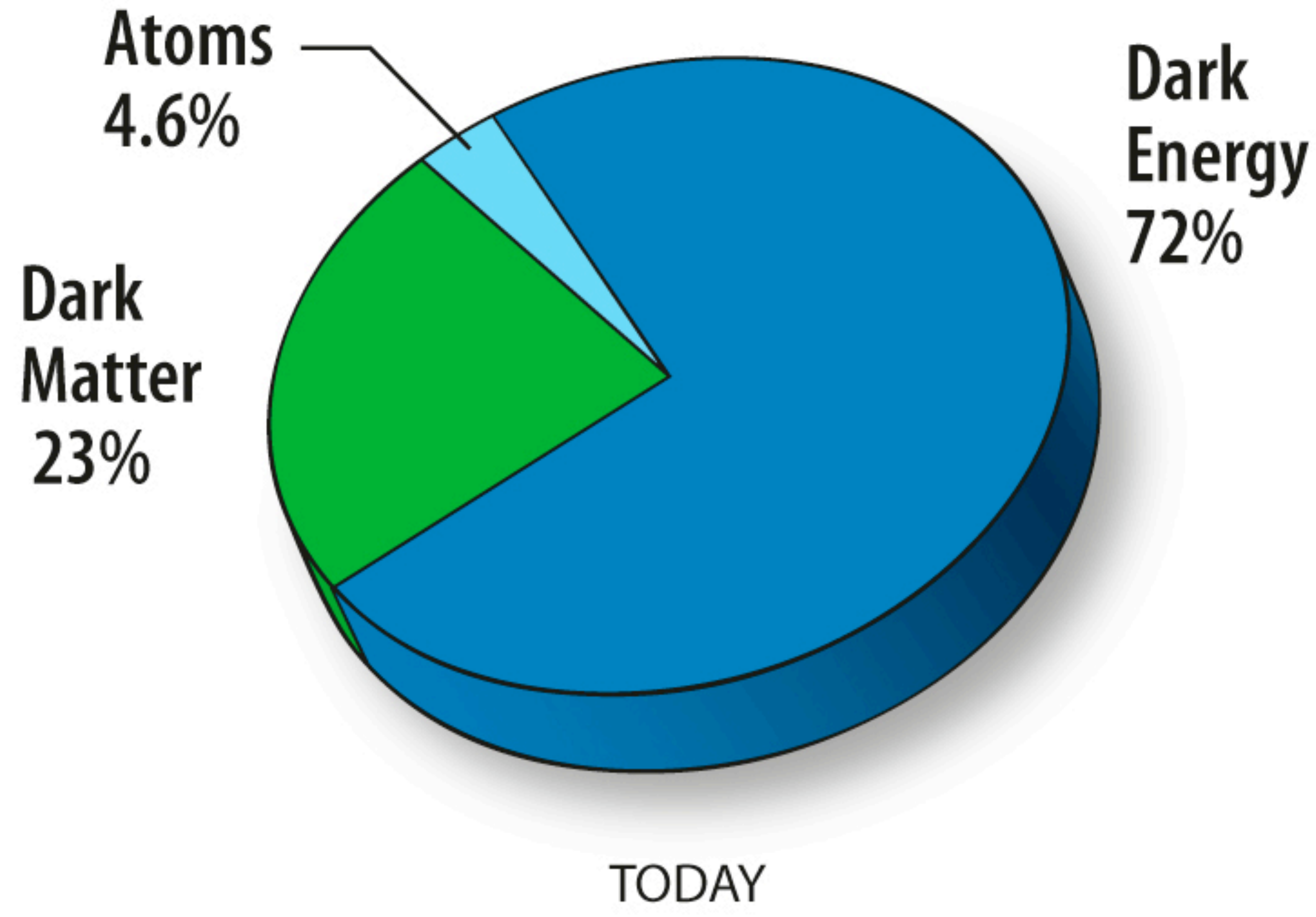
WMAP to Planck

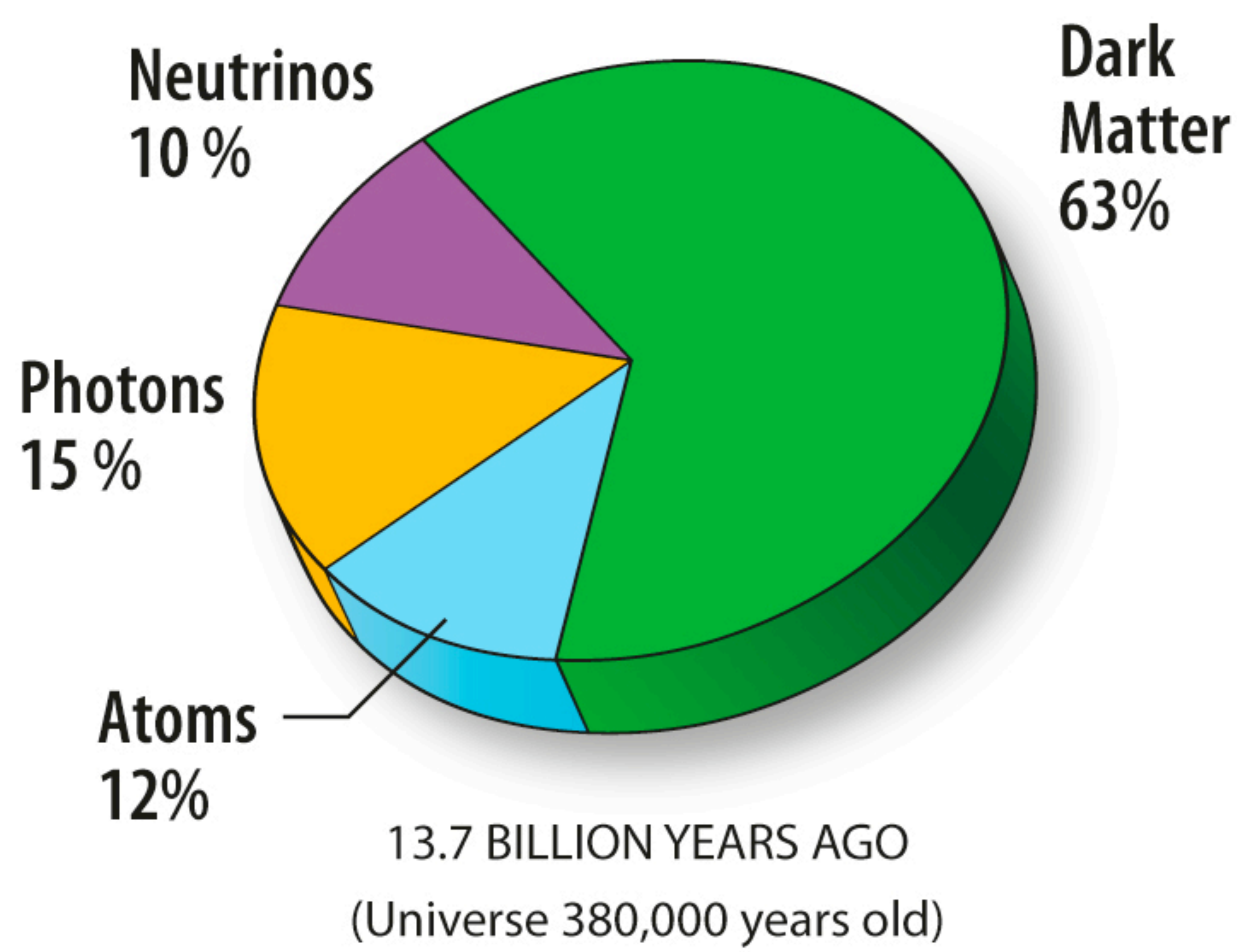
June 2001:
WMAP launched
August 2010:
WMAP 9-year full-sky survey completed

March 2011:
WMAP 9-year data release (tentative)
~***July 2012:***
Planck 1-year data release (tentative)



- Planck will also be at L2
- Block microwaves from the Sun, Earth and Moon





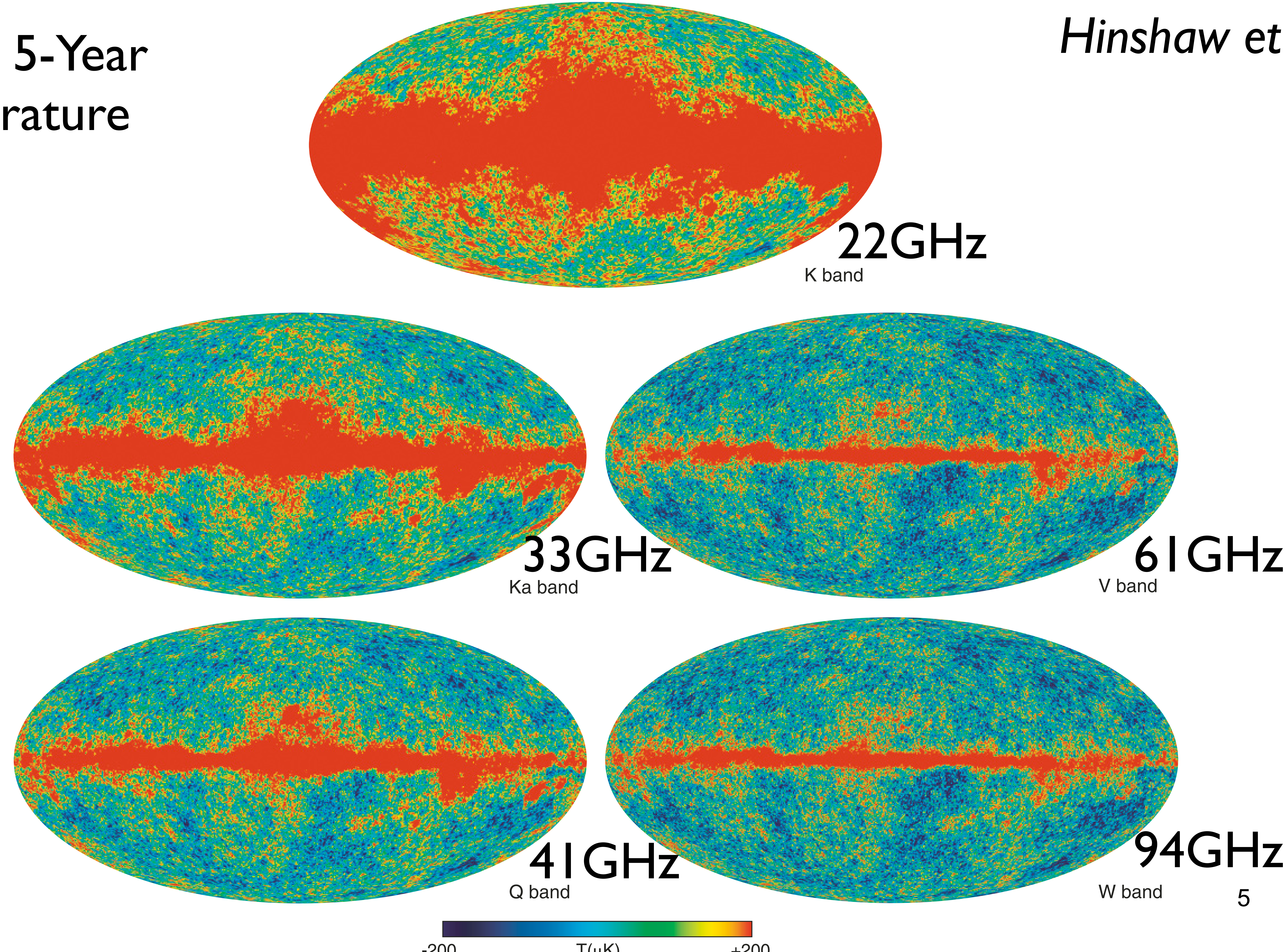
Komatsu et al.

~WMAP 5-year~ Cosmic Pie Chart

- Present-day Universe
  - Age: **13.72 ± 0.12** Gyr
  - Baryon: **4.56 ± 0.15 %**
  - Dark Matter: **22.8 ± 1.3%**
  - Vacuum Energy: **72.6 ± 1.5%**

WMAP 5-Year Temperature

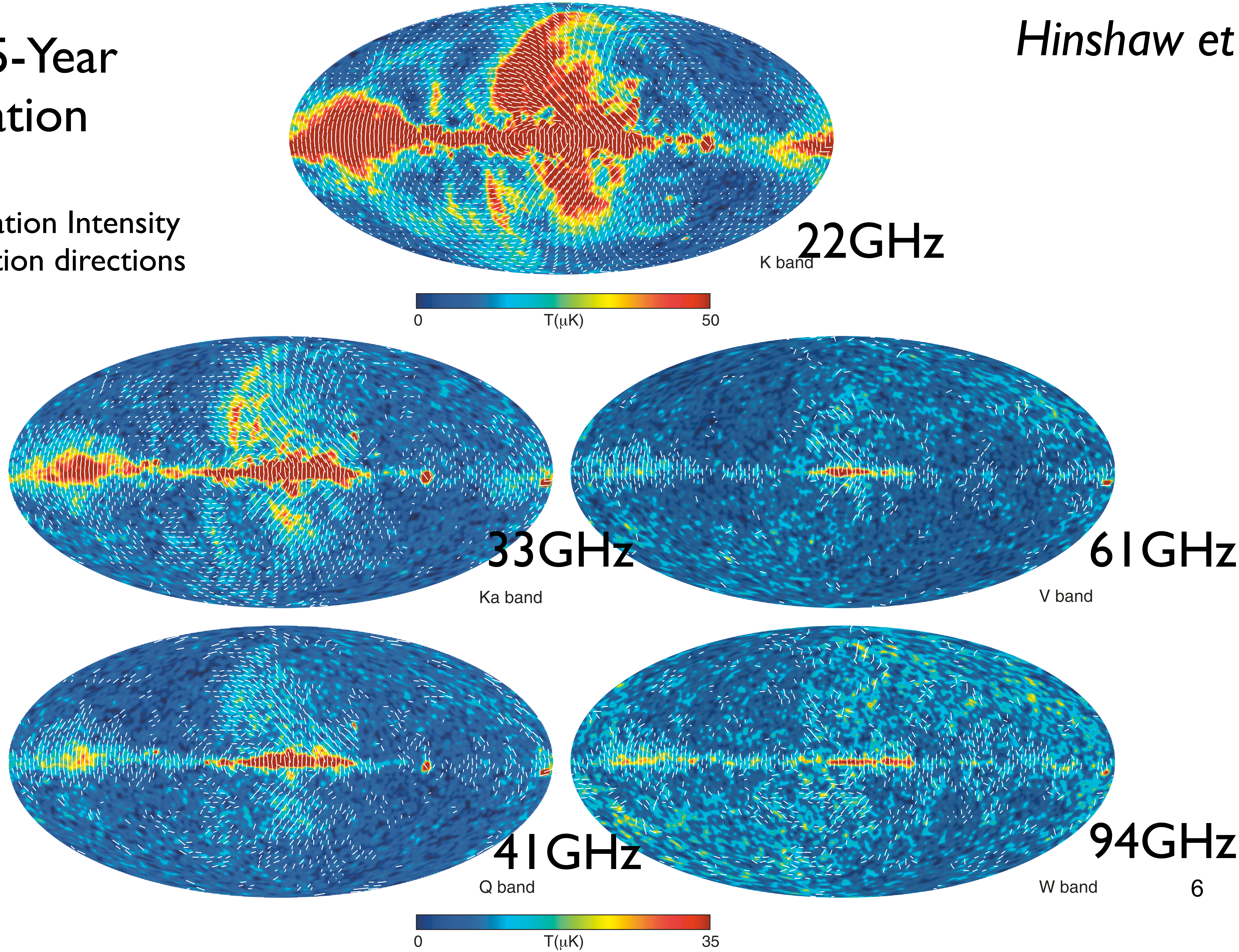
Hinshaw et al.



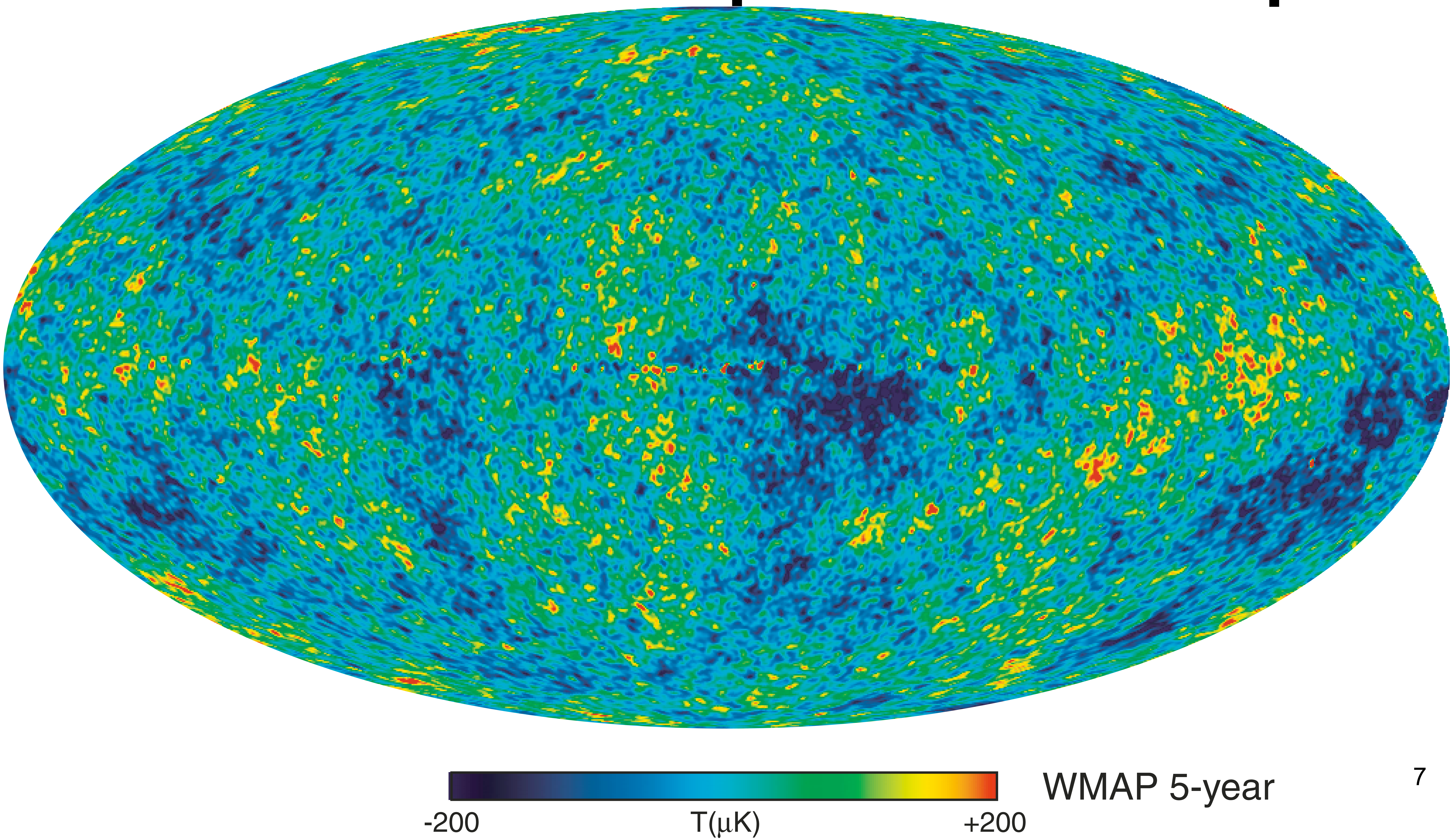
WMAP 5-Year Polarization

Hinshaw et al.

