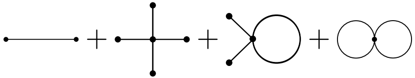
12.6 Diagrammatic Expectation Values

$$\mathcal{H}(\phi) = \frac{1}{2}\phi^\dagger D^{-1}\phi - j^\dagger\phi + \frac{1}{4!}\lambda\phi^4$$

Logarithm of partition function:

$\ln \mathcal{Z}(j) = \ln \mathcal{Z}_{\mathcal{G}}(0) +$



$+ \mathcal{O}(\lambda^2)$

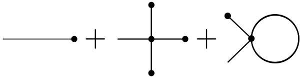
$$\Rightarrow \ln \mathcal{Z}(j) = \ln \mathcal{Z}_{\mathcal{G}}(0) + \frac{1}{2}j^\dagger D j - \frac{\lambda^\dagger}{4!}(Dj)^4 - \frac{1}{4}\lambda^\dagger (Dj)^2\widehat{D} - \frac{1}{8}\lambda^\dagger\widehat{D}^2 + \mathcal{O}(\lambda^2)$$

12.6 Diagrammatic Expectation Values

$$\ln \mathcal{Z}(j) = \ln \mathcal{Z}_{\mathcal{G}}(0) + \frac{1}{2} j^\dagger D j - \frac{\lambda^\dagger}{4!}(Dj)^4 - \frac{1}{4}\lambda^\dagger (Dj)^2 \widehat{D} - \frac{1}{8}\lambda^\dagger \widehat{D}^2 + \mathcal{O}(\lambda^2)$$

Expectation value:

$$\begin{aligned}
\langle \phi \rangle &= \frac{\delta \ln \mathcal{Z}}{\delta j} \\
&= D\, j - \frac{\lambda}{3!} D (Dj)^3 - \frac{1}{2} D \lambda (Dj) \widehat{D} + \mathcal{O}(\lambda^2) \\
\langle \phi^x \rangle &= D^{xy} j_y - \frac{\lambda}{3!} \int dy\, D^{yx} (D^{yz} j_z)^3 - \frac{\lambda}{2} \int dy\, D^{yx} (D^{yz} j_z) D^{yy} + \mathcal{O}(\lambda^2)
\end{aligned}$$

$\langle \phi^x \rangle =$  $+ \mathcal{O}(\lambda^2)$

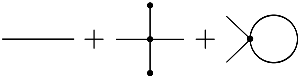
12.6 Diagrammatic Expectation Values

$$\ln \mathcal{Z}(j) = \ln \mathcal{Z}_{\mathcal{G}}(0) + \frac{1}{2} j^\dagger D j - \frac{\lambda^\dagger}{4!}(Dj)^4 - \frac{1}{4}\lambda^\dagger (Dj)^2 \widehat{D} - \frac{1}{8}\lambda^\dagger \widehat{D}^2 + \mathcal{O}(\lambda^2)$$

Covariance:

$$\begin{aligned} \langle \phi \phi^\dagger \rangle^c &= \frac{\delta \ln \mathcal{Z}}{\delta j \delta j^\dagger} \\ &= D - \frac{\lambda}{2} D (Dj)^2 D - \frac{\lambda}{2} D (\widehat{D}) D + 0 + \mathcal{O}(\lambda^2) \\ \langle \phi^x \phi^y \rangle^c &= D^{xy} - \frac{\lambda}{2} \int dz\, D^{zx} (D^{zu} j_u)^2 D^{zy} - \frac{\lambda}{2} \int dz\, D^{xz} D^{zz} D^{zy} + \mathcal{O}(\lambda^2) \end{aligned}$$

$\langle \phi^x \phi^y \rangle^c =$



$+ \mathcal{O}(\lambda^2)$

12.7 Log-Normal Poisson Model Diagrammatically

Log-normal Poisson model *(repetition)*

$$
\begin{aligned}
\mathcal{H}(d,\, s) &\;\widehat{=}\; \frac{1}{2} s^\dagger S^{-1} s - d^\dagger s + \kappa^\dagger e^s \\
&\;\widehat{=}\; \frac{1}{2} s_x \left((S^{-1})^{xy} + \kappa \delta^{xy} \right) s^y - (d_x - \kappa_x) s^x + \kappa_x \sum_{n=3}^{\infty} \frac{1}{n!} (s^n)^x \\
&\;\widehat{=}\; \frac{1}{2} s^\dagger (S^{-1} + \widehat{\kappa}) s - j^\dagger s + \kappa^\dagger \sum_{n=3}^{\infty} \frac{1}{n!} s^n \\
&\;\widehat{=}\; \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + \sum_{n=3}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n\, \kappa_{x_1}\, \delta(x_1 - x_2) \ldots \delta(x_1 - x_n)\, s^{x_1} \ldots s^{x_n} \\
&\;\widehat{=}\; \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + \sum_{n=0}^{\infty} \frac{1}{n!} \Lambda^{(n)}_{x_1 \ldots x_n} s^{x_1} \ldots s^{x_n}
\end{aligned}
$$

