Exercise sheet 6

Exercise 6 - 1

The numbers quantifying the degree of industrialization $\tilde{\iota}$ of a society, its fertility rate $\tilde{f}$, and the stork population $\tilde{s}$ on its territory are random variables assumed to belong to a joint three-dimensional Gaussian distribution. Consider the fluctuations around the respective mean values $\iota = \tilde{\iota} - \langle\, \tilde{\iota}\, \rangle_{(\tilde{\iota})}$, $f = \tilde{f} - \langle\, \tilde{f}\, \rangle_{(\tilde{f})}$, and $s = \tilde{s} - \langle\, \tilde{s}\, \rangle_{(\tilde{s})}$. It is known that both the stork index s and the fertility index f are anticorrelated with the degree of industrialization. The normalized correlation coefficients are $c_{s\iota} = -0.85$ and $c_{f\iota} = -0.70$, where

$$c_{ab} = \frac{\langle ab\rangle_{(a,b)}}{\sqrt{\langle aa\rangle_{(a)}\, \langle bb\rangle_{(b)}}}. \tag{1}$$

Assume further that there is no direct correlation between f and s, i.e., $\mathcal{P}(s|f,\iota) = \mathcal{P}(s|\iota)$ and $\mathcal{P}(f|s,\iota) = \mathcal{P}(f|\iota)$. Derive an expression for $\langle sf\rangle_{(s,f,\iota)}$. Use this to calculate the normalized correlation coefficient c_{sf} (2 points).

Exercise 6 - 2

Given a field $s : \mathcal{S}^2 \to \mathbb{C}$ on the two-dimensional sphere, assume that it is statistically homogeneous and isotropic, i.e., $S(\hat{n}, \hat{n}') = \langle s(\hat{n})s^*(\hat{n}')\rangle = S(\hat{n}\cdot\hat{n}')$, where $\hat{n}$ and $\hat{n}'$ are unit vectors that give directions or, equivalently, points on $\mathcal{S}^2$. Prove that the covariance matrix S is diagonal in the basis given by the spherical harmonic functions and its entries are independent of m, i.e.,

$$S_{(\ell m)(\ell' m')} := \langle s_{\ell m} s^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell. \tag{2}$$

(3 points)
<u>Hint</u>: Use the following properties of the spherical harmonic functions $Y_{\ell m}$ and the Legendre polynomials P_ℓ:

$$s(\hat{n}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} s_{\ell m}Y_{\ell m}(\hat{n}), \qquad s_{\ell m} = \int_{\mathcal{S}^2} \mathrm{d}\Omega\; s(\hat{n})Y^*_{\ell m}(\hat{n}) \tag{3}$$

$$\int_{\mathcal{S}^2} \mathrm{d}\Omega\; Y^*_{\ell m}(\hat{n})Y_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'} \tag{4}$$

$$P_\ell(\hat{n}\cdot\hat{n}') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{n})Y_{\ell m}(\hat{n}') \tag{5}$$

Exercise 6 - 3

Imaging devices can probe continuous fields, such as physical flux or matter densities, by pixel averaged measurements. Assume the value d_i in the ith pixel of the obtained image satisfies

$$d_i = \int_\Omega \mathrm{d}x\; R_i(x)s(x) + n_i \qquad \text{with} \qquad i \in I = \{1,\ldots,u\}, \tag{6}$$

where $R : I\times\Omega \to \mathbb{R}$ denotes the instrument response function, $s : \Omega \to \mathbb{R}$ the signal field, and $n \in \mathbb{R}^u$ the noise vector. Both, signal and noise, are a priori assumed to be independent of each other and to follow Gaussian distributions,

$$s \curvearrowleft \mathcal{G}(s-t, S) \qquad \text{and} \qquad n \curvearrowleft \mathcal{G}(n-r, N), \tag{7}$$

with known non-zero means, $t : \Omega \to \mathbb{R}$ and $r \in \mathbb{R}^u$, as well as known covariances, $S : \Omega\times\Omega \to \mathbb{R}$ and $N \in \mathcal{M}_{u\times u}(\mathbb{R})$.

a) Derive an expression for the likelihood, $P(d|s)$. Which quantity needs to be marginalized over? (1 point)

b) Compute the full information Hamiltonian (including constant terms); i.e., $H(d, s) = -\log P(d, s)$. Identify the information propagator, D and the information source, j. (2 points)

c) Derive an expression for the posterior mean field, $m = \langle s \rangle_{(s|d)}$, in terms of the given image, d. To do so use the maximum a posterior Ansatz. (2 points)

d) Say the field $\widetilde{m}$ was inferred from a modified data set $\widetilde{d}$ applying a Wiener filter that solely uses S, R, and N. Given the following relation between the image data d and $\widetilde{d}$,

$$\widetilde{d}_i = d_i - \int_\Omega \mathrm{d}x \; R_i(x) t(x) - r_i; \tag{8}$$

Find the relation between the posterior mean field m derived in **c)** and the field $\widetilde{m}$. (2 points)

This exercise sheet will be discussed during the exercises.
Group 01, Wednesday 18:00 - 20:00, Theresienstr. 37, A 449,
Group 02, Thursday, 10:00 - 12:00, Theresienstr. 37, A 249,

https://wwwmpa.mpa-garching.mpg.de/ ensslin/lectures/lectures.html