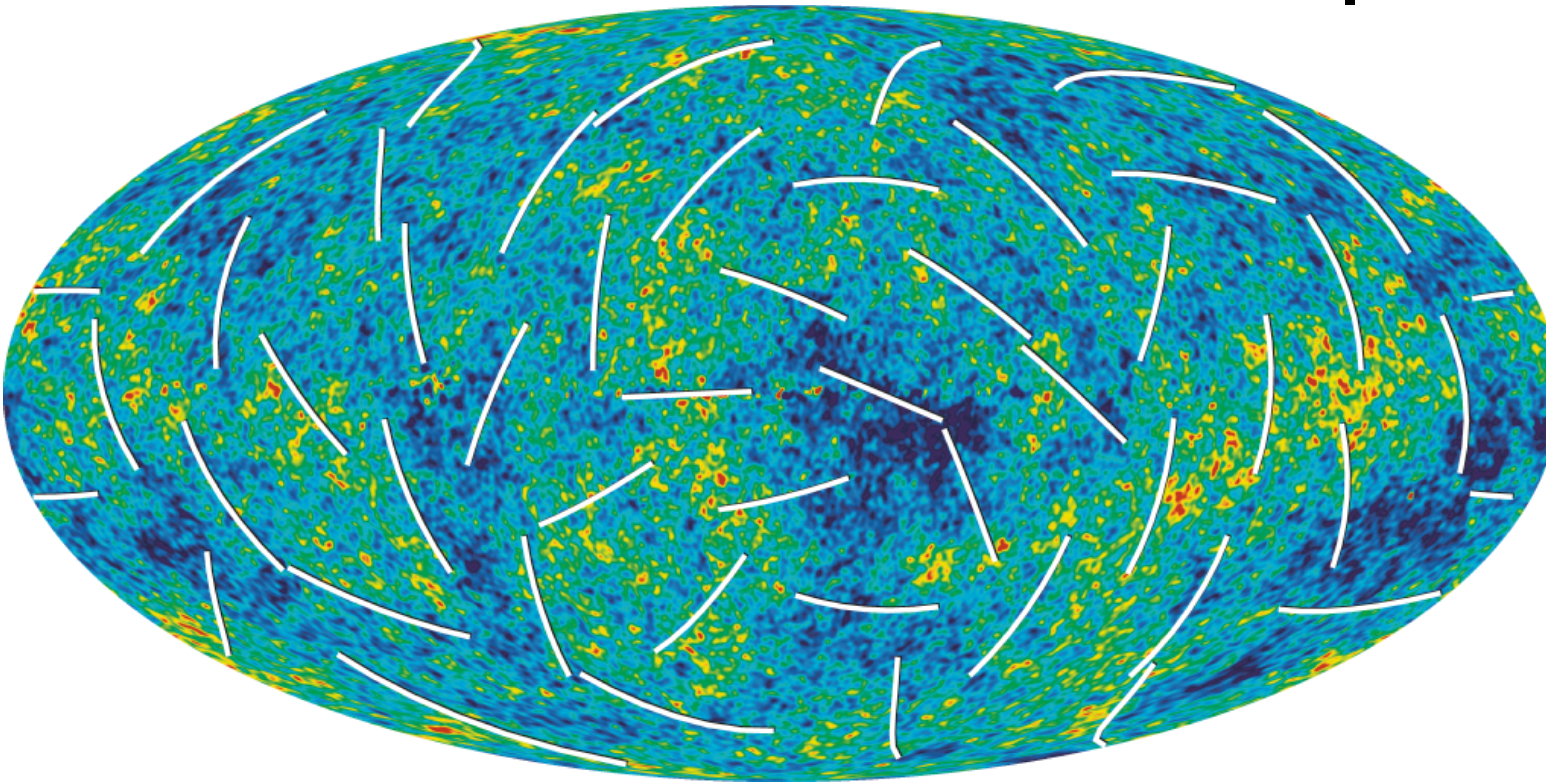
Color: Polarization Intensity
Lines: Polarization directions

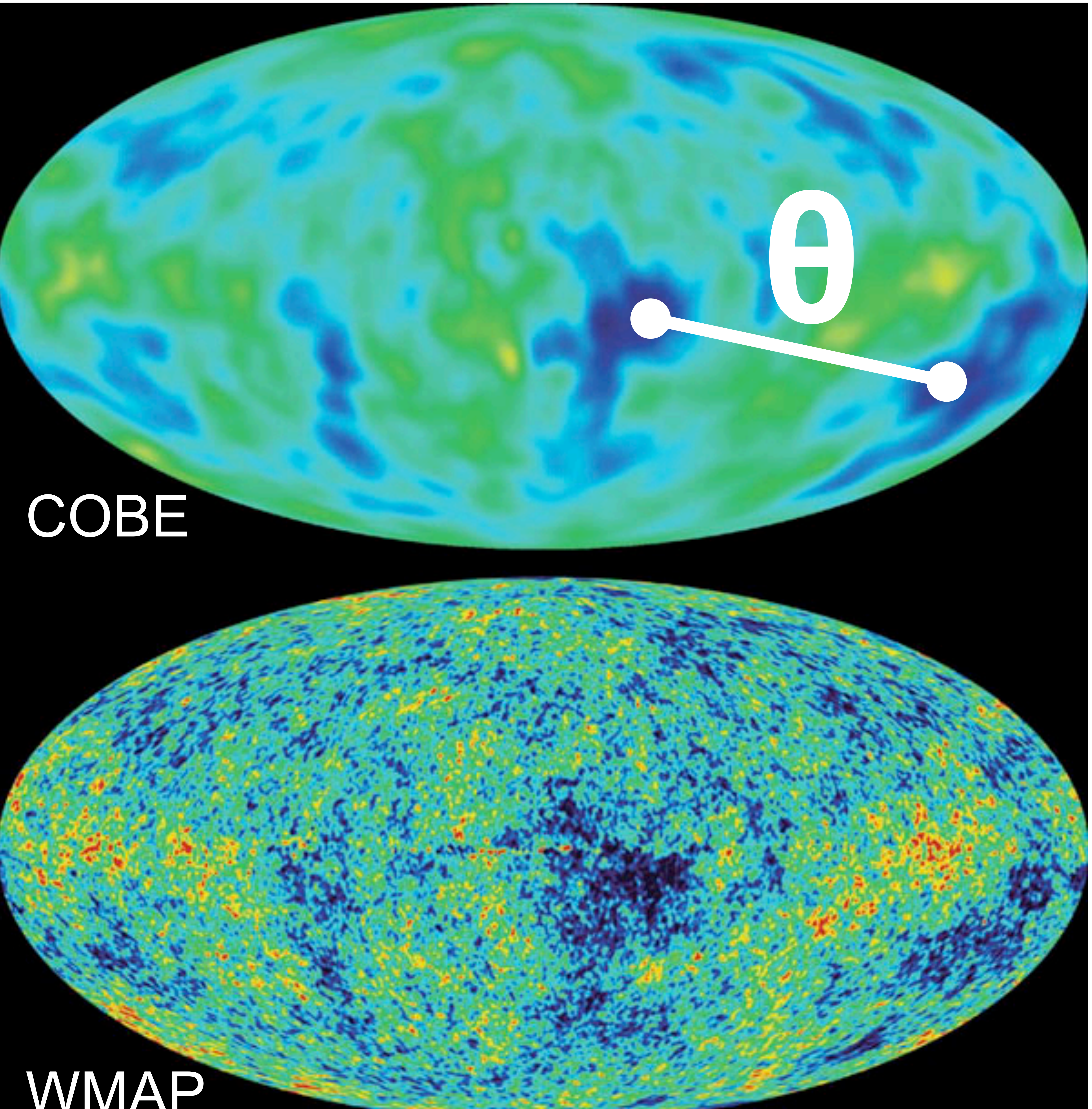


Cleaned Temperature Map



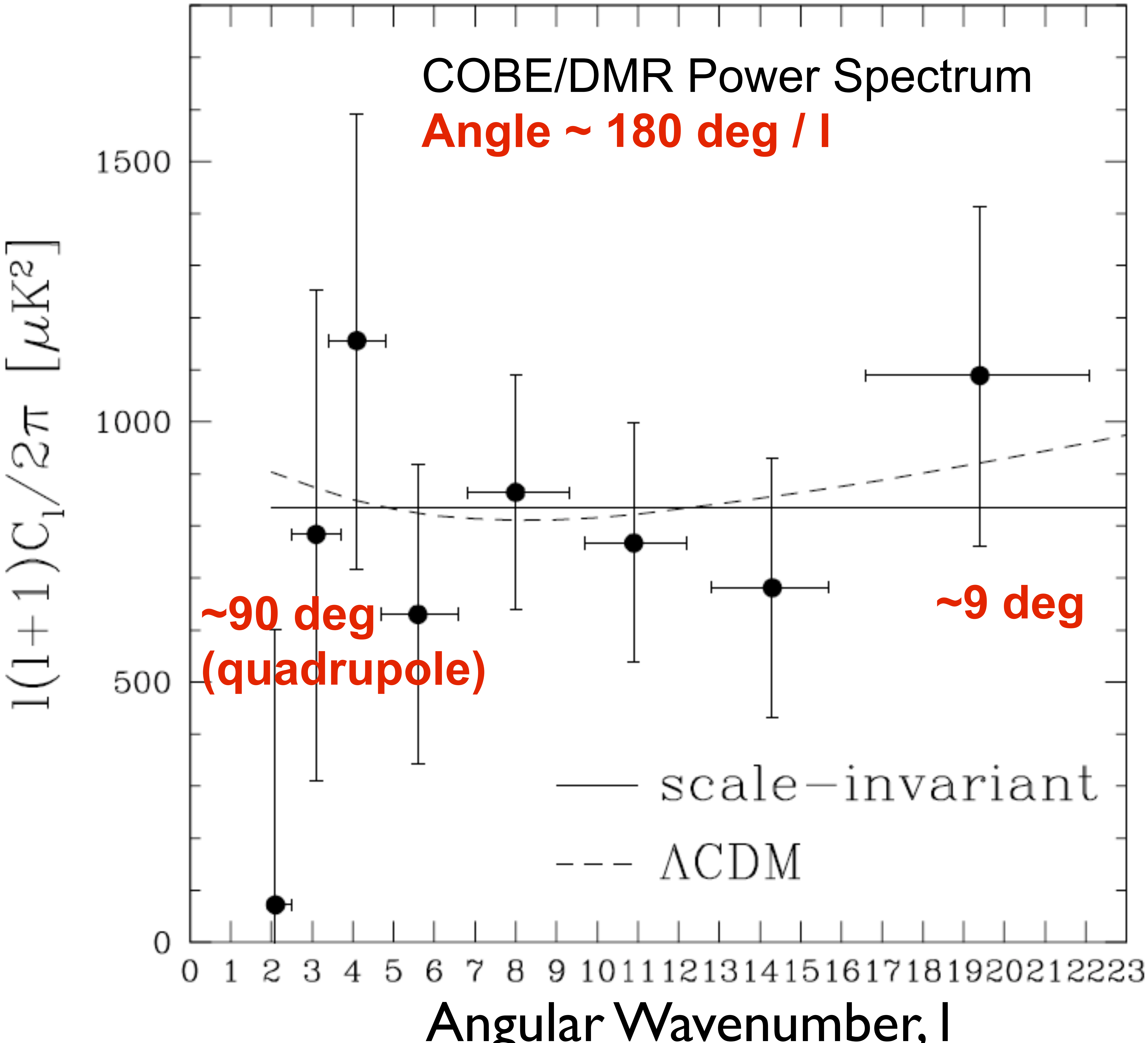
Cleaned Polarization Map

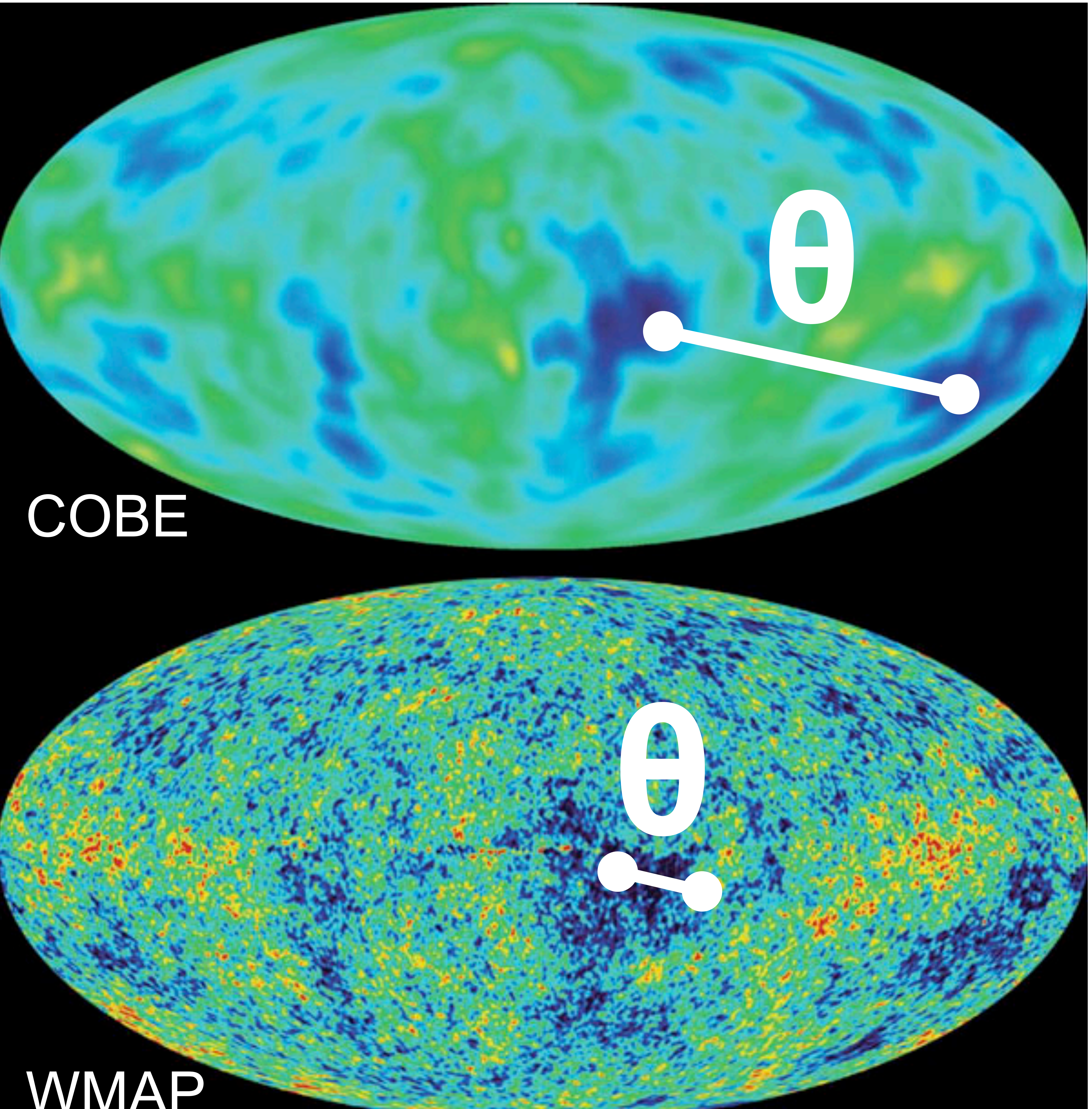




Analysis: 2-point Correlation

- $C(\theta)=(1/4\pi)\sum(2l+1)C_l P_l(\cos\theta)$
- How are temperatures on two points on the sky, separated by θ, are correlated?
- **"Power Spectrum," C_l**
  - How much fluctuation power do we have at a given angular scale?
  - **$l \sim 180$ degrees / θ**

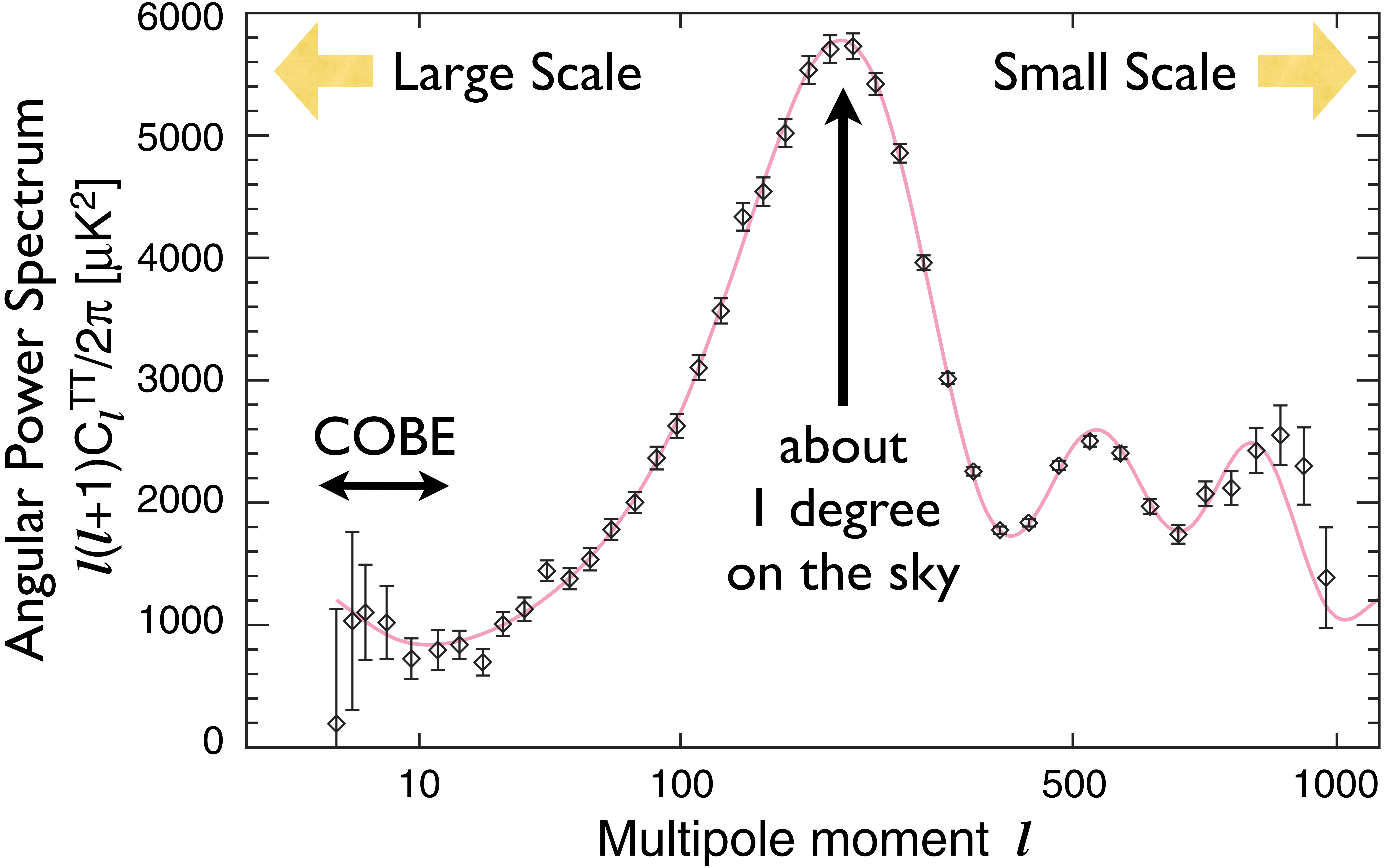

COBE/DMR Power Spectrum
Angle ~ 180 deg / l
~90 deg
(quadrupole)
~9 deg
scale−invariant
ΛCDM
l(l+1)C_l/2π [μK²]
1500
1000
500
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
Angular Wavenumber, l



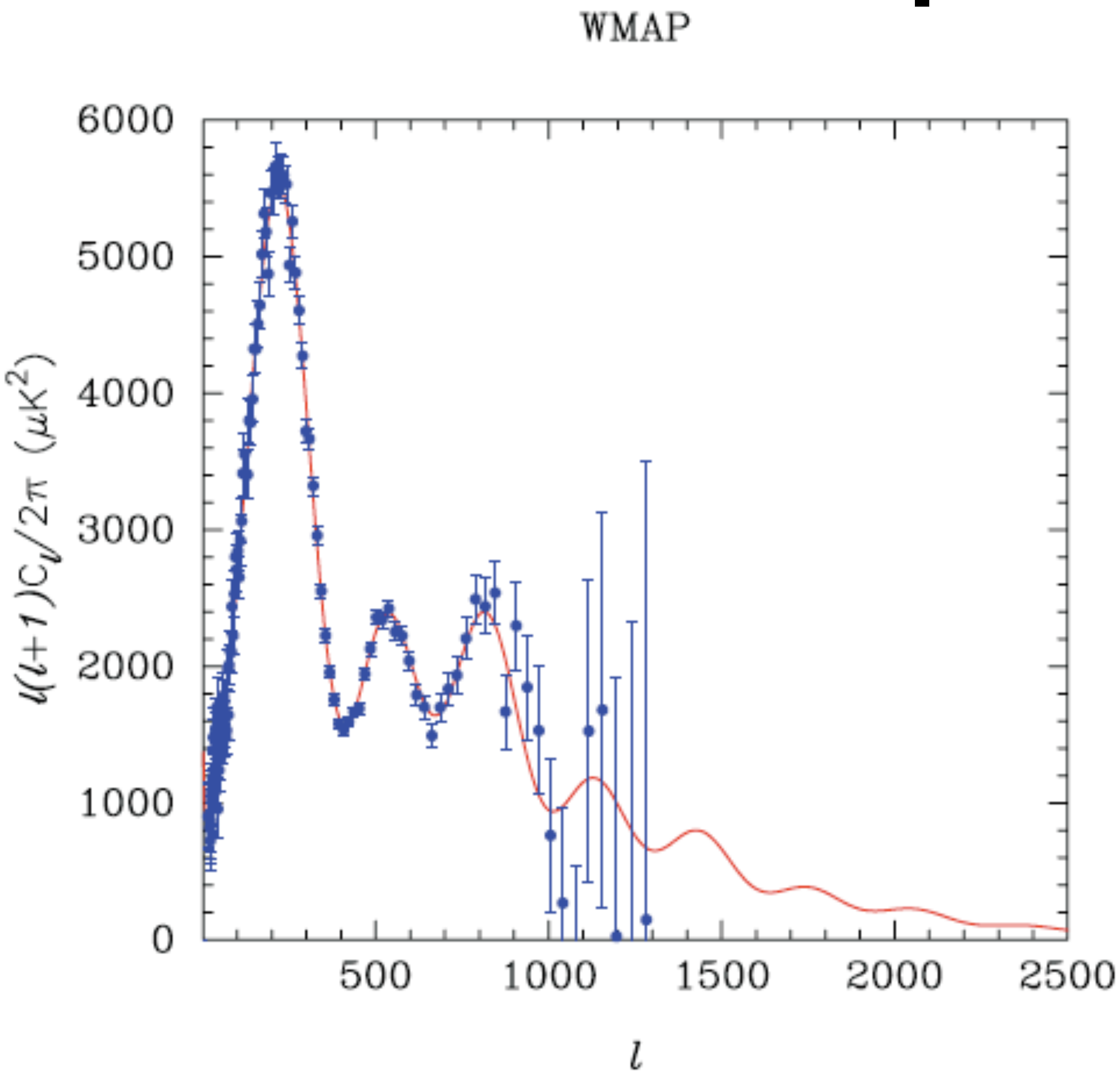
COBE To WMAP

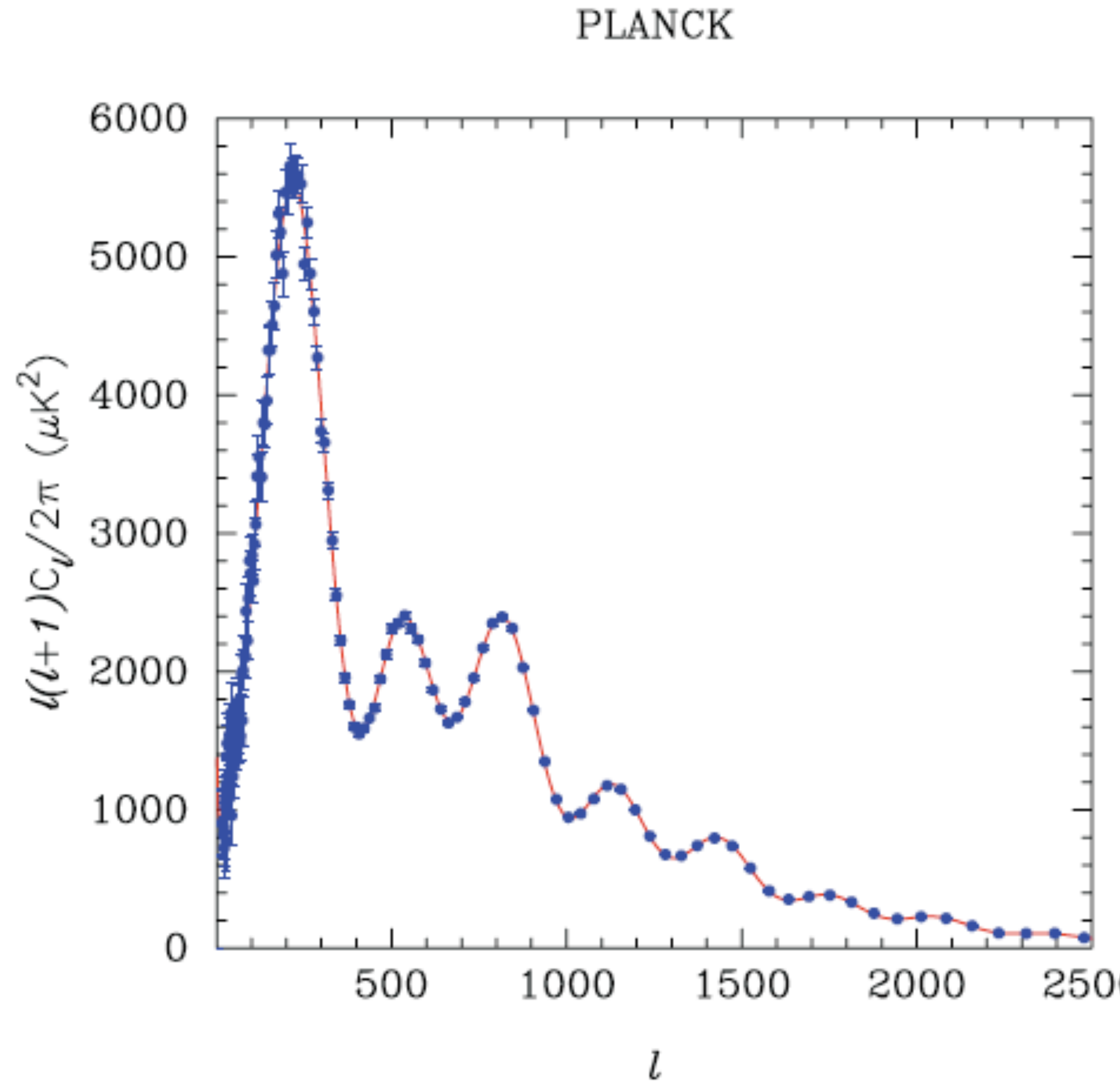
- COBE is unable to resolve the structures below ~7 degrees
- WMAP's resolving power is 35 times better than COBE.
- What did WMAP see?