12.7 Log-Normal Poisson Model Diagrammatically

$$\mathcal{H}(d, s) \;\widehat{=}\; \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + \kappa^\dagger \sum_{n=3}^{\infty} \frac{1}{n!} s^n$$

$$\textbf{MAP:}\; \left.\frac{\delta H}{\delta s}\right|_{s=m} \overset{!}{=} 0$$

$$= D^{-1} s - j + \kappa \sum_{n=3}^{\infty} \left.\frac{s^{n-1}}{(n-1)!}\right|_{s=m}$$

$$\Rightarrow m = D\left(j - \kappa \sum_{n=2}^{\infty} \frac{m^n}{n!}\right)$$

Iteration:

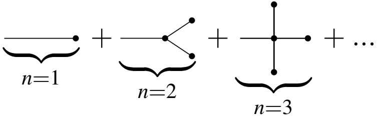
- $i = 0$: $m_0 = 0$
- $i = 1$:

$m_1 = Dj =$ 

12.7 Log-Normal Poisson Model Diagrammatically

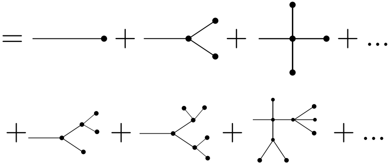
- $i = 2$:

$$m_2 = D\left(j - \kappa \sum_{n=2}^{\infty} \frac{(Dj)^n}{n!}\right) =$$



- $i = 3$

$$m_3 = D\left(j - \kappa \sum_{n=2}^{\infty} \frac{(m_2)^n}{n!}\right) =$$

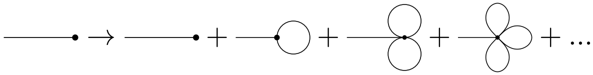


⇒The classical/ MAP estimate m_∞ is always given by the sum of all tree diagrams with one external point.

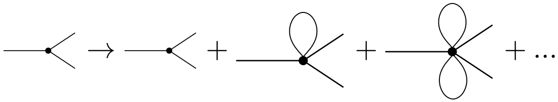
12.7.1 Consideration of uncertainty loops

$$\langle s \rangle_{(s|d)} \quad = \underbrace{\sum \text{tree diagrams}}_{\text{MAP}} + \underbrace{\sum \text{loop diagrams}}_{\text{uncertainty corrections}}$$

- source loops:

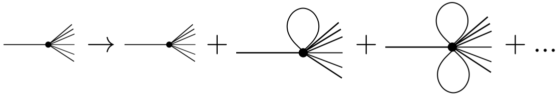


- 3-vertex:



12.7.1 Consideration of uncertainty loops

- n-vertex:



$$\Rightarrow -\kappa_x \quad \to \quad -\left[\kappa_x + \frac{1}{2}\kappa_x D^{xx} + \frac{1}{8}\kappa_x \left(D^{xx}\right)^2 + \ldots + \frac{1}{n!2^n}\kappa_x \left(\widehat{D}^x\right)^n\right] \text{ no summation}$$

$$-\kappa \quad \to \quad -\kappa\, e^{\widehat{D}/2}$$

12.7.1 Consideration of uncertainty loops

Loop-normalized MAP:

$$\begin{aligned} m &= S(d - \kappa e^{m}) \\ &= S(d - \kappa_m) \end{aligned}$$

$\kappa_m \to \kappa_m e^{\widehat{D}/2} = \kappa_{m+\widehat{D}/2}$:

loop normalized solution:

$$\begin{aligned} m &= S\left(d - \kappa_{m+\widehat{D}/2}\right) \\ D &= \left(S^{-1} + \widehat{\kappa}_{m+\widehat{D}/2}\right) \end{aligned}$$

13 Thermodynamical Inference

Tempered posterior: $T = \frac{1}{\beta}$

$$\mathcal{P}(s|d,\, T,\, J) = \frac{e^{-\beta(\mathcal{H}(d,s)+J^\dagger s)}}{\mathcal{Z}(d,\, \beta,\, J)} = \frac{\left(\mathcal{P}(d,\, s)e^{-J^\dagger s}\right)^\beta}{\underbrace{\int \mathcal{D}s \left(\mathcal{P}(d,\, s)e^{-J^\dagger s}\right)^\beta}_{=\mathcal{Z}(d,\beta,J)}}$$