WMAP Power Spectrum



Planck: Expected $C_l^{Temperature}$

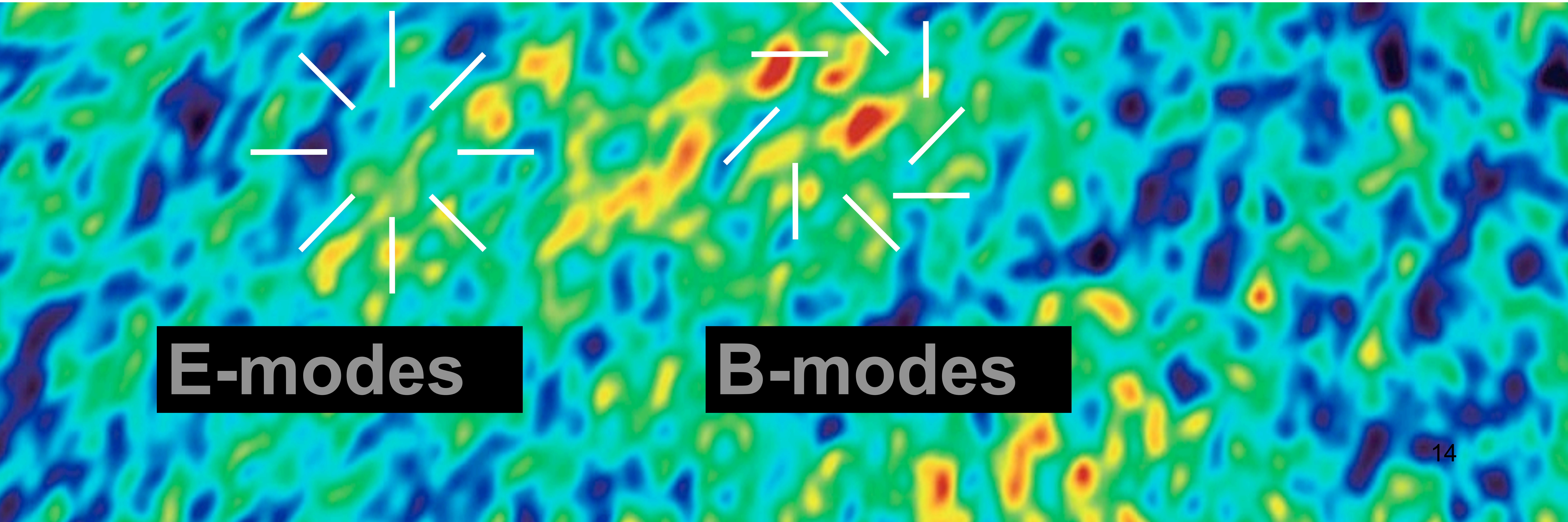


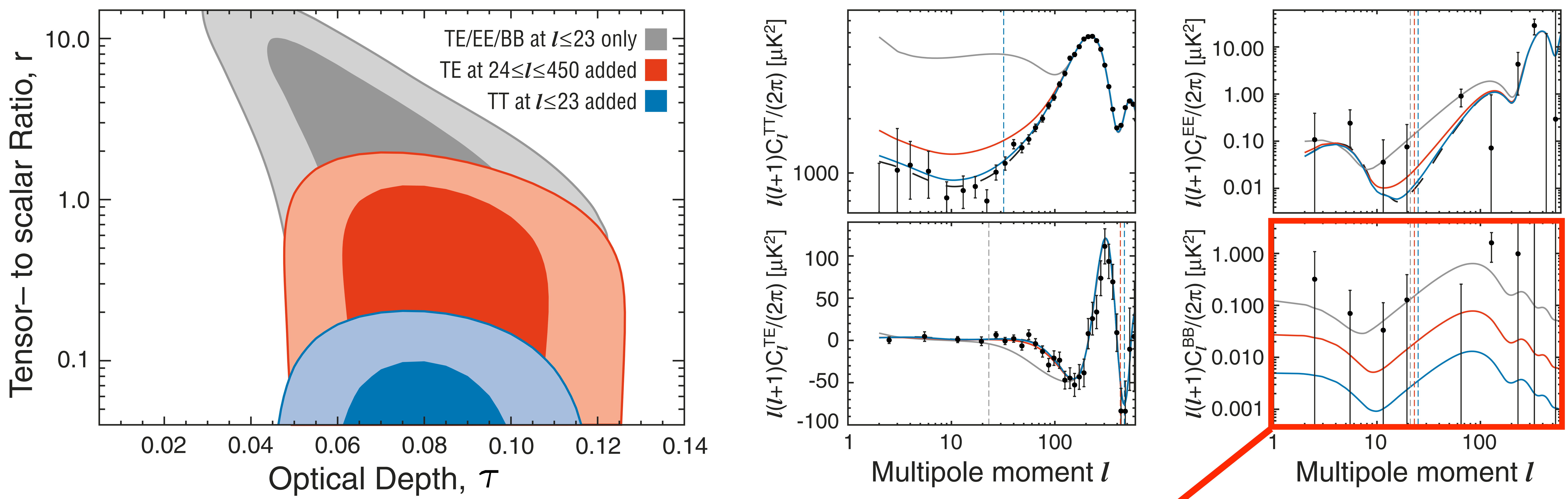


- WMAP: l~1000 => Planck: l~3000

E-mode & B-mode Polarization

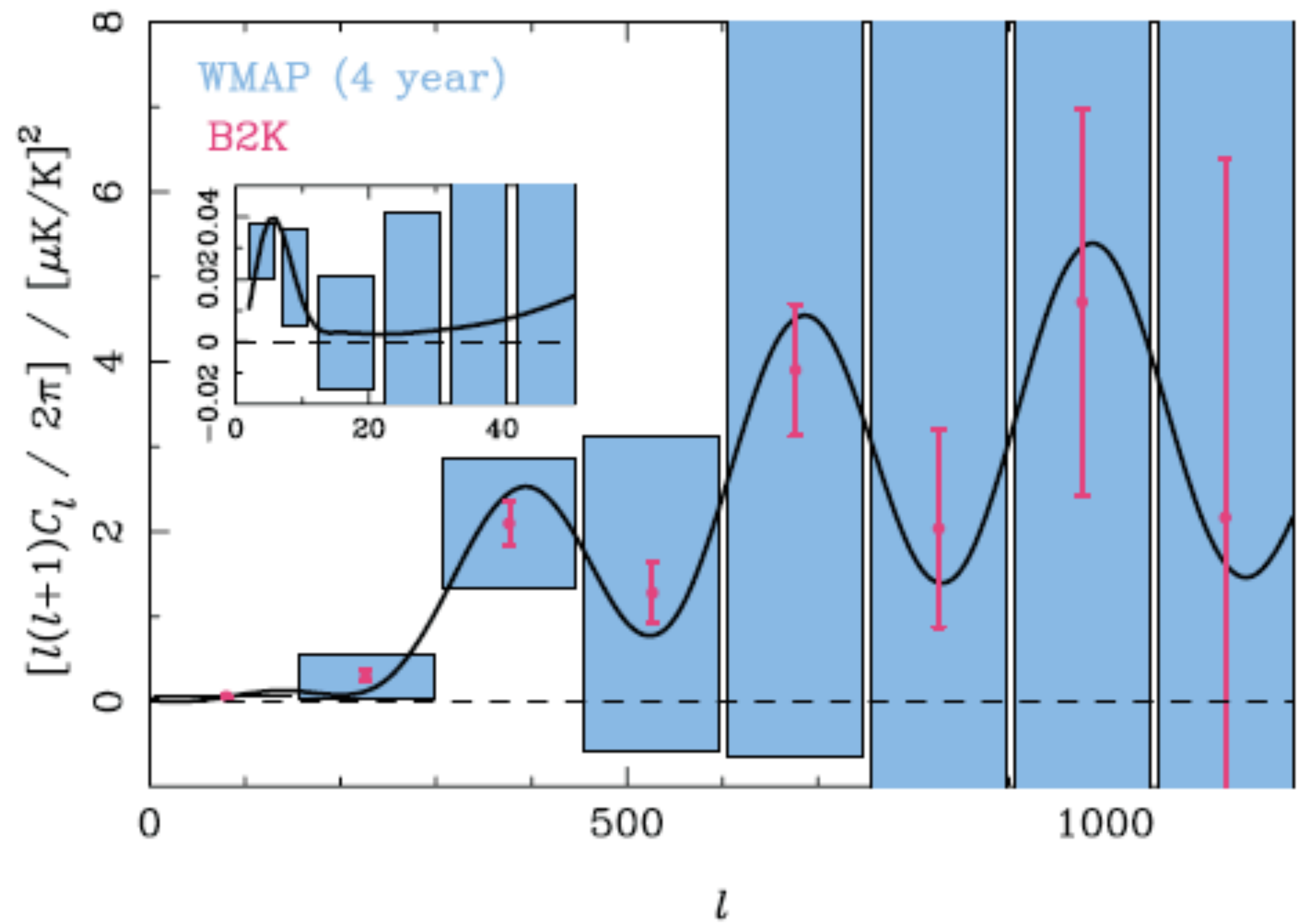
- Polarization has directions:
  - Divergence-like Pattern: E-modes
  - Curl-like Pattern: B-modes

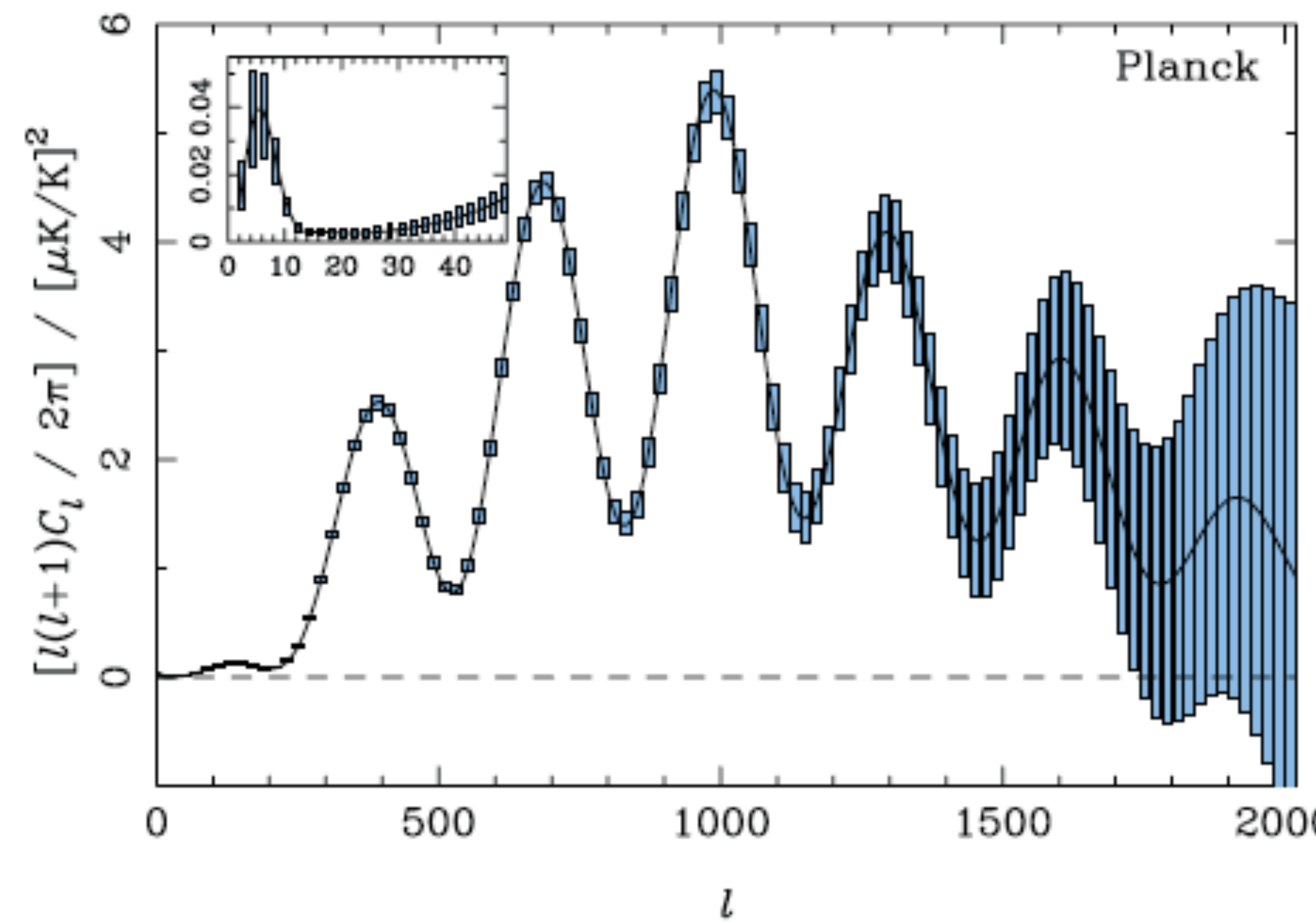


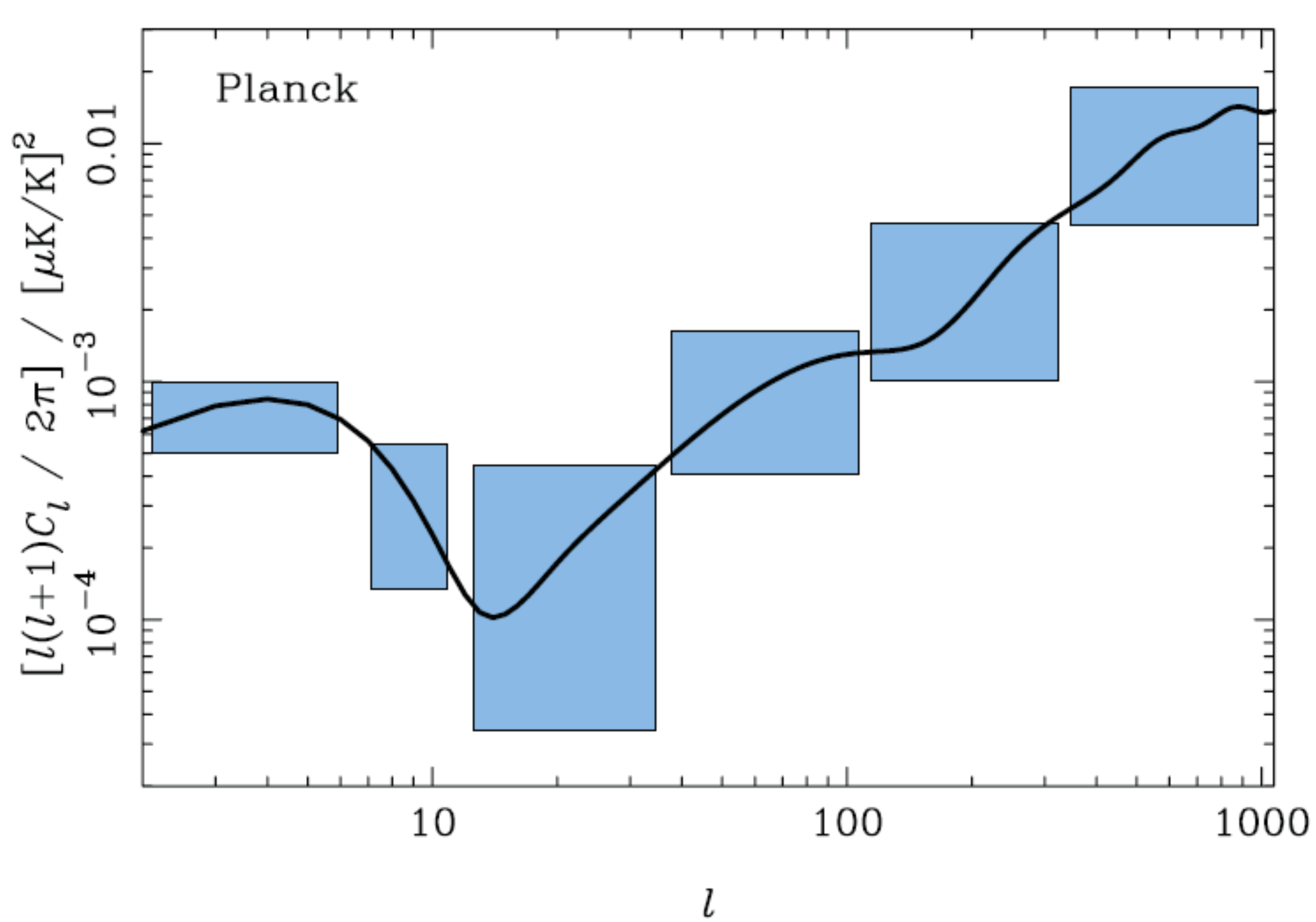


- **No detection of B-mode polarization yet.**
  - Upper limit on the primordial gravitational waves:
    - (Gravitational Wave)^2 / (Density Fluctuation.)^2 < 0.22
    - B-mode only limit: < 4.5 (95%CL)

Planck: Expected $C_l^{Polarization}$







- (Above) E-modes
- (Left) B-modes (r=0.3)

CMB: Achievements So Far

- What has CMB done so far?
  - Proof of the Big Bang (Penzias&Wilson; 1965)
  - Discovery of fluctuations (COBE; 1992)
  - Determination of cosmic composition (WMAP; 2003)
  - Deviation from scale invariance (WMAP, 3σ; 2008)
- What is next?
  - Toward earlier universe!
  - **Primordial gravitational waves (B-mode) & non-Gaussianity**

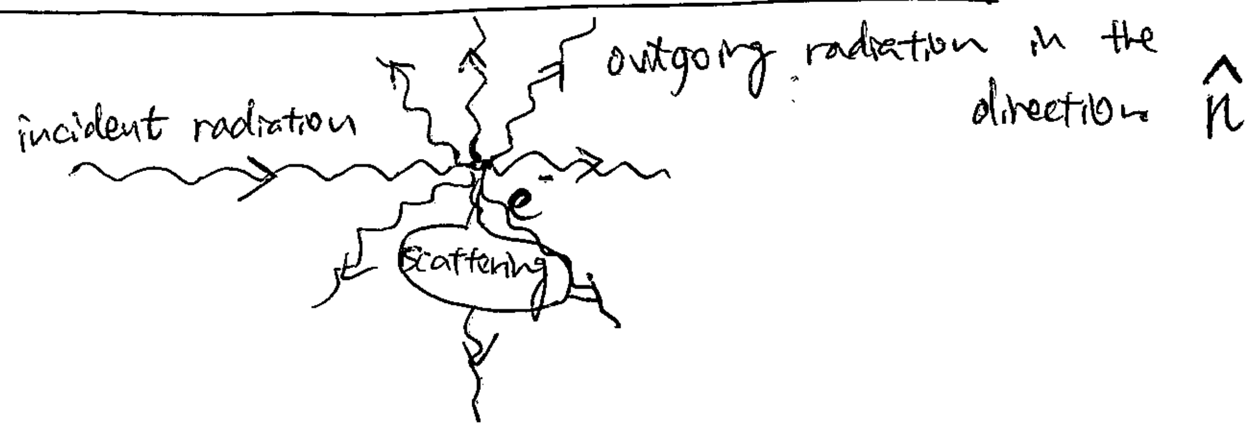
CMB Polarization Lecture

E. Komatsu, U Texas

Q. What's required for generating CMB polarization ?

A. Scattering & Anisotropic (quadrupolar) radiation around an electron.

$\boxed{\text{Thomson scattering is anisotropic !}}$

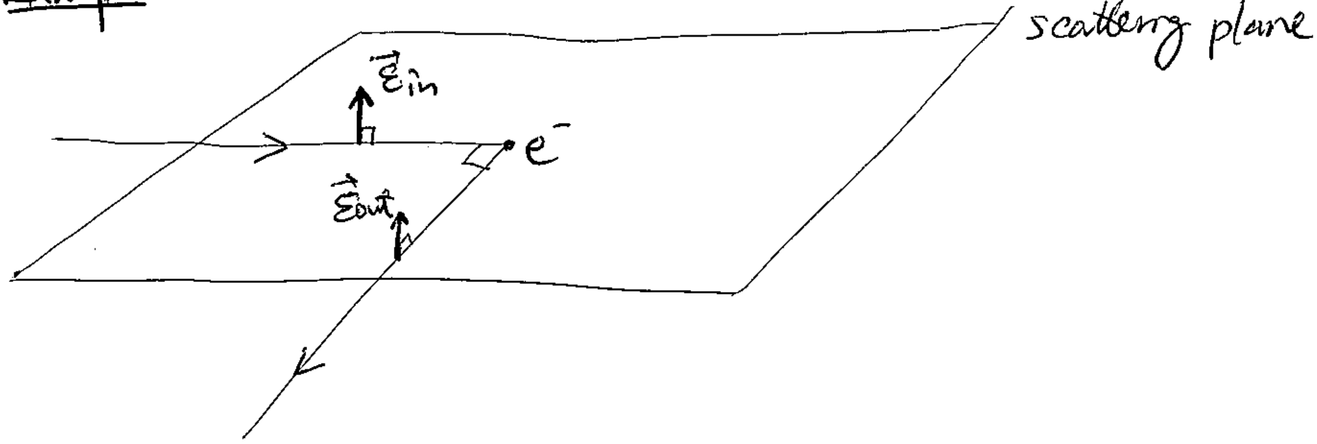


Probability of the scattering into a direction $\hat{n}$ is proportional to :

$$\boxed{\frac{d\sigma}{d\Omega_{\hat{n}}} = \frac{3}{8\pi}\sigma_T\left(\vec{\varepsilon}_{in}\cdot\vec{\varepsilon}_{out}\right)^2}$$

where
- $\vec{\varepsilon}_{in}$: polarization vector of the incident photon
- $\vec{\varepsilon}_{out}$: polarization vector of the outgoing photon

<u>Example</u> 1 : 90° scattering with polarization vector is perpendicular to the scattering plane.

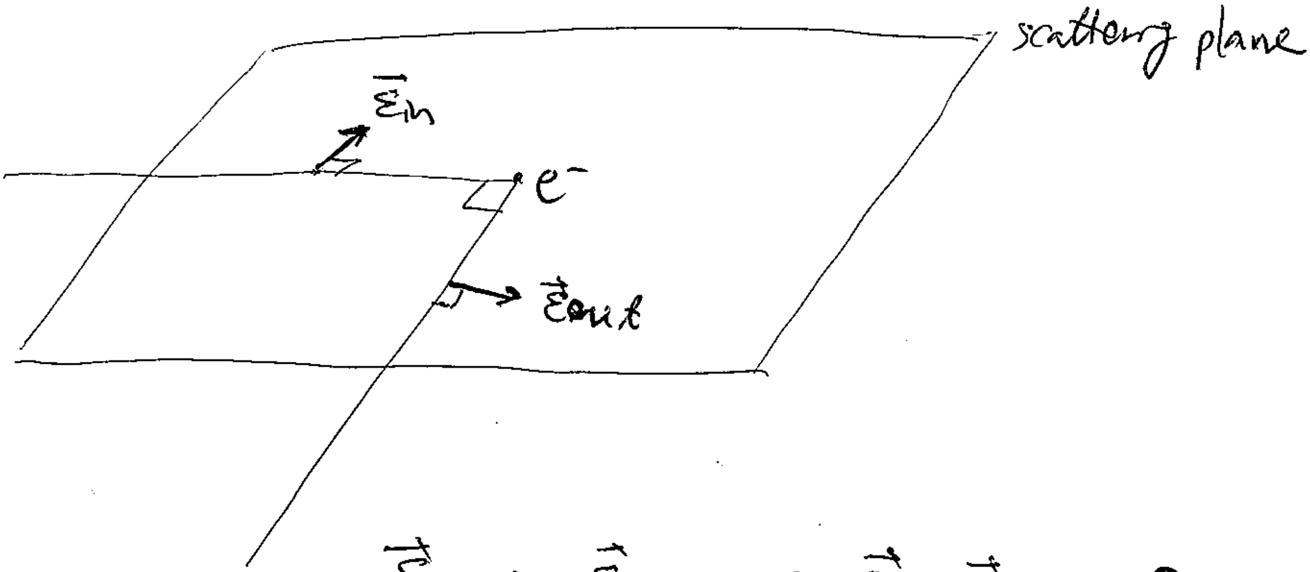


$\vec{\varepsilon}_{in} \parallel \vec{\varepsilon}_{out} \quad \therefore \vec{\varepsilon}_{in} \cdot \vec{\varepsilon}_{out} = 1.$

therefore,

$$\boxed{\frac{d\sigma}{d\Omega_{\hat{n}}} = \frac{3}{8\pi}\sigma_T.}$$

<u>Example</u> 2 : 90° scattering with polarization vector is <u>on</u> the scattering plane

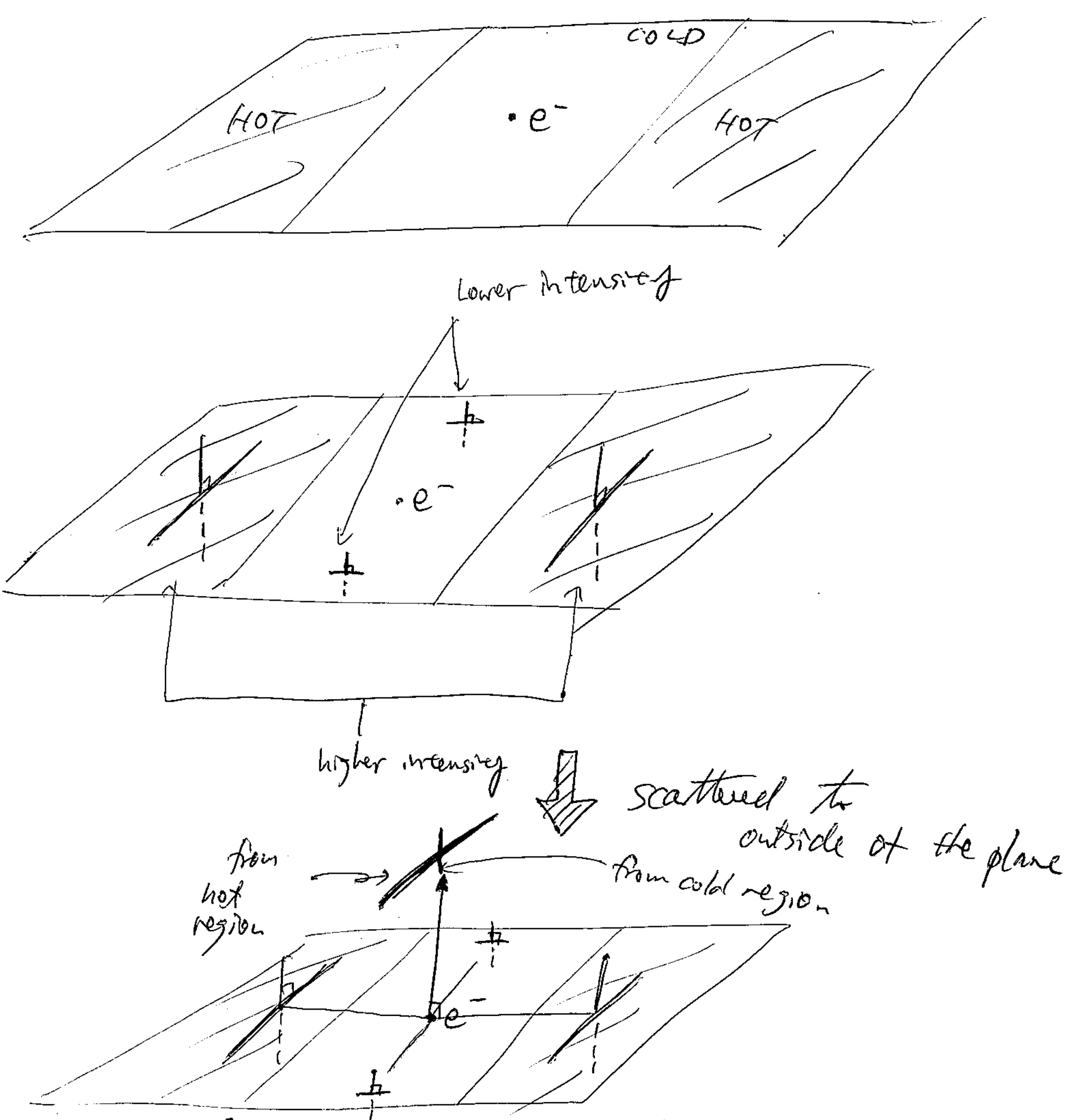


$\vec{\varepsilon}_{in} \perp \vec{\varepsilon}_{out} \quad \therefore \vec{\varepsilon}_{in} \cdot \vec{\varepsilon}_{out} = 0.$

therefore,

$$\boxed{\frac{d\sigma}{d\Omega_{\hat{n}}} = 0\ !}$$

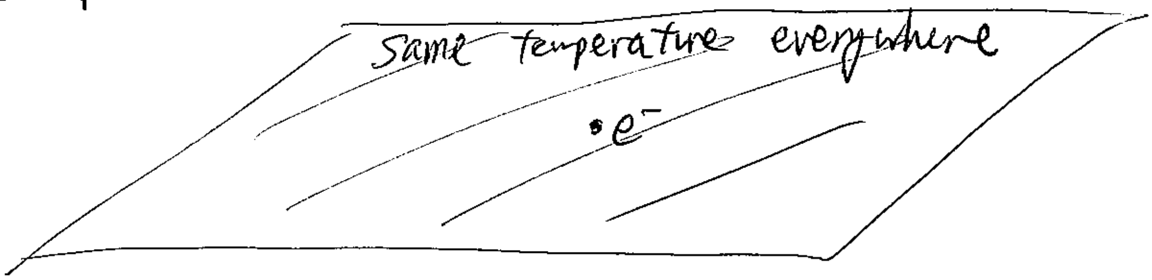
So, this gives the following picture :

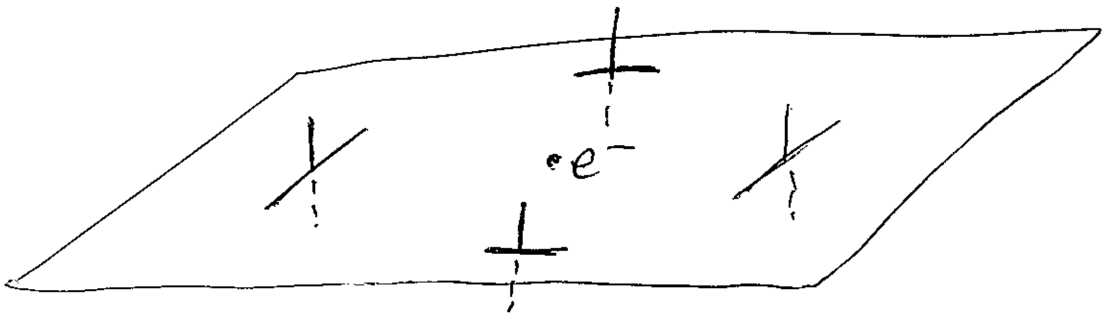


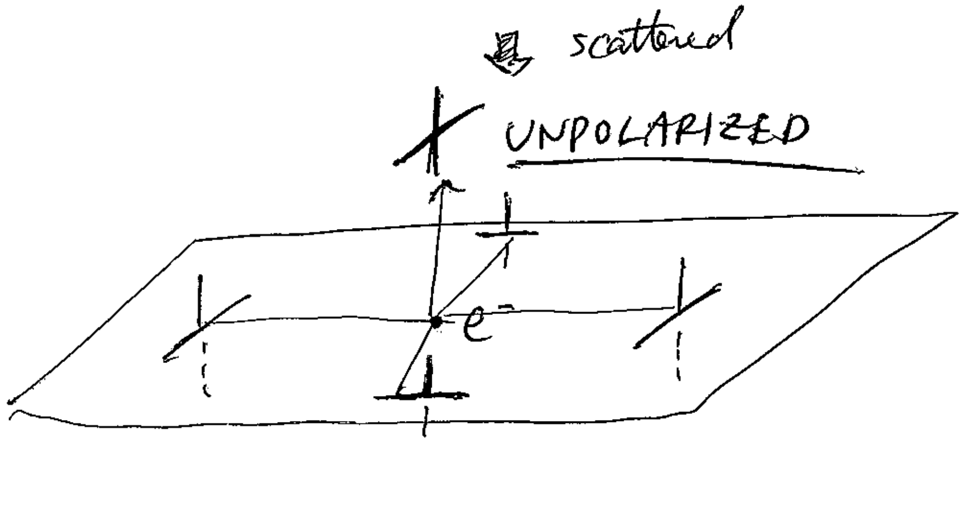
$\because$ $(\vec{\varepsilon}_{in} \cdot \vec{\varepsilon}_{out})^2$ dependence, coupled with the existence of the quadrupole, will produce polarization !!

What if there is **no** quadrupole?

Example Monopole



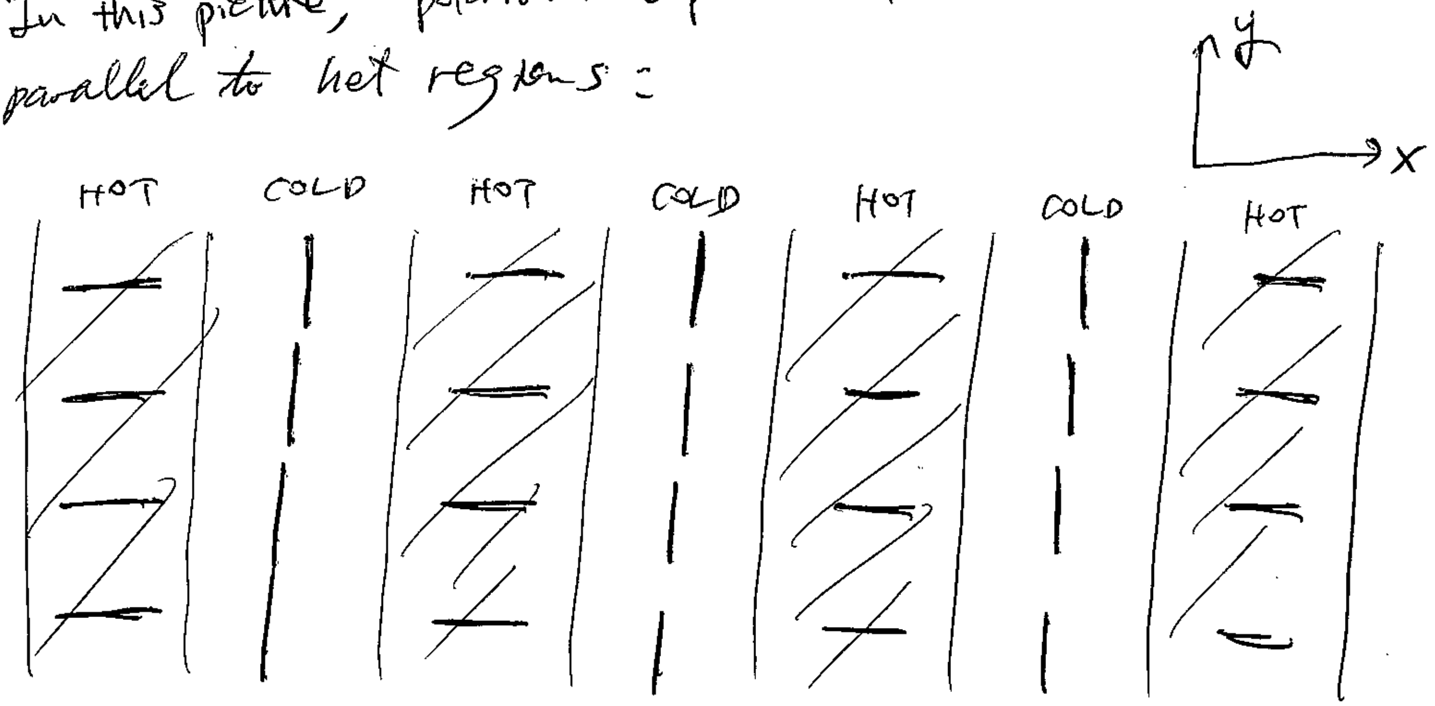




Similarly, one can show that the dipole does not produce polarization. We need quadrupole!!

Polarization Pattern

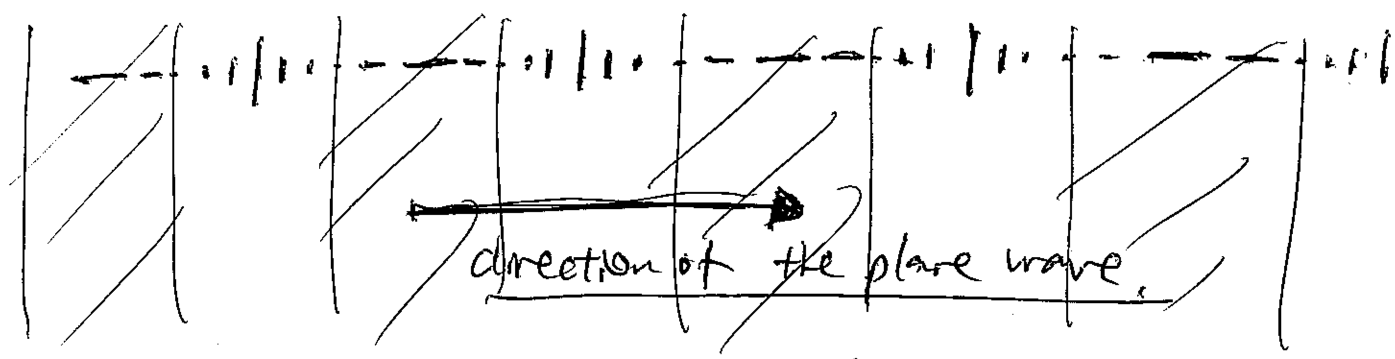
In this picture, polarization pattern tends to be parallel to hot regions :



$\vec{k}$ ⇨ direction of a plane-wave temperature fluctuation,

$$\frac{\delta T}{T} \propto e^{i\vec{k}\cdot\vec{x}}$$

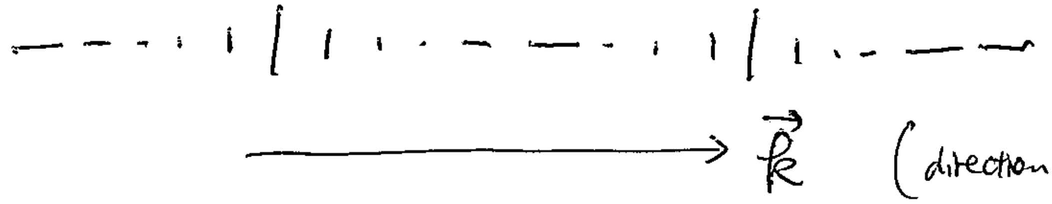
Looking at this more closely, one finds :



therefore, in this picture, the polarization direction is either parallel or perpendicular to $\vec{k}$ (of temperature fluctuation.)

$\boxed{\text{E-MODE polarization}}$

This particular pattern, in which the polarization directions are either parallel or perpendicular to $\vec{k}$, is called <u>E-mode</u> (or grad-mode) polarization.



(direction of a plane wave perturbation)

$\boxed{\text{B-MODE ?}}$

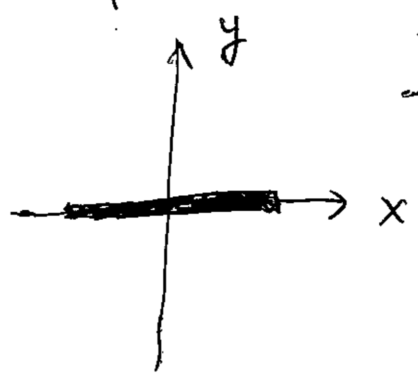
(direction of a plane wave perturbation)

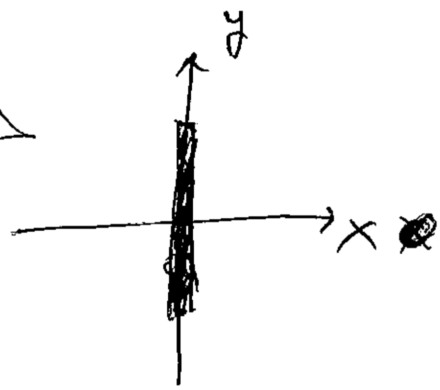
When the polarization directions are 45° tilted against $\vec{k}$, we have a pure <u>"B-mode"</u> polarization.

Precise, mathematical definition of E/B decomposition (in the flat-sky approximation).

Define "Stokes parameters" to characterize the linear polarization.

Stokes Q





$Q > 0$
$U = 0$
[purely positive Q]

$Q < 0$
$U = 0$
[purely negative Q]

$$\boxed{Q \equiv I_x - I_y}$$

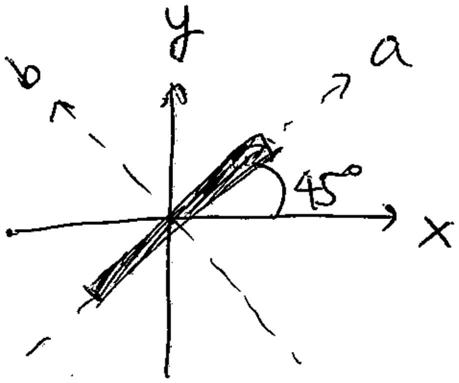
※ coordinate dependent !!

By rotating the coordinate by 90°, one can transform Q to $-Q$.

In terms of the electric field,

$$Q = |E_x|^2 - |E_y|^2$$

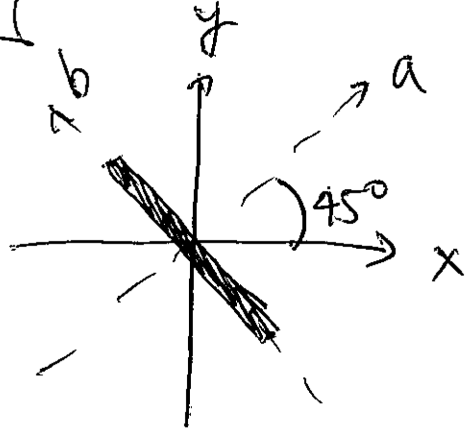
Stokes U



$Q = 0$

$U > 0$

[purely positive U]



$Q = 0$

$U < 0$

[purely negative U]

$$\boxed{U \equiv I_a - I_b}$$

By rotating the coordinate by 45° at a time, (in a clockwise rotation)
we have the following sequence of the transformation

$$\cdots \to +U \to -Q \to -U \to +Q \to +U \to \cdots$$

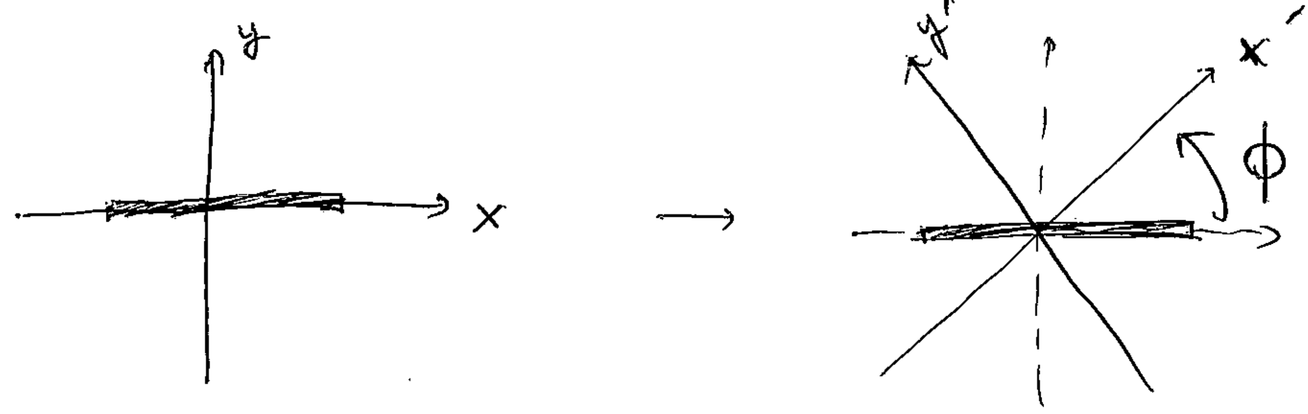
In terms of the electric field,

$$U = |E_a|^2 - |E_b|^2$$
$$= 2\mathrm{Re}(E_x E_y^*)$$

where $E_x = \frac{1}{\sqrt{2}}(E_a + iE_b)$

$E_y = \frac{1}{\sqrt{2}}(E_a - iE_b)$

For counter-clockwise rotations :



In this example, $\phi = 45^\circ$, and

pure Q (>0) transformed into pure U (<0).

$$\begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ -\sin 2\phi & \cos 2\phi \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix}$$

For the coordinate trans. of

$$\begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

It is often more convenient to work with a complex linear combination :

$$\boxed{Q + iU}$$

For counter-clockwise rotations,

$$Q' + iU' = e^{-2i\phi}(Q+iU)$$
$$Q' - iU' = e^{+2i\phi}(Q+iU)$$

This property shows that $Q \pm iU$ has the spin of ± 2 (or ∓ 2, depending on one's definition)

⇒ This is important : this means that we cannot decompose $Q \pm iU$ into the ordinary plane-wave decomposition, a.k.a. Fourier transform.

We need the spin-2 decomposition!!

Spherical / Spin-2 harmonics decomposition

As we measure temperature & polarization on the sky, which is the surface of a sphere, we need to use the spherical harmonics & spin 2 tensor harmonics to decompose T & $Q \pm iU$.

I. e.,

$$\underset{\text{temperature at a given pixel } \hat{n}}{T(\hat{n})} = \sum_{\ell m} a_{\ell m} \underset{\text{spherical harmonics}}{Y_{\ell m}(\hat{n})}$$

$$[Q \pm iU](\hat{n}) = \sum_{\ell m} {}_{\mp 2}a_{\ell m} \, \underset{\text{spin 2 harmonics}}{{}_{\mp 2}Y_{\ell m}(\hat{n})}$$

Then, E- / B-modes are defined as:

$$\boxed{\begin{aligned} E_{\ell m} &\equiv -\frac{1}{2}\left({}_2a_{\ell m} + {}_{-2}a_{\ell m}\right) \\ B_{\ell m} &= -\frac{i}{2}\left({}_2a_{\ell m} - {}_{-2}a_{\ell m}\right) \end{aligned}}$$

※ physical meanings will be given later!!

Flat-sky Approximation

Zaldarriaga & Seljak 1997. PRD 55, 1830 [astro-ph/9609170]

The harmonics decomposition must be used for the actual data analysis, but it is more convenient to use the "flat-sky approximation" for obtaining the physical intuition.

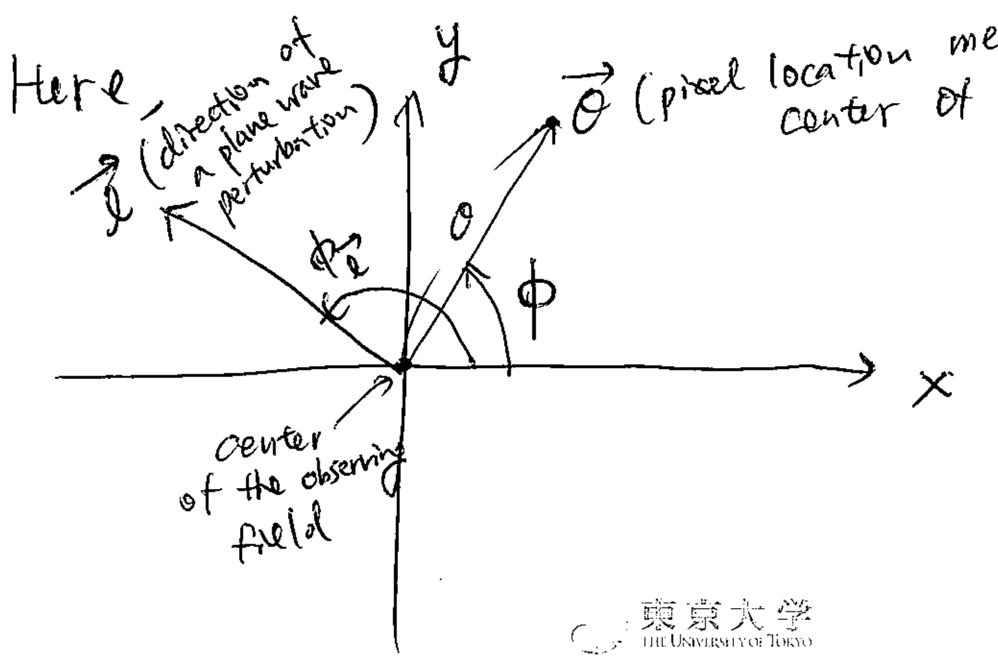
In this app., we can continue to use the ordinary (2d) Fourier transform for $T(\hat{n})$:

$$T(\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2}\, a_{\vec{\ell}}\, e^{i\vec{\ell}\cdot\vec{\theta}}$$

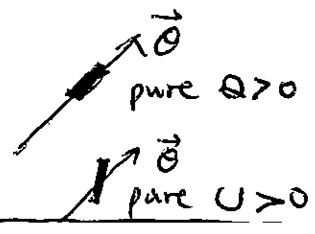
The corresponding expansion for spin-2 fields is

$$[Q \pm iU](\vec{\theta}) = -\int \frac{d^2\vec{\ell}}{(2\pi)^2}\, {}_{\mp2}a_{\vec{\ell}}\, e^{\pm 2i(\phi-\phi_{\vec{\ell}})}\, e^{i\vec{\ell}\cdot\vec{\theta}}$$

Here,



In this setup, Q & U are defined with respect to the polar coordinate:



With E/B decomposition,

$$\boxed{[Q \pm iU](\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2}\left[E_{\vec{\ell}} \mp iB_{\vec{\ell}}\right] e^{\pm 2i(\phi-\phi_{\vec{\ell}})} e^{i\vec{\ell}\cdot\vec{\theta}}}$$

which gives :

$$Q(\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2}\left\{E_{\vec{\ell}}\cos[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}}\sin[2(\phi-\phi_{\vec{\ell}})]\right\} e^{i\vec{\ell}\cdot\vec{\theta}}$$

$$U(\vec{\theta}) = \int \frac{d^2\vec{\ell}}{(2\pi)^2}\left\{-E_{\vec{\ell}}\sin[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}}\cos[2(\phi-\phi_{\vec{\ell}})]\right\} e^{i\vec{\ell}\cdot\vec{\theta}}$$

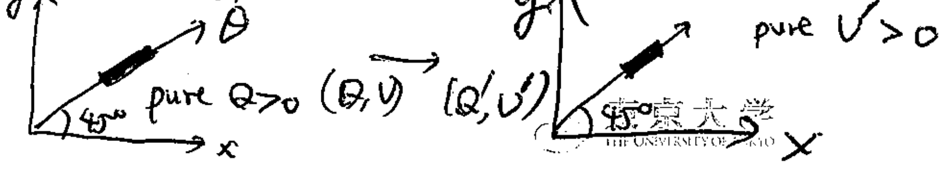
This formula is very useful for picturing E&B modes.

Let's take a single plane wave, $\vec{\ell}$. Then this wave generates $Q(\vec{\theta})$ & $U(\vec{\theta})$ as

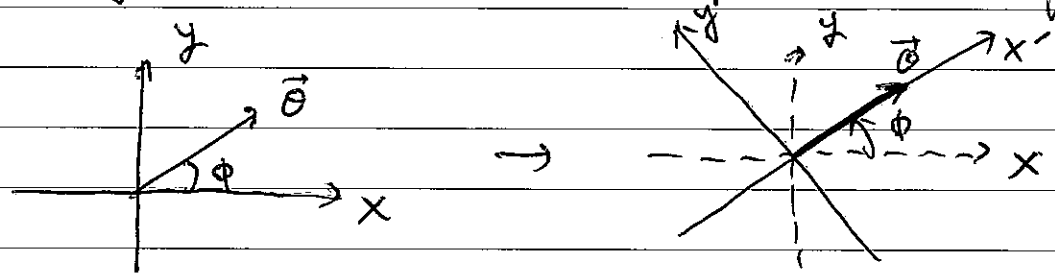
$$Q(\vec{\theta}) = \left\{E_{\vec{\ell}}\cos[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}}\sin[2(\phi-\phi_{\vec{\ell}})]\right\} e^{i\ell\theta\cos\varphi}$$

$$U(\vec{\theta}) = \left\{-E_{\vec{\ell}}\sin[2(\phi-\phi_{\vec{\ell}})] + B_{\vec{\ell}}\cos[2(\phi-\phi_{\vec{\ell}})]\right\} e^{i\ell\theta\cos\varphi}$$

Now, we need to convert these results to Q & U defined in the Cartesian coordinate :



We can do the coordinate transformation by rotating the coordinate (counter-clock wise) by ϕ:



In other words, we choose the coordinates such that $\vec{\theta}$ is always on the x-axis.

$$Q' \pm iU' = e^{\mp 2i\phi} (Q \pm iU)$$

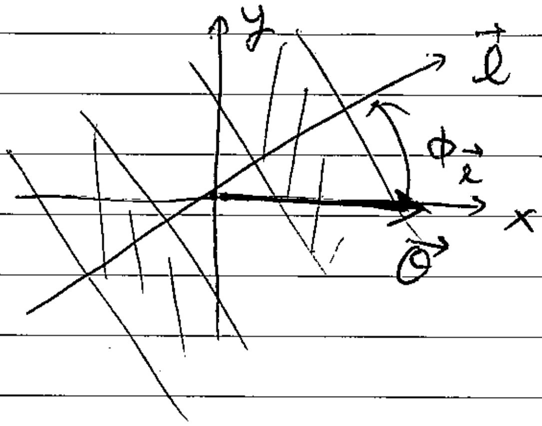
Therefore (Zaldarriaga & Seljak 1997)

$$\begin{cases} Q'(\vec{\theta}) = + \displaystyle\int \frac{d^2\vec{l}}{(2\pi)^2} \left[E_{\vec{l}} \cos(2\phi_{\vec{l}}) - B_{\vec{l}} \sin(2\phi_{\vec{l}}) \right] e^{i\vec{l}\cdot\vec{\theta}} \\ U'(\vec{\theta}) = + \displaystyle\int \frac{d^2\vec{l}}{(2\pi)^2} \left[E_{\vec{l}} \sin(2\phi_{\vec{l}}) + B_{\vec{l}} \cos(2\phi_{\vec{l}}) \right] e^{i\vec{l}\cdot\vec{\theta}} \end{cases}$$

$\boxed{\text{From now on, we use } Q \,\&\, U \text{ for } Q' \,\&\, U'.}$

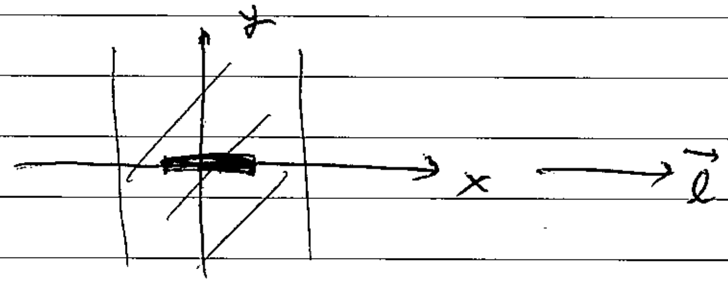
pure E-modes

Consider a pure E-mode from a single plane wave.

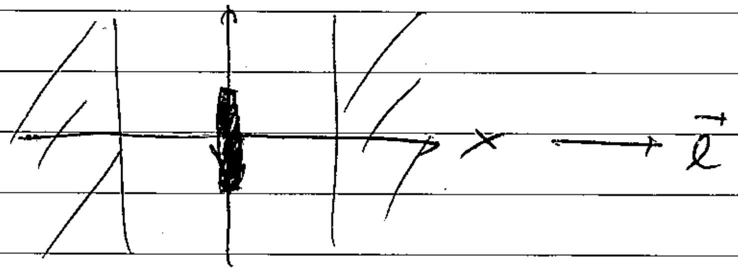


$$\begin{cases} Q(\vec{\theta}) = E_{\vec{\ell}} \cos(2\phi_{\vec{\ell}})\, e^{i\ell\theta\cos\phi_{\vec{\ell}}} \\ U(\vec{\theta}) = E_{\vec{\ell}} \sin(2\phi_{\vec{\ell}})\, e^{i\ell\theta\cos\phi_{\vec{\ell}}} \end{cases}$$

$\boxed{\phi_{\vec{\ell}} = 0}$ $E_{\vec{\ell}} = +1 \rightarrow \begin{cases} Q = +1 \\ U = 0 \end{cases}$ (we take $\ell\theta \ll 1$)

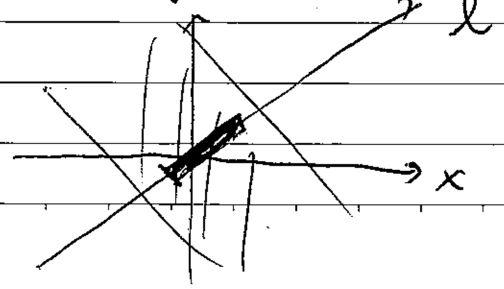


$E_{\vec{\ell}} = -1 \rightarrow \begin{cases} Q = -1 \\ U = 0 \end{cases}$

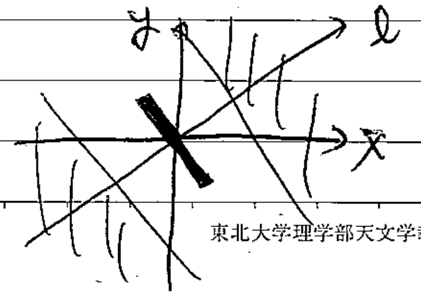


$\boxed{\phi_{\vec{\ell}} = 45^\circ}$

$E_{\vec{\ell}} = +1$ $\begin{cases} Q = 0 \\ U = +1 \end{cases}$



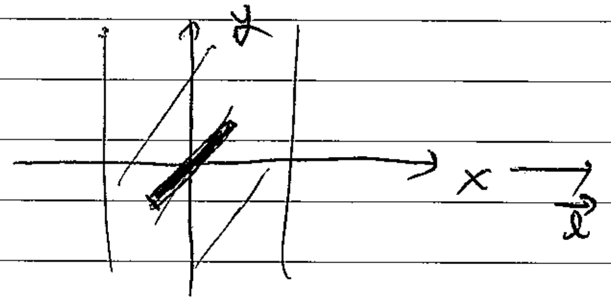
$E_{\vec{\ell}} = -1$ $\begin{cases} Q = 0 \\ U = +1 \end{cases}$

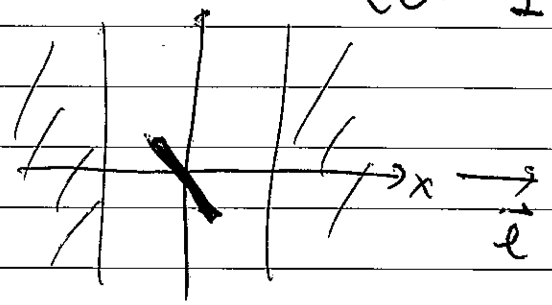


$\boxed{\text{pure B-modes}}$ Single plane wave

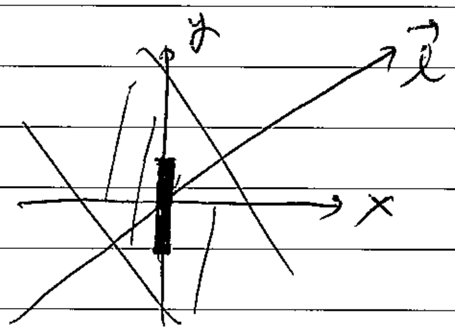
$$\begin{cases} Q(\vec{\theta}) = -B_{\vec{\ell}} \sin(2\phi_{\vec{\ell}})\, e^{i\ell\theta\cos\phi_{\vec{\ell}}} \\ U(\vec{\theta}) = +B_{\vec{\ell}} \cos(2\phi_{\vec{\ell}})\, e^{i\ell\theta\cos\phi_{\vec{\ell}}} \end{cases}$$

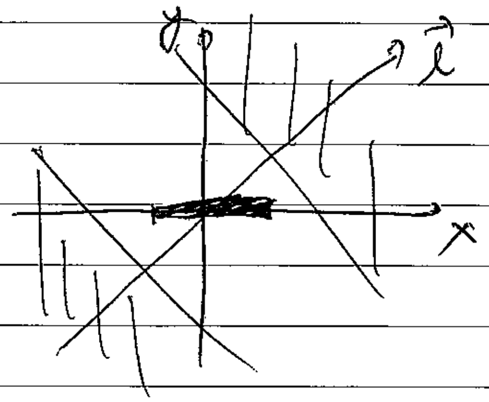
$\boxed{\phi_{\vec{\ell}} = 0}$ $B_{\vec{\ell}} = +1$ $\begin{cases} Q=0 \\ U=+1 \end{cases}$ $B_{\vec{\ell}} = -1$ $\begin{cases} Q=0 \\ U=-1 \end{cases}$





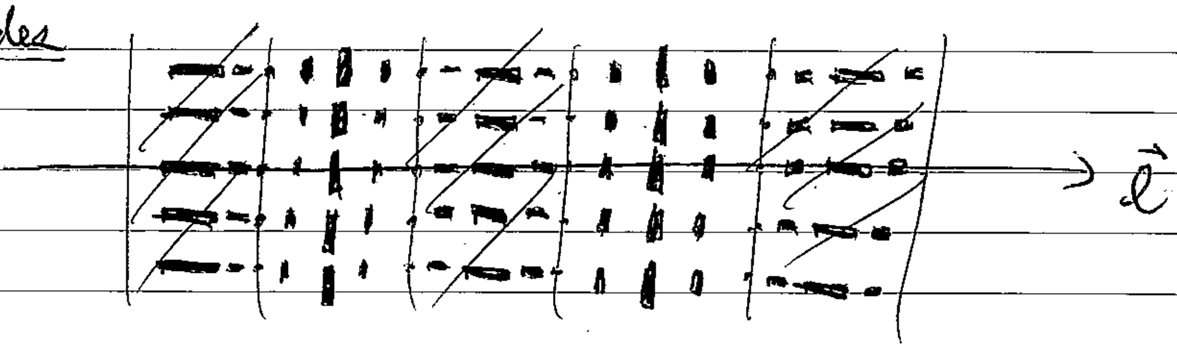
$\boxed{\phi_{\vec{\ell}} = 45°}$ $B_{\vec{\ell}} = +1$ $\begin{cases} Q=-1 \\ U=0 \end{cases}$ $B_{\vec{\ell}} = -1$ $\begin{cases} Q=+1 \\ U=0 \end{cases}$



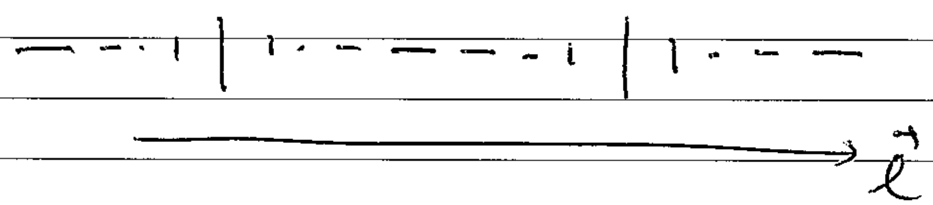


Therefore, we obtain the following picture :

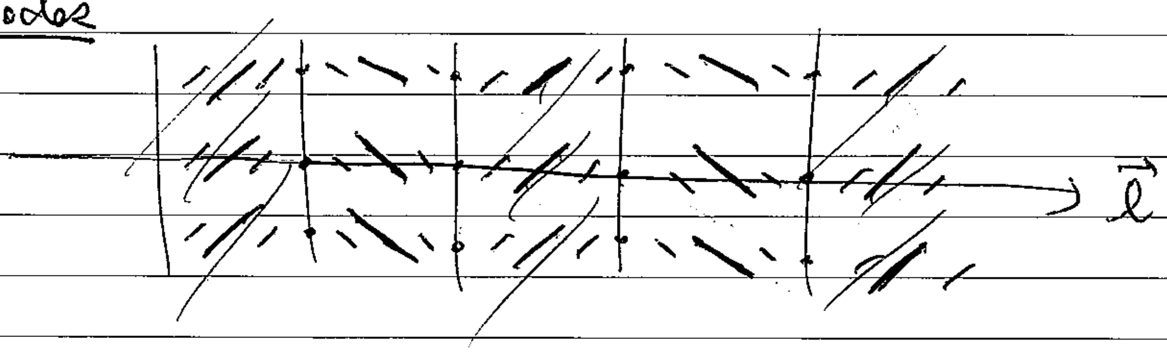
E-modes



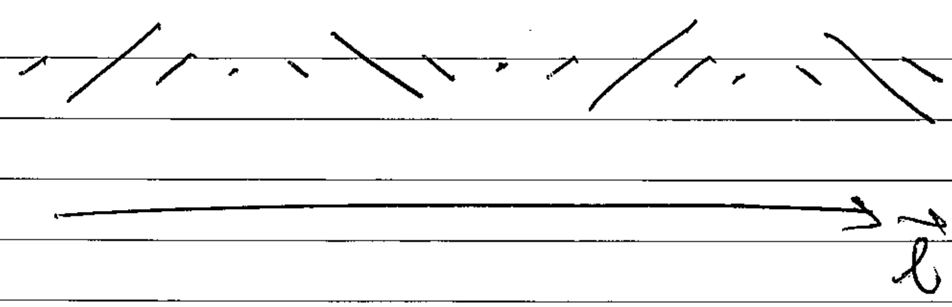
i.e.,



B-modes



i.e.,

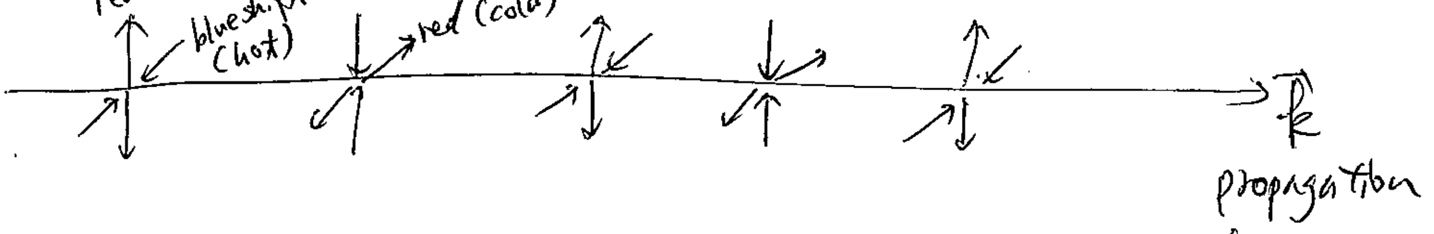


How do we generate B-modes?

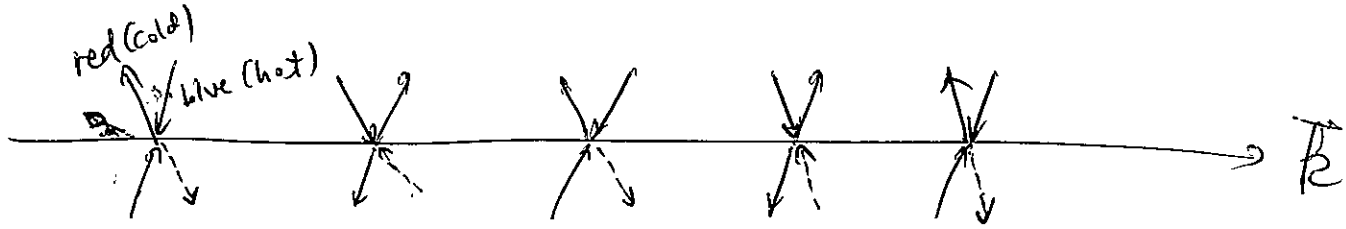
One way: gravitational waves !!

Gravitational waves can produce polarization, as it can create the quadrupolar temperature anisotropy.

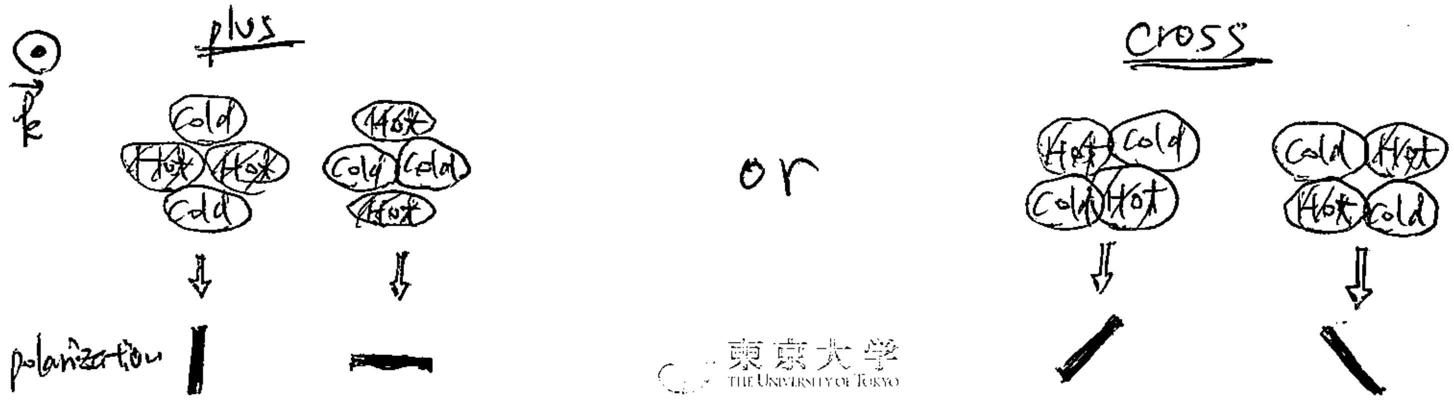
"plus (+) mode"



"cross (×) mode"



So, when GW is coming toward us:

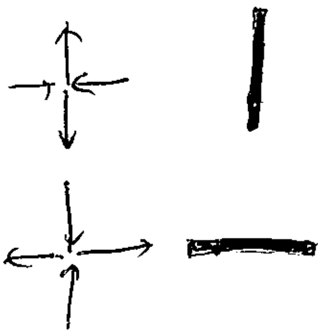


Gravitational waves can generate BOTH E & B modes.

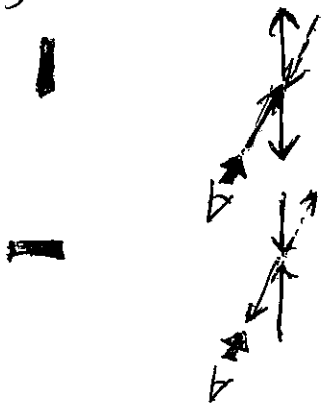
because they have two modes: + & X.

"+" mode

Face-on



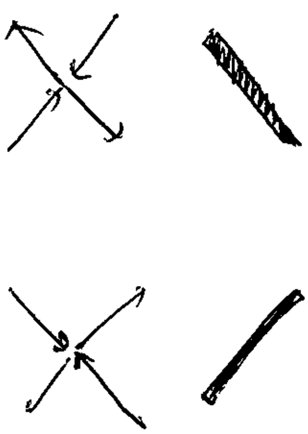
Edge-on



[Calculations show that
(the edge-on amplitude) = 1/2 (face-on amplitude)]

"X" mode

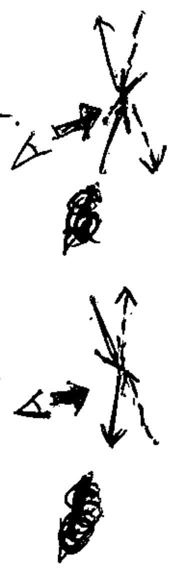
Face-on



Edge-on

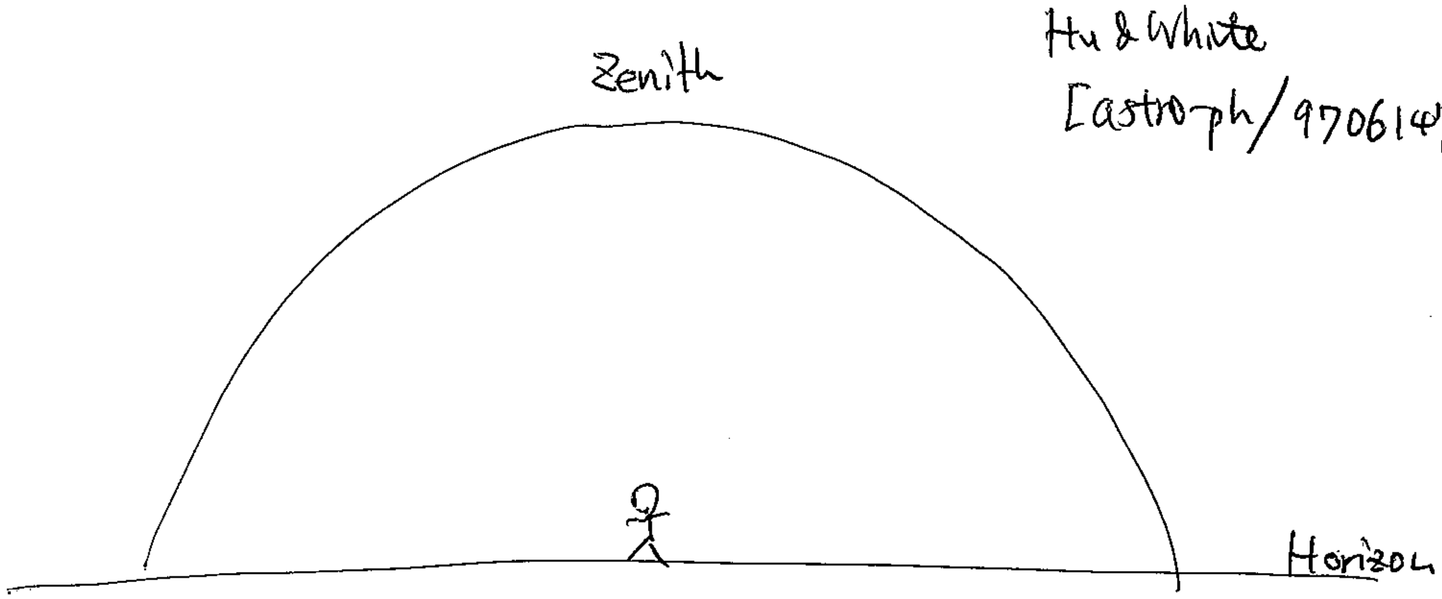
NO POL.

NO POL.

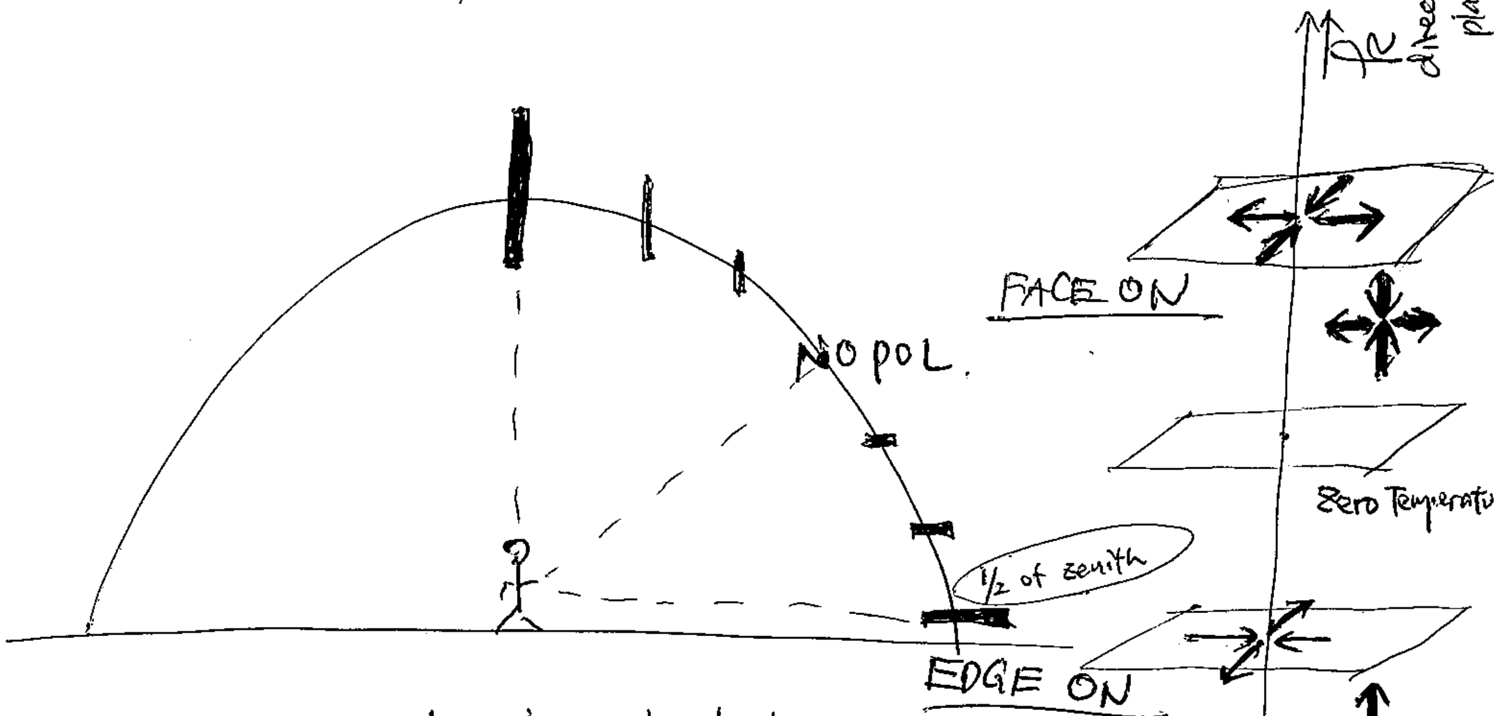


Now, let's project this on the sky.

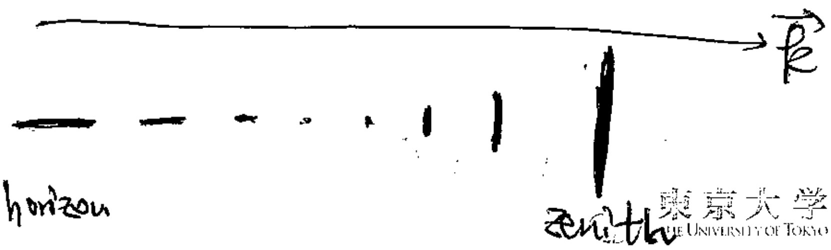
Hu & White [astroph/9706147]



For "+" mode (1/2 wavelength from Horizon to zenith)

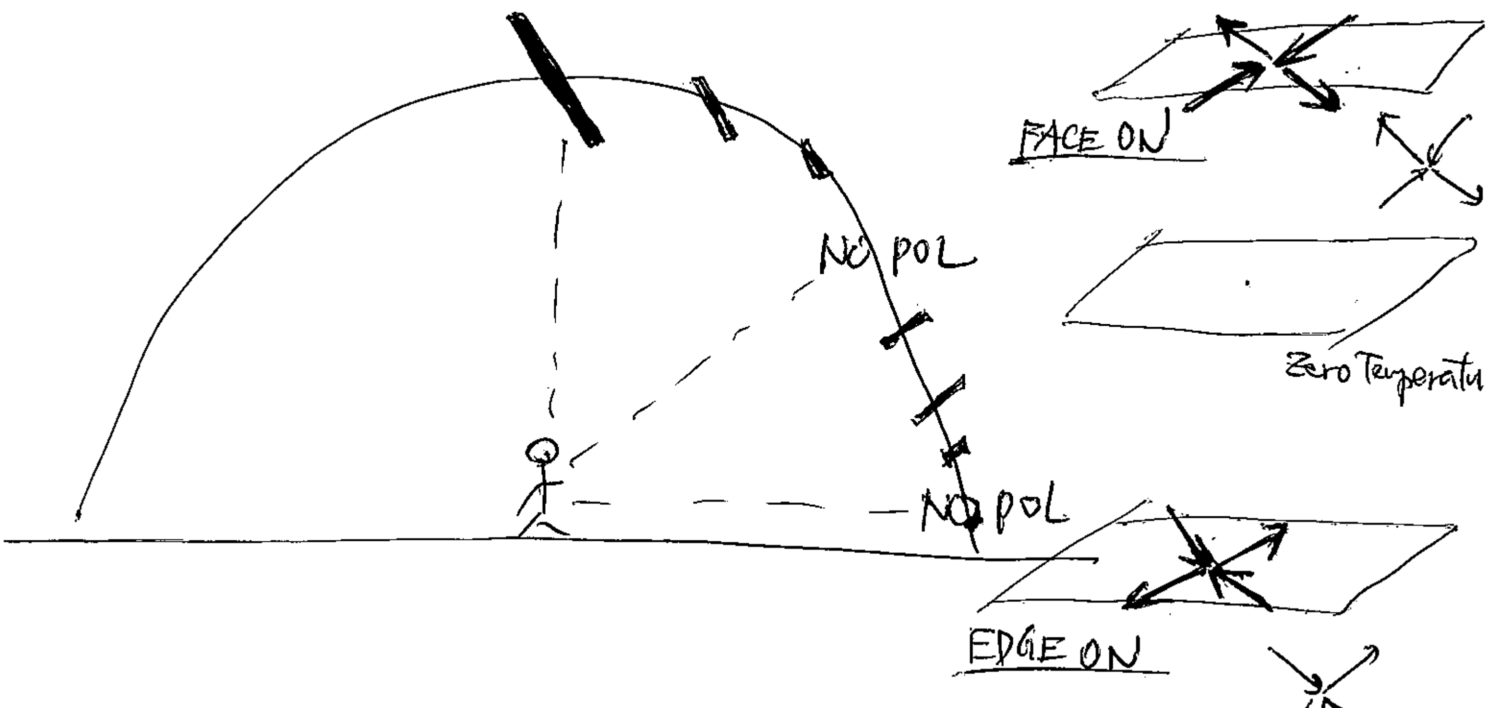


Therefore, on the sky, it looks:

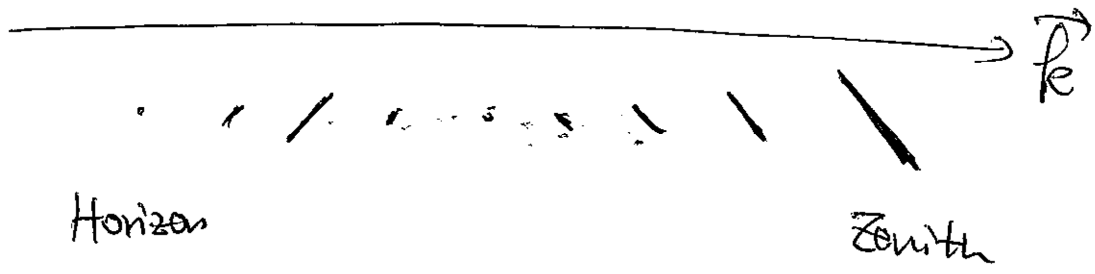


This is E mode.

For "X" mode (1/2 wavelength from Horizon to zenith)



Therefore, on the sky, it looks:



This is B mode.

Because there is no polarization on the horizon for B modes, the amplitude of B mode power spectrum is smaller than that of E modes!

$\therefore C_\ell^{BB} < C_\ell^{EE}$.

Pedagogical Review : astro-ph/0206039
Executive Summary : 0902.4759

Non-Gaussianity Lecture

E. Komatsu, U Texas

Q1. Can we test the hypothesis that primordial fluctuations were generated by quantum fluctuations during inflation?

Q2. If so, can we observationally determine which model describes the physics of inflation correctly?

For a long time, the two-point correlation functions of the observed cosmological fluctuations such as:

- CMB
- galaxies
- weak lensing
- Inter Galactic Medium (IGM) gas

have been used to test both of these (Q1 & Q2.)

What have we learned?

2

(also, a perturbation in the 3d Ricci tensor, $R^{(3)}$, is $\boxed{\delta R^{(3)} = -4\nabla^2\Phi}$)

Let's consider a perturbation in spatial curvature, Φ.

(For the Schwarzschild geometry, $\Phi = +\frac{GM}{R}$.
The Newtonian potential, Ψ_N, is $\Psi_N = -\frac{GM}{R} = -\Phi$)

☆ On large scales, CMB temperature anisotropy at a give location on the sky, $\frac{\Delta T}{T}(\hat{n})$, is

$$\boxed{\frac{\Delta T}{T}(\hat{n}) = -\frac{1}{3}\Phi(\hat{n}r_*, r_*)}$$

where r_* is the distance to the photon decoupling epoch, or the last scattering surface. $z_* = 1090$.
($\therefore r_* \sim 14$ Gpc.)

☆ On small scales, (smaller than the horizon size) the matter density fluctuation at a given location in space, $\delta(\vec{x}) \equiv \frac{\delta\rho}{\bar{\rho}}(\vec{x})$, is given by the Poisson equation:
← mean mass density

$$\boxed{-\nabla^2\Phi(\vec{x}) = 4\pi G\bar{\rho}\,\delta(\vec{x})\,a^2}$$

Therefore, by analyzing $\delta T/T$ or δ, we can learn statistical properties of Φ.

Now, the two-point correlation function:

$$\langle \Phi(\vec{x})\Phi(\vec{y}) \rangle$$

depends only on the separation $|\vec{r}| \equiv |\vec{x}-\vec{y}|$ when the Universe is statistically homogeneous & isotropic. So,

$$\xi(r) \equiv \langle \Phi(\vec{x})\Phi(\vec{x}+\vec{r}) \rangle$$

It is often more convenient to work in Fourier space,

$$\tilde{\Phi}(\vec{k}) = \int d^3\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\, \Phi(\vec{x}).$$

Then, the two-point function of $\tilde{\Phi}(\vec{k})$ is

$$\langle \tilde{\Phi}(\vec{k})\tilde{\Phi}^*(\vec{k}') \rangle = (2\pi)^3 \delta^D(\vec{k}-\vec{k}')\, P_\Phi(k)$$

where $P(k)$ is the "power spectrum" and is related to $\xi(r)$ as:

$$\xi(r) = \int \frac{dk}{k} \times \frac{k^3 P_\Phi(k)}{2\pi^2} \times \frac{\sin(kr)}{kr} \quad \left(\Leftarrow \int \frac{d^3\vec{k}}{(2\pi)^3} P_\Phi(k)\, e^{i\vec{k}\cdot\vec{r}} \right).$$

For a good approximation,

$$\xi(r) \sim \left.\frac{k^3 P_\Phi(k)}{2\pi^2}\right|_{k\sim 1/r}.$$

The <u>scale-invariant spectrum</u> is defined as

$$k^3 P_\Phi(k) = \text{constant}$$

(independent of k).

$$\boxed{\therefore \; P_\Phi(k) = \frac{A}{k^3}}$$

where A is a constant.

Inflation predicts :

$$P_\Phi(k) = \frac{A}{k^{4-n_s}}$$

where $n_s \sim 1$. (typically, not exactly one.)

WMAP 5-year result shows $\boxed{n_s = 0.960 \pm 0.013}$

Komatsu et al. (2009)
ApJS, 180, 330
[0803.0547]

Inflation likes this result !!

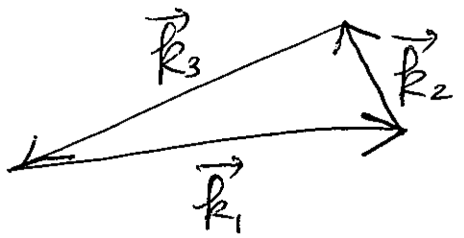
How about a 3-point function?

$$\zeta(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \langle \Phi(\vec{x}_1)\Phi(\vec{x}_2)\Phi(\vec{x}_3) \rangle$$

or, we can define the "bispectrum," $B_\Phi(k_1, k_2, k_3)$, as

$$\langle \tilde{\Phi}(\vec{k}_1)\tilde{\Phi}(\vec{k}_2)\tilde{\Phi}(\vec{k}_3) \rangle = B_\Phi(k_1, k_2, k_3) \times (2\pi)^3 \underline{\delta^D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)}.$$

The delta function ensures that we have a closed triangle:



What does inflation predict for $B_\Phi(k_1, k_2, k_3)$?

The form & magnitude of $B_\Phi(k_1, k_2, k_3)$ is highly model-dependent

⇒ Therefore, the bispectrum is an extremely powerful probe of the physics of inflation!!

Modeling 3pt function : f_{NL} parameter

Let's assume a local form of non-linear function, expanded into a quadratic form :

$$\Phi(\vec{x}) = \Phi_L(\vec{x}) + f_{NL}\left[\Phi_L^2(\vec{x}) - \langle \Phi_L^2 \rangle\right].$$

Gaussian field.

(Komatsu & Spergel 2001, PRD 63, 063002) [astroph/0005036]

※ The physical motivation, meaning, & usefulness of this form will be explained later in this lecture — that's the main goal of this lecture!!

The f_{NL} term, which is non-linear in Φ_L and therefore non-Gaussian, can generate non-Gaussianity in CMB, galaxy distribution, etc.

The bispectrum ?

$$\boxed{B_\Phi(k_1, k_2, k_3) = (2 f_{NL})\left[P_\Phi(k_1)P_\Phi(k_2) + P_\Phi(k_2)P_\Phi(k_3) + P_\Phi(k_3)P_\Phi(k_1)\right]}$$

$f_{NL} \sim 40 \pm 20$ (1σ error) Smith et al. (2009) [0901.2572]

→ implications for inflation?

Let's understand the local form bispectrum better.

For a scale-invariant spectrum, $P_\Phi(k) = A/k^3$,

$$B_\Phi(k_1, k_2, k_3) = (2 f_{NL}) A^2 \left(\frac{1}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_3^3} + \frac{1}{k_3^3 k_1^3} \right)$$

Let's order k_i as $k_1 \geq k_2 \geq k_3$.

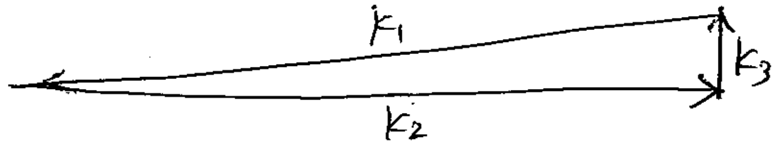
So, for a <u>given k_1</u>, $B_\Phi(k_1, k_2, k_3)$ is maximized when k_3 is the smallest.

In $k_3 \to 0$ limit, $k_2 \to k_1$. Thus,

$$B_\Phi(k_1, \overset{\to k_1}{k_2}, k_3 \to 0)$$

$$\to (2 f_{NL}) A^2 \times 2 \frac{1}{k_1^3 k_3^3} = \underline{\underline{(4 f_{NL}) P_\Phi(k_1) P_\Phi(k_3)}}$$

This configuration is the "squeezed-limit" triangle:



$\underline{\therefore k_3 \ll k_1 \sim k_2}$

So, "f_{NL} non-Gaussianity" is maximized in the squeezed limit. Babich et al. (2004) JCAP 0408:009 [astroph/0405356]

What does inflation predict?

in the squeezed limit?

★ Before we go in there, let's write down a relation between Φ that we measure from CMB, and the primordial curvature perturbation, ζ, during inflation.

That is, the perturbation to the 3d Ricci tensor is

$$\delta R^{(3)} = -4\nabla^2\Phi \quad \text{(during matter era)} \ (z<3000)$$

$$\delta R^{(3)} = -4\nabla^2\zeta \quad \text{(during inflation era)}$$

and, the relation is (Kodama & Sasaki 1984, Prog. Theor. Phys. Suppl. 78, 1)

$$\boxed{\Phi = \frac{3}{5}\zeta}$$

Therefore, f_{NL} formula becomes

$$\Phi = \Phi_L + f_{NL}\Phi_L^2$$

$$\Downarrow$$

$$\zeta = \zeta_L + \frac{3}{5}f_{NL}\zeta_L^2$$

$$\boxed{\therefore B_\zeta(k_1,k_2,k_3) = \left(\frac{6}{5}f_{NL}\right)\left[P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_2)P_\zeta(k_3) + P_\zeta(k_3)P_\zeta(k_1)\right]}$$

$\boxed{\text{What does inflation predict?}}$

CASE. I SINGLE-FIELD INFLATION

☆ There is a **beautiful** theorem, first found by:

- Maldacena 2003, JHEP 05, 013 [astro-ph/0210603]

and later generalized by:

- Creminelli & Zaldarriaga, JCAP, 10, 006 (2004) [astro-ph/0407059]
- Seery & Lidsey, JCAP, 06, 003 (2005) [astro-ph/0503692]
- Cheung et al., JCAP, 02, 021 (2008) [0709.0295]

saying: "For a general SINGLE-FIELD model*, irrespective of the form of potential, kinetic term, or vacuum state, $B_\zeta(k_1, k_2, k_3)$ in the squeezed limit is ALWAYS given by:

$$\boxed{B_\zeta(k_1, k_2 \to k_1, k_3 \to 0) = (1-n_s) P_\zeta(k_1) P_\zeta(k_3)}$$

"

(*) Here, single-field models refer to the models in which the single field is solely responsible for driving inflation, AND generating observed fluctuations.

So, let's compare the two :

★ f_{NL} form in the squeezed limit :

$$B_\zeta \to \left(\frac{12}{5} f_{NL}\right) P_\zeta(k_1) P_\zeta(k_3)$$

★ Any single-field models in the same limit =

$$B_\zeta \to (1-n_s) P_\zeta(k_1) P_\zeta(k_3)$$

Therefore, ANY single-field models predict

$$\boxed{f_{NL} = \frac{5}{12}(1-n_s) = 0.017 \quad (\text{for } n_s = 0.96)}$$

In other words, if $f_{NL} \sim 40$ (± 20) is confirmed by the Planck sattelite (which is expected to reach $\Delta f_{NL} \sim 5$ (1σ)), ALL inflation models will be ruled out convincingly !!

→ Breakthrough in Cosmology.

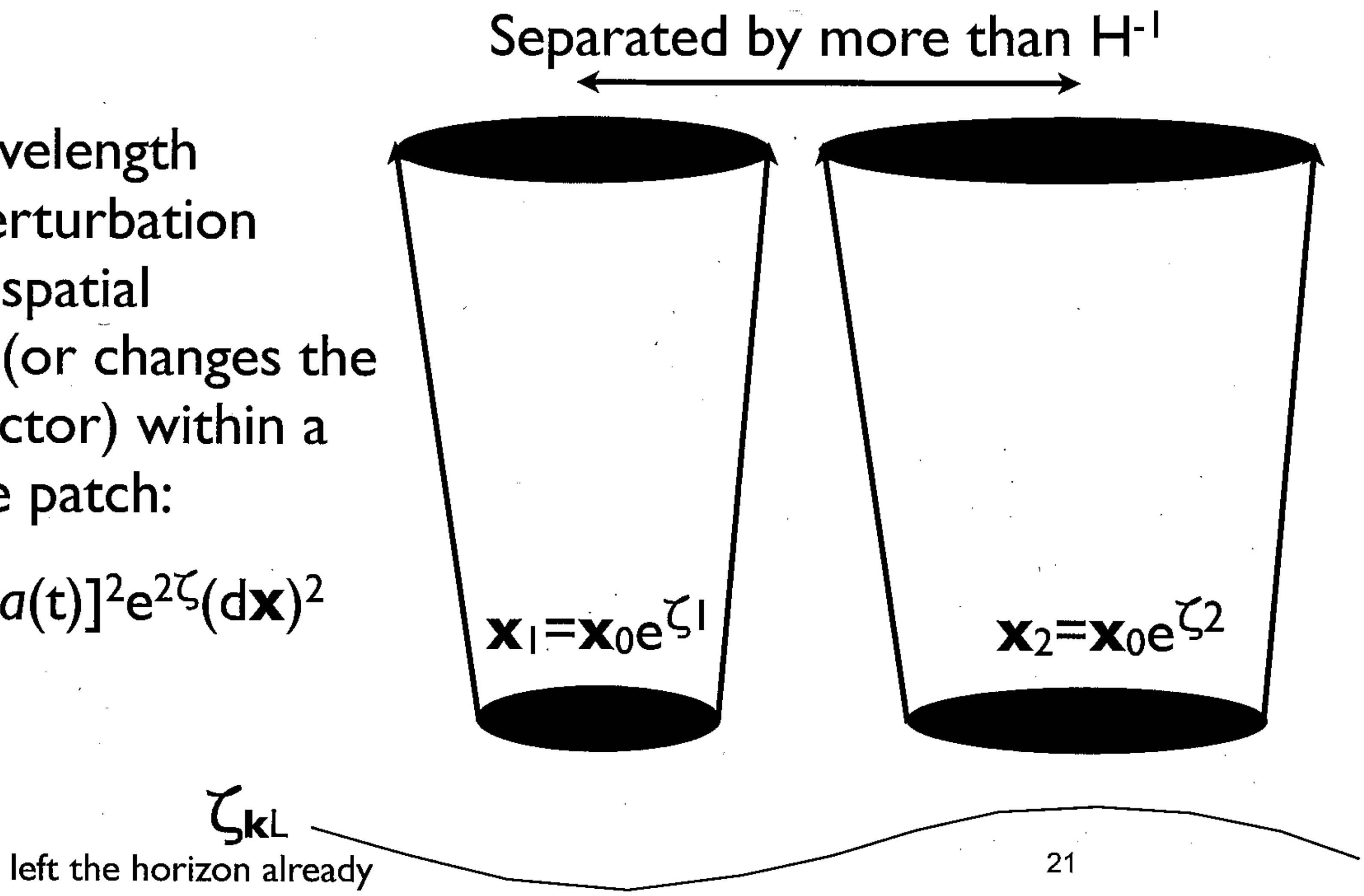
Understanding the Theorem

- First, the squeezed triangle correlates one very long-wavelength mode, k_L ($=k_3$), to two shorter wavelength modes, k_S ($=k_1 \approx k_2$):
    - $\langle \zeta_{\mathbf{k}1} \zeta_{\mathbf{k}2} \zeta_{\mathbf{k}3} \rangle \approx \langle (\zeta_{\mathbf{k}S})^2 \zeta_{\mathbf{k}L} \rangle$
- Then, the question is: "why should $(\zeta_{\mathbf{k}S})^2$ ever care about $\zeta_{\mathbf{k}L}$?"
    - The theorem says, "it doesn't care, if $\zeta_{\mathbf{k}}$ is exactly scale invariant."

=

$\zeta_{\mathbf{k}L}$ rescales coordinates

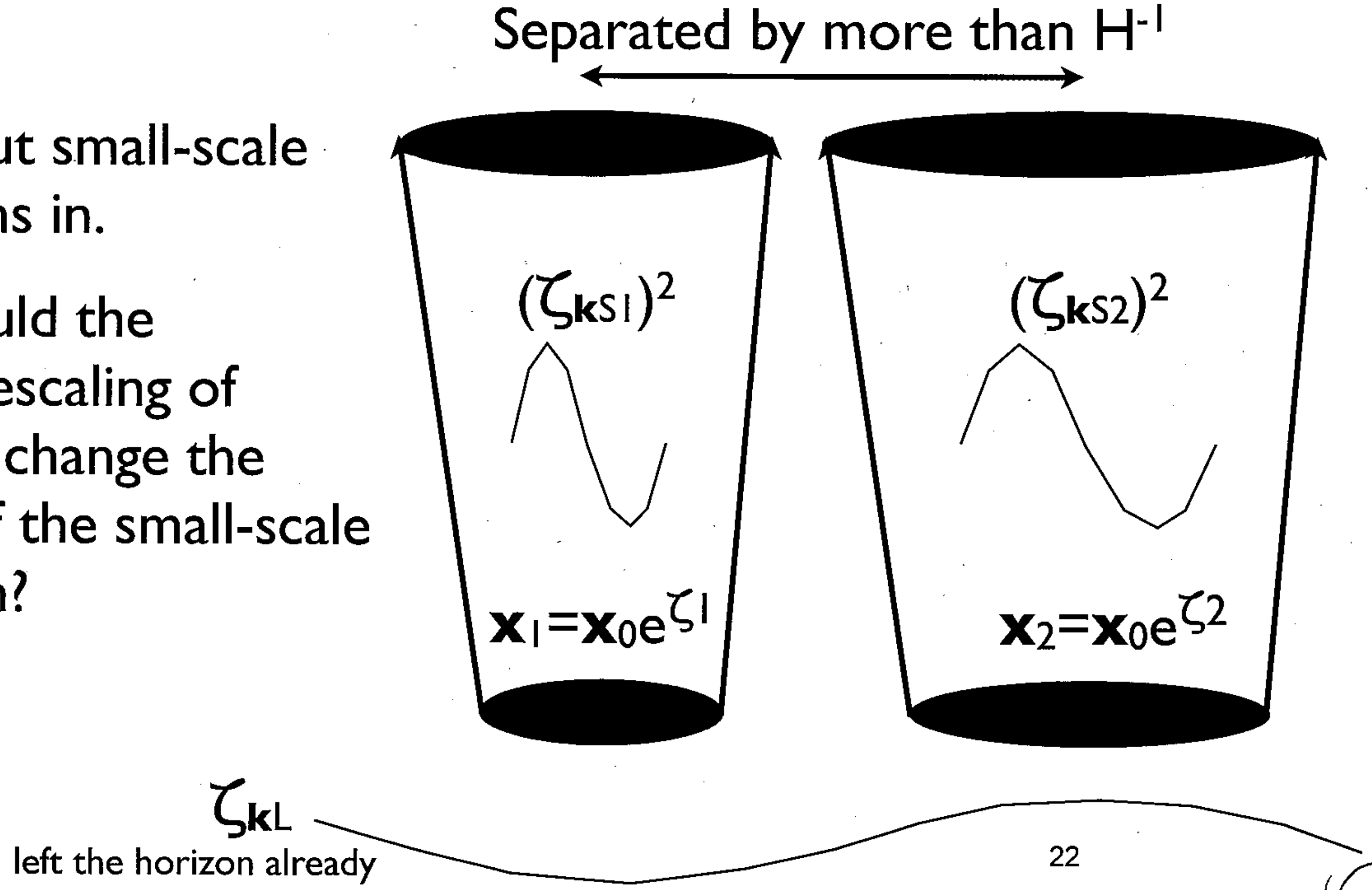
- The long-wavelength curvature perturbation rescales the spatial coordinates (or changes the expansion factor) within a given Hubble patch:
  - $ds^2 = -dt^2 + [a(t)]^2 e^{2\zeta} (d\mathbf{x})^2$



12

$\zeta_{\mathbf{k}L}$ rescales coordinates

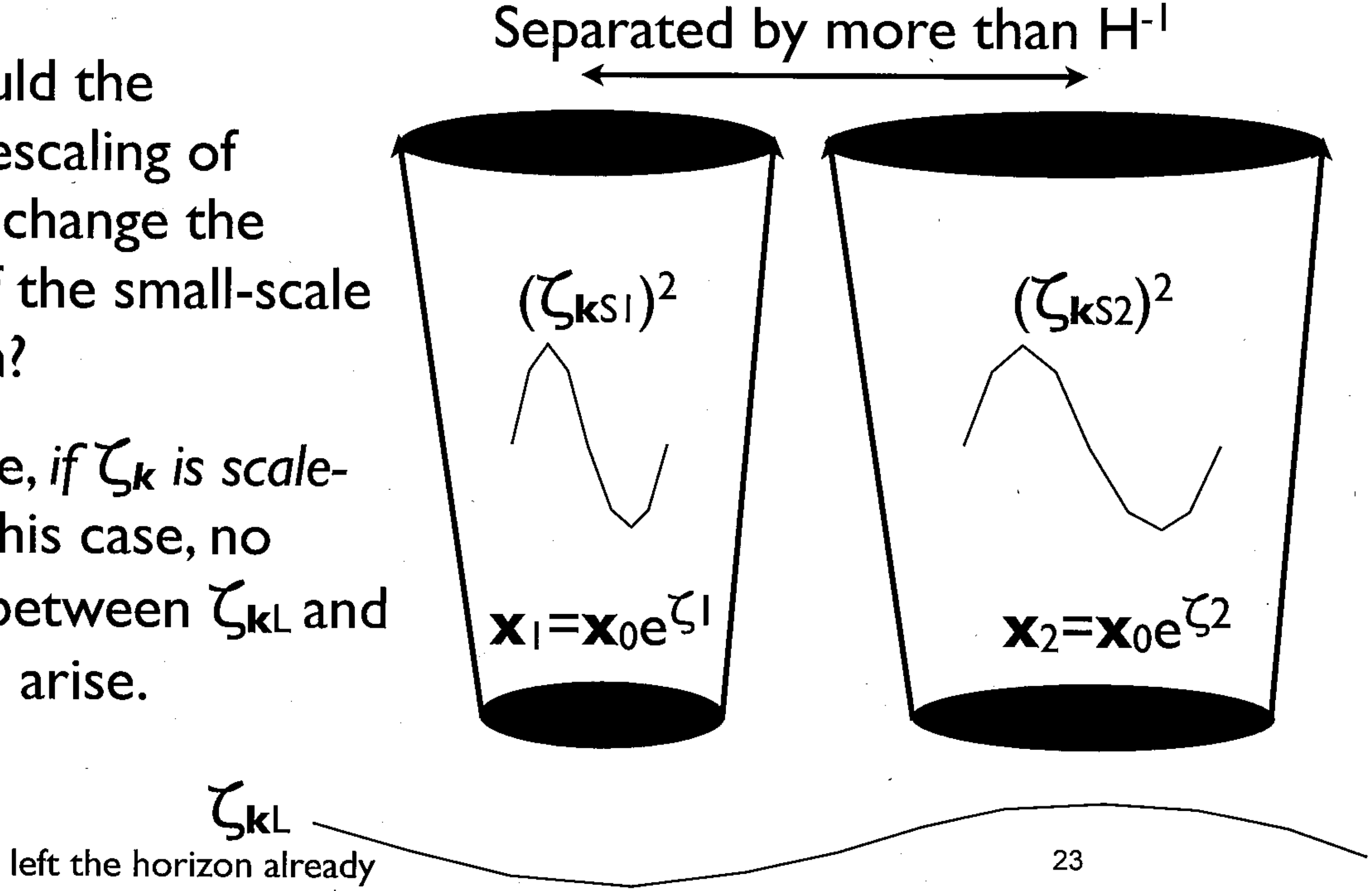
- Now, let's put small-scale perturbations in.
- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?



13

$\zeta_{\mathbf{k}L}$ rescales coordinates

- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?
- A. No change, *if* $\zeta_{\mathbf{k}}$ *is scale-invariant*. In this case, no correlation between $\zeta_{\mathbf{k}L}$ and $(\zeta_{\mathbf{k}S})^2$ would arise.

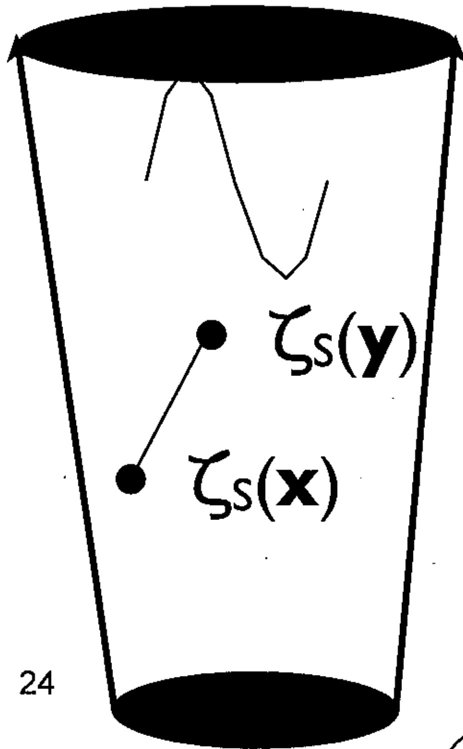


14

Creminelli & Zaldarriaga (2004); Cheung et al. (2008)

Real-space Proof

- The 2-point correlation function of short-wavelength modes, $\xi = \langle \zeta_S(\mathbf{x})\zeta_S(\mathbf{y}) \rangle$, within a given Hubble patch can be written in terms of its vacuum expectation value (in the absence of ζ_L), ξ_0, as:
  - $\xi_{\zeta_L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L \, [d\xi_0(|\mathbf{x}-\mathbf{y}|)/d\zeta_L]$
  - $\xi_{\zeta_L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L \, [d\xi_0(|\mathbf{x}-\mathbf{y}|)/d\ln|\mathbf{x}-\mathbf{y}|]$
  - $\xi_{\zeta_L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L \, (1-n_s)\xi_0(|\mathbf{x}-\mathbf{y}|)$



$$\text{3-pt func.} = \langle (\zeta_S)^2 \zeta_L \rangle = \langle \xi_{\zeta_L} \zeta_L \rangle = (1-n_s)\xi_0(|\mathbf{x}-\mathbf{y}|)\langle \zeta_L^2 \rangle$$

(5)

Where was "Single-field"?

- Where did we assume "single-field" in the proof?
- For this proof to work, it is crucial that there is only one dynamical degree of freedom, i.e., it is only ζ_L that modifies the amplitude of short-wavelength modes, and nothing else modifies it.
- Also, ζ must be constant outside of the horizon (otherwise anything can happen afterwards). This is also the case for single-field inflation models.

Therefore...

- A convincing detection of $f_{NL} > 1$ would rule out ***all*** of the single-field inflation models, regardless of:
  - the form of potential
  - the form of kinetic term (or sound speed)
  - the initial vacuum state
- A convincing detection of f_{NL} would be a breakthrough.

61

Large Non-Gaussianity from Single-field Inflation

- $S=(1/2)\int d^4x \sqrt{-g}\,[R-(\partial_\mu\varphi)^2-2V(\varphi)]$
- 2nd-order (which gives P_ζ)
  - $S_2=\int d^4x\, \varepsilon\, [a^3(\partial_t\zeta)^2-a(\partial_i\zeta)^2]$
- 3rd-order (which gives B_ζ)
  - $S_3=\int d^4x\, \boldsymbol{\varepsilon^2}\, [\ldots a^3(\partial_t\zeta)^2\zeta+\ldots a(\partial_i\zeta)^2\zeta+\ldots a^3(\partial_t\zeta)^3] + O(\varepsilon^3)$

Cubic-order interactions are suppressed by an additional factor of ε.
(Maldacena 2003)

Large Non-Gaussianity from Single-field Inflation

- $S=(1/2)\int d^4x \sqrt{-g}\,\{R-2P[(\partial_\mu\varphi)^2,\varphi]\}$ [general kinetic term]
- 2nd-order
  - $S_2=\int d^4x\, \varepsilon\, [a^3(\partial_t\zeta)^2/c_s^2-a(\partial_i\zeta)^2]$

"Speed of sound"
$c_s^2=P_{,X}/(P_{,X}+2XP_{,XX})$

- 3rd-order
  - $S_3=\int d^4x\, \boldsymbol{\varepsilon^2}\, [\ldots a^3(\partial_t\zeta)^2\zeta/\mathbf{c_s^2} + \ldots a(\partial_i\zeta)^2\zeta + \ldots a^3(\partial_t\zeta)^3/\mathbf{c_s^2}] + O(\varepsilon^3)$

Some interactions are enhanced for $c_s^2<1$.

(Seery & Lidsey 2005; Chen et al. 2007)

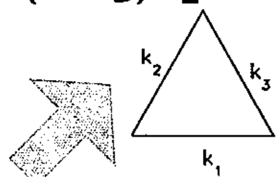
19

Large Non-Gaussianity from Single-field Inflation

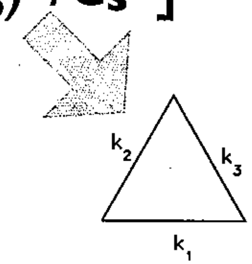
- $S=(1/2)\int d^4x\,\sqrt{-g}\,\{R-2P[(\partial_\mu\varphi)^2,\varphi]\}$ [general kinetic term]
- 2nd-order
  - $S_2=\int d^4x\,\varepsilon\,[a^3(\partial_t\zeta)^2/c_s^2-a(\partial_i\zeta)^2]$

"Speed of sound"
$c_s^2=P_{,X}/(P_{,X}+2XP_{,XX})$

- 3rd-order



  - $S_3=\int d^4x\,\boldsymbol{\varepsilon^2}\,[\ldots a^3(\partial_t\zeta)^2\zeta/\mathbf{c_s^2}+\ldots a(\partial_i\zeta)^2\zeta+\ldots a^3(\partial_t\zeta)^3/\mathbf{c_s^2}]+O(\varepsilon^3)$

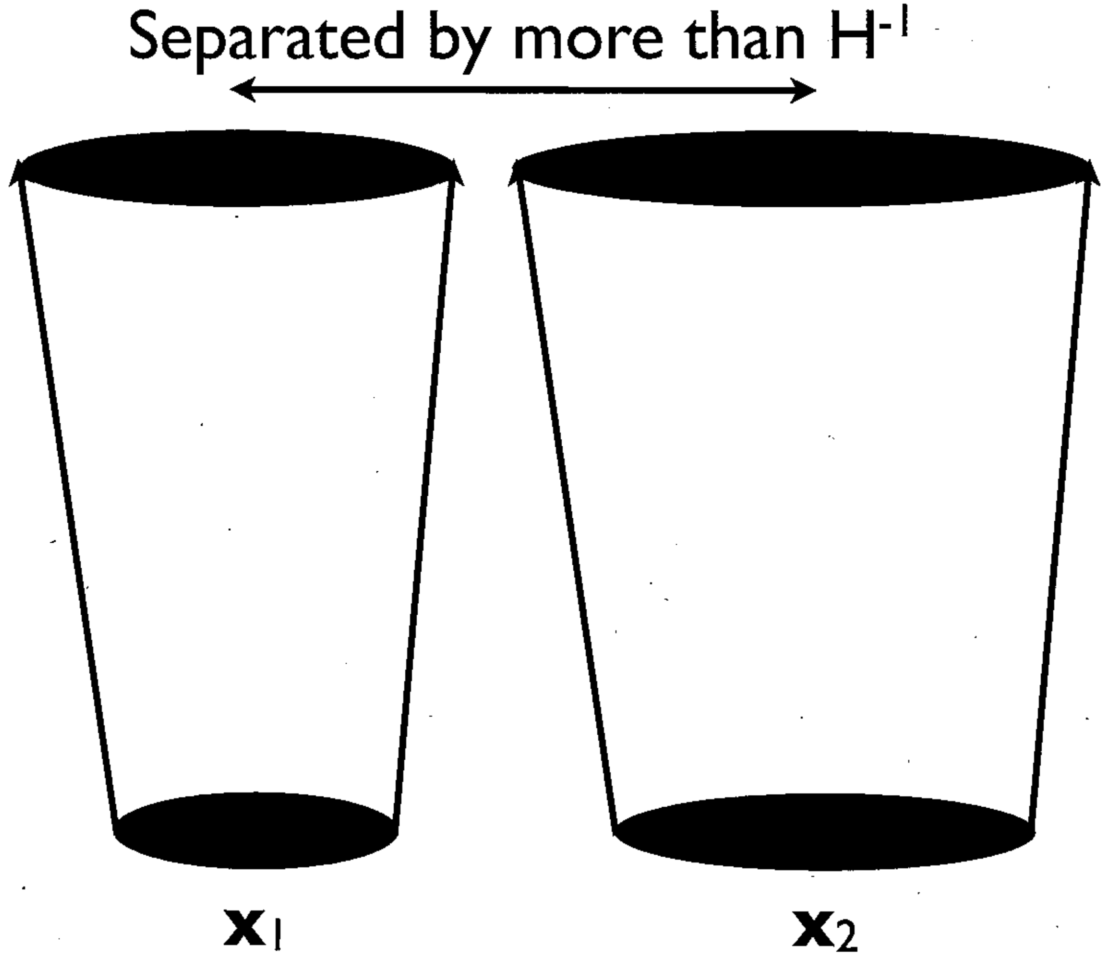


Some interactions are enhanced for $c_s^2<1$.
(Seery & Lidsey 2005; Chen et al. 2007)

20

Another Motivation For f_{NL}

- In multi-field inflation models, ζ_k can evolve outside the horizon.
- This evolution can give rise to non-Gaussianity; however, causality demands that the form of non-Gaussianity must be local!



$$\zeta(\mathbf{x})=\zeta_g(\mathbf{x})+(3/5)f_{NL}[\zeta_g(\mathbf{x})]^2+A\chi_g(\mathbf{x})+B[\chi_g(\mathbf{x})]^2+\ldots$$

(12)

Multi-field Case

the locality demands that $\zeta(\vec{x})$ should be given by

$$\zeta(\vec{x}) = F\left[\delta\varphi_1(\vec{x}),\ \delta\varphi_2(\vec{x}),\ \cdots,\ \delta\varphi_N(\vec{x})\right]$$

If F is a smooth function [see 0903.3407 for non-smooth case],

we can expand this form and obtain :

$$\zeta(\vec{x}) = \frac{\partial F}{\partial\varphi_1}\delta\varphi_1(\vec{x}) + \frac{1}{2}\frac{\partial^2 F}{\partial\varphi_1^2}\delta\varphi_1^2(\vec{x}) + \cdots$$

$$+ \frac{\partial F}{\partial\varphi_2}\delta\varphi_2(\vec{x}) + \frac{1}{2}\frac{\partial^2 F}{\partial\varphi_2^2}\delta\varphi_2^2(\vec{x}) + \cdots$$

$$+ \cdots$$

$$+ \frac{\partial F}{\partial\varphi_N}\delta\varphi_N(\vec{x}) + \frac{1}{2}\frac{\partial^2 F}{\partial\varphi_N^2}\delta\varphi_N^2(\vec{x}) + \cdots$$

$\boxed{\text{What determines } F\ ??}$

δN Formalism

Salopek & Bond, PRD, 42, 3936 (1990)
Sasaki & Stewart, Prog. Theor. Phys. 95, 71 (1996) [astro-ph/9507001]
Lyth, Malik & Sasaki, JCAP, 05, 004 (2005) [astro-ph/0411220]

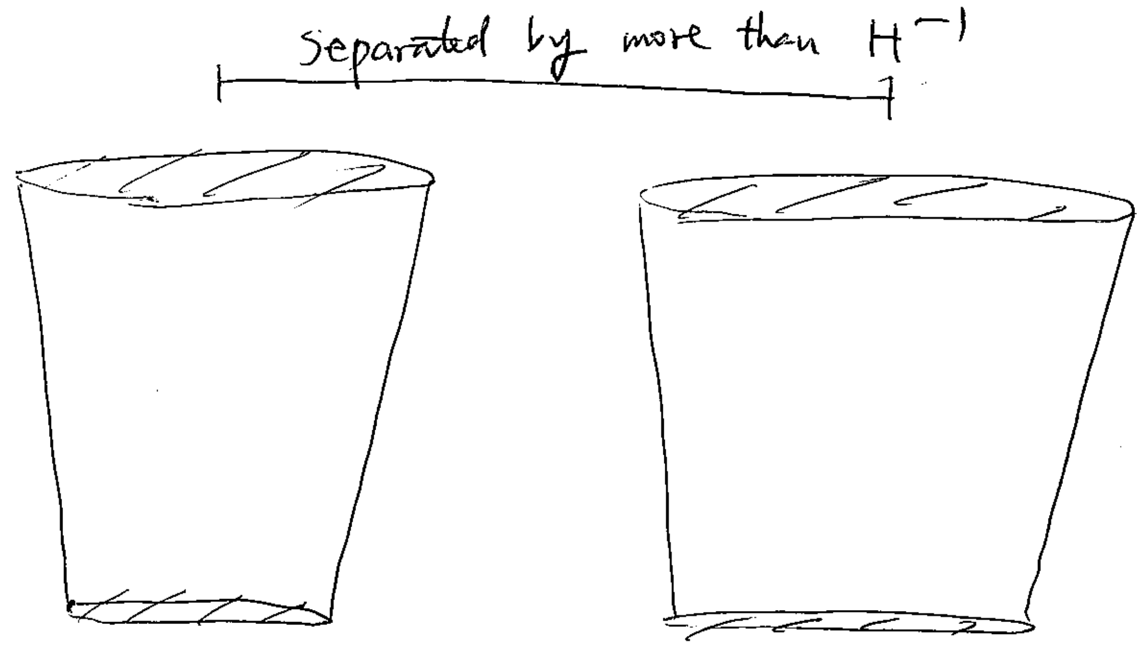
have shown :

$$F = N \quad (\equiv \text{\# of e-foldings of expansion,}$$

$$= \int_{t_{\text{horizon-crossing}}}^{t_{\text{today}}} H\,dt$$

$$= \ln\left[\frac{a(t_{\text{today}})}{a(t_{\text{horizon-crossing}})}\right]$$

<u>**Why so?**</u> Let's go back to this picture :



$$ds^2 = -dt^2 + a^2(t)\, e^{2\zeta(\vec{x})} (d\vec{x})^2 .$$

$$= -dt^2 + [\tilde{a}(\vec{x},t)]^2 (d\vec{x})^2$$

where $\underline{\tilde{a}(\vec{x},t) = a(t)\, e^{\zeta(\vec{x})}}$

Therefore, we can interpret $\boxed{\ln \tilde{a}}$ as the curvature perturbation :

$$\boxed{\ln \tilde{a} = \zeta + \ln a(t)}$$

Then, only thing we have to care about is "How much has each horizon patch expanded relative to the others?"

※ Single-field example

For single-field,

$$N = \int_{H.C.}^{NOW} H\, dt = \int_{\phi_{H.C.}}^{\phi_{NOW}} H \frac{dt}{d\phi}\, d\phi$$

$$= \int_{\bar{\phi}_{H.C.} + \delta\phi_{H.C.}}^{\bar{\phi}_{NOW}} \frac{H}{\dot{\phi}}\, d\phi$$

$$\simeq \underbrace{\int_{\bar{\phi}_{H.C.}}^{\bar{\phi}_{NOW}} \frac{H}{\dot{\phi}}\, d\phi}_{\bar{N}} + \left.\frac{H}{\dot{\phi}}\right|_{\phi=\bar{\phi}_{H.C.}} \delta\phi_{H.C.}$$

$$\therefore\ \delta N = \left.\frac{H}{\dot{\phi}}\right|_{\phi=\bar{\phi}_{H.C.}} \delta\phi_{H.C.} \ \Rightarrow\ \zeta = \left.\frac{H}{\dot{\phi}}\right|_{\phi=\bar{\phi}_{H.C.}} \delta\phi_{H.C.}$$

⇒ This is the famous result for ζ, obtained by

(Guth & Pi, PRL, 49, 1110 (1982)
Hawking, PLB, 115, 295 (1982)
Starobinsky, PLB, 117, 175 (1982)
Bardeen, Steinhardt & Turner, PRD, 28, 679 (1983)

※ Multi-field generalization

Lyth & Rodriguez, PRL, 95, 121302 (2005)
[astro-ph/0504045]

$$\zeta = N(\overline{\varphi_1}+\delta\varphi_1, \overline{\varphi_2}+\delta\varphi_2, + \cdots, \overline{\varphi_N}+\delta\varphi_N) - N(\overline{\varphi_1}, \overline{\varphi_2}, \cdots, \overline{\varphi_N})$$

$$\simeq \sum_i \frac{\partial N}{\partial \varphi_i}\delta\varphi_i + \frac{1}{2}\sum_{ij}\frac{\partial^2 N}{\partial\varphi_i \partial\varphi_j}\delta\varphi_i \delta\varphi_j + \cdots$$

Now, let's remind us of the fact that:

$$\langle \delta\varphi(\vec{k})\,\delta\varphi^*(\vec{k}')\rangle = (2\pi)^3 \delta(\vec{k}-\vec{k}')\, P_{\delta\varphi}(k)$$

where

$$P_{\delta\varphi}(k) = \left(\frac{H}{2\pi}\right)^2 \frac{2\pi^2}{k^3}$$

for a scale-invariant spectrum.

So - - -

For uncorrelated φ_i, i.e., $\langle \delta\varphi_i \delta\varphi_j \rangle \propto \delta_{ij}$,

$$P_\zeta(k) = \left(\frac{H}{2\pi}\right)^2 \frac{2\pi^2}{k^3}\left[\sum_i \left(\frac{\partial N}{\partial \varphi_i}\right)^2\right] + \cdots$$

and the bispectrum is :

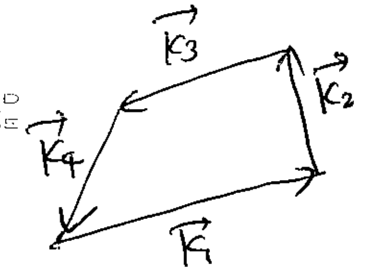
$$B_\zeta(k_1,k_2,k_3) = \left(\frac{H}{2\pi}\right)^4 \left[\sum_{ij}\left(\frac{\partial N}{\partial\varphi_i}\right)\left(\frac{\partial N}{\partial\varphi_j}\right)\left(\frac{\partial^2 N}{\partial\varphi_i\partial\varphi_j}\right)\right]$$
$$\times (2\pi^2)^2\left[\frac{1}{k_1^3k_2^3}+\frac{1}{k_2^3k_3^3}+\frac{1}{k_3^3k_1^3}\right]$$
$$+ \cdots$$

Therefore, REMARKABLY, we recover the local form bispectrum :

$$B_\zeta(k_1,k_2,k_3) = \frac{\sum_{ij}\left(\frac{\partial N}{\partial\varphi_i}\right)\left(\frac{\partial N}{\partial\varphi_j}\right)\left(\frac{\partial^2 N}{\partial\varphi_i\partial\varphi_j}\right)}{\left[\sum_i (\partial N/\partial\varphi_i)^2\right]^2}\left[P_\zeta(k_1)P_\zeta(k_2) + \text{cyclic}\right]$$

$$= \frac{6}{5} f_{NL}$$

$f_{NL} \gg 1$ is possible for many models !!



Further reading ---

What about 4-pt function ?

It's finally hot now !!

- Observational prospects :

CMB • Kogo & Komatsu 2006 PRD, 73, 083007 [astro-ph/0602099]

galaxies • Jeong & Komatsu 2009 [0904.0497]

- 4pt function from single field
  - Seery, Lidsey & Sloth 2007, JCAP 01, 027 [astro-ph/0610210]
  - Chen, Huang & Shiu 2006 [hep-th/0610235]
  - Arroja & Koyama 2008, PRD 77, 083517 [0802.1167]
  - Arroja, Mizuno, Koyama & Tanaka 2009 [0905.3641]
  - Chen, Hu, Huang, Shiu & Wang 2009 [0905.3494]

- 4pt function from multi field
  - Seery & Lidsey 2007, JCAP, 01, 008 [astroph/0611034]
  - Byrnes, Sasaki & Wands 2006, PRD, 74, 123519 [astroph/0611075]
  - Boubekeur & Lyth 2006 PRD 73, 021301 [astro-ph/0504046]

Trispectrum : Local Form

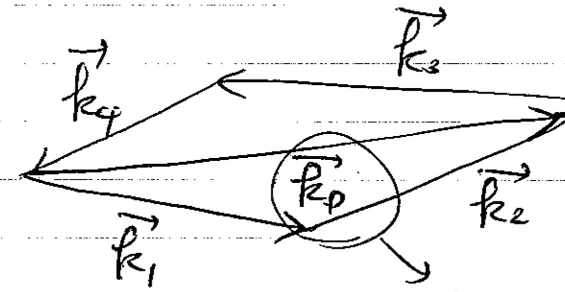
For $\Phi(\vec{x}) = \Phi_L(\vec{x}) + f_{NL}\Phi_L^2(\vec{x}) + g_{NL}\Phi_L^3(\vec{x})$, or

$$\left(\zeta(\vec{x}) = \zeta_L(\vec{x}) + \frac{3}{5} f_{NL}\zeta_L^2(\vec{x}) + \frac{9}{25} g_{NL}\zeta_L^3(\vec{x}) \right)$$

$$\langle \Phi(\vec{k}_1)\Phi(\vec{k}_2)\Phi(\vec{k}_3)\Phi(\vec{k}_4) \rangle$$

$$\left.\begin{aligned} = &\langle \Phi(\vec{k}_1)\Phi(\vec{k}_2)\rangle\langle\Phi(\vec{k}_3)\Phi(\vec{k}_4)\rangle \\ + &\langle \Phi(\vec{k}_1)\Phi(\vec{k}_3)\rangle\langle\Phi(\vec{k}_2)\Phi(\vec{k}_4)\rangle \\ + &\langle \Phi(\vec{k}_1)\Phi(\vec{k}_4)\rangle\langle\Phi(\vec{k}_2)\Phi(\vec{k}_3)\rangle \end{aligned}\right\} \text{"un-connected" terms.}$$

$$+ (2\pi)^3 \delta^D(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\, \underbrace{T_\Phi(k_1, k_2, k_3, k_4; k_D)}_{\text{"connected" trispectrum}}$$



Diagonal

$\therefore\ T_\Phi(k_1, k_2, k_3, k_4; k_D)$

$$= 6 g_{NL}\left[P_\Phi(k_1)P_\Phi(k_2)P_\Phi(k_3) + (3 \text{ cyclic terms})\right]$$

$$+ \frac{25}{18}\tau_{NL}\left[P_\Phi(k_1)P_\Phi(k_2)\left\{P_\Phi(|\vec{k}_1+\vec{k}_3|) + P_\Phi(|\vec{k}_1+\vec{k}_4|)\right\} + (11 \text{ cyclic terms})\right]$$

$$= \frac{36}{25} f_{NL}^2 \times \frac{25}{18}$$

$$= \underline{\underline{2 f_{NL}^2}}$$

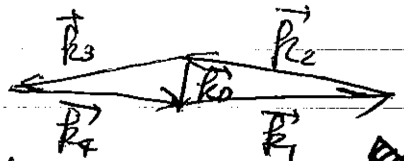
or

$$T_\zeta(k_1, k_2, k_3, k_4; k_D)$$

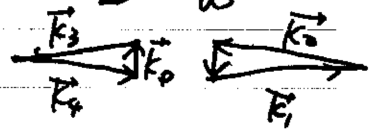
$$= \frac{54}{25} g_{NL} \left[P_\zeta(k_1) P_\zeta(k_2) P_\zeta(k_3) + (3 \text{ cyclic terms}) \right]$$

$$+ \underbrace{\tau_{NL}}_{= \frac{36}{25} f_{NL}^2} \left[P_\zeta(k_1) P_\zeta(k_2) \left\{ P_\zeta(|\vec{k}_1 + \vec{k}_3|) + P_\zeta(|\vec{k}_1 + \vec{k}_4|) \right\} + (11 \text{ cyclic terms}) \right].$$

The "folded" limit, $k_D \to 0$.



can be understood as...



☆ The single-field inflation gives (Seery et al. 2009)

$$T_\zeta(k_1, k_2, k_3, k_4; k_D \to 0)$$

$$= (1-n_s)^2 P_\zeta(k_D) P_\zeta(k_1) P_\zeta(k_3)$$

[NB: this formula ignored contributions from gravitational waves.]

☆ The local form gives

$$T_\zeta(k_1, k_2, k_3, k_4; k_D \to 0)$$

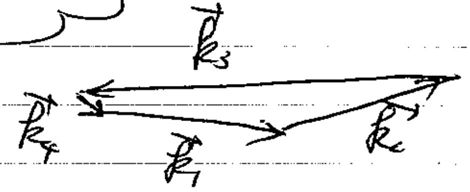
$$= 4 \tau_{NL} P_\zeta(k_D) P_\zeta(k_1) P_\zeta(k_3)$$

$$= \frac{144}{25} f_{NL}^2 P_\zeta(k_D) P_\zeta(k_1) P_\zeta(k_3)$$

$$= \left(\frac{12}{5} f_{NL}\right)^2 P_\zeta(k_D) P_\zeta(k_1) P_\zeta(k_3).$$

$\boxed{\therefore\ f_{NL} = \frac{5}{12}(1-n_s) \text{ is also satisfied for the trispectrum.}}$

The "squeezed" limit, $k_4 \to 0$.



☆ The single-field inflation gives (Seery et al. 2007)

$$T_\zeta(k_1, k_2, k_3, k_4 \to 0 \,;\, k_D)$$
$$= -P_\zeta(k_4)\frac{d}{d\ln a} B_\zeta(k_2, k_3, k_1)$$

☆ The local form gives

$$T_\zeta(k_1, k_2, k_3, k_4 \to 0 \,;\, k_D)$$
$$= \left(2\tau_{NL} + \frac{54}{25} g_{NL}\right) P_\zeta(k_4)$$
$$\times \left[P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3) \right],$$

$$= \frac{2\tau_{NL} + \frac{54}{25} g_{NL}}{\frac{6}{5} f_{NL}} P_\zeta(k_4)\, B_\zeta(k_1, k_2, k_3).$$

For $g_{NL} = 0$,

$$= \frac{12}{5} f_{NL}\, P_\zeta(k_4) B_\zeta(k_1, k_2, k_3)$$

$$= \underline{(1-n_s)}\, P_\zeta(k_4)\, B_\zeta(k_1, k_2, k_3).$$