- $T = \beta = 1$: usual inference
- $T \to 0$, $\beta \to \infty$: enlarged contrast $\Rightarrow \mathcal{P}(s|d,\, T) \to \delta(s - s_{\mathrm{MAP}})$
- $T \to \infty$, $\beta \to 0$: weaker contrast $\Rightarrow \mathcal{P}(s|d,\, T) \to \mathrm{const}$

Boltzmann Entropy

$$
\begin{aligned}
\mathcal{P}(s|d,\,T,\,J) &= \frac{e^{-\beta(\mathcal{H}(d,s)+J^\dagger s)}}{\mathcal{Z}(d,\,\beta,\,J)} \\
S_B &= -\int \mathcal{D}s\,\mathcal{P}(s|d,\,T,\,J)\,\ln\left(\frac{\mathcal{P}(s|d,\,T,\,J)}{\text{const}}\right) \\
\Delta S_B &= \int \mathcal{D}s\,\mathcal{P}(s|d,\,T,\,J)\,\left[\beta\left(\mathcal{H}(d,\,s)+J^\dagger s\right)+\ln\mathcal{Z}(d,\,\beta,\,J)\right] \\
&= \beta\left[\langle\mathcal{H}(d,\,s)\rangle_{(s|d,T,J)}+J^\dagger\langle s\rangle_{(s|d,T,J)}+\frac{1}{\beta}\ln\mathcal{Z}(d,\,\beta,\,J)\right] \\
&= \beta\left[U(d,\,T,\,J)+J^\dagger m(d,\,T,J)-F(d,\,\beta,\,J)\right]
\end{aligned}
$$

Boltzmann Entropy

$$T\Delta S_B = U(d,\, T,\, J) + J^{\dagger} m(d,\, T, J) - F(d,\, \beta,\, J)$$

- internal energy: $U(d,\, T,\, J) = \langle \mathcal{H}(d,\, s) \rangle_{(s|d,\, T,\, J)}$
- Helmholtz free energy: $F(d,\, \beta,\, J) = -\frac{1}{\beta} \ln \mathcal{Z}(d,\, \beta,\, J)$
- mean field:

$$m(d,\, 1,\, J) = \langle s \rangle_{(s|d,\, 1,\, \mathrm{J})} = -\frac{\partial}{\partial J} \ln \mathcal{Z}(d,\, \beta = 1,\, J)\Big|_{J=0} = \frac{\partial F(d, \beta, J)}{\partial J}\Big|_{J=0,\, \beta=1}$$

13. Thermodynamical Inference

Ansatz:

$$
\begin{aligned}
\mathcal{P}(s|d,\, T,\, J) &\approx \widetilde{\mathcal{P}}\,(s|m,\, D) = \mathcal{G}(s-m,\, D) \\
T\Delta\, \widetilde{S}_B\,(d,\, T,\, J) &= \widetilde{U}\,(d,\, T,\, J) + J^{\dagger} m(d,\, T,\, J) - \widetilde{F}\,(d,\, \beta,\, J) \\
\widetilde{U}\,(d,\, T,\, J) &= \langle \mathcal{H}(d,\, s) \rangle_{\mathcal{G}(s-m,\, D)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \widetilde{S}_B\,(d,\, T,\, J) &= -\langle \ln \widetilde{\mathcal{P}} \rangle_{\widetilde{\mathcal{P}}} \\
&= +\int \mathcal{D}s\, \mathcal{G}(s-m,\, D) \left[\frac{1}{2}(s-m)^{\dagger} D^{-1} (s-m) + \frac{1}{2} \ln |2\pi D| \right] \\
&= \frac{1}{2} \left[\int \mathcal{D}\varphi \left(\mathcal{G}(\varphi,\, D)\, \mathrm{Tr}(\varphi \varphi^{\dagger} D^{-1}) \right) + \ln |2\pi D| \right]
\end{aligned}
$$

13. Thermodynamic Inference

$$
\begin{aligned}
\Rightarrow \widetilde{S_B}\,(d,\,T,\,J) &= \frac{1}{2}\left[\int \mathcal{D}\varphi \left(\mathcal{G}(\varphi,\,D)\,\mathrm{Tr}(\varphi\varphi^\dagger D^{-1})\right) + \ln|2\pi D|\right] \\
&= \frac{1}{2}\mathrm{Tr}\left(\langle\varphi\varphi^\dagger\rangle_{\mathcal{G}(\varphi,\,D)} D^{-1}\right) + \frac{1}{2}\ln|2\pi D| \\
&= \frac{1}{2}\mathrm{Tr}(D D^{-1}) + \frac{1}{2}\mathrm{Tr}(\ln(2\pi D)) \\
&= \frac{1}{2}\mathrm{Tr}(\mathbb{1} + \ln(2\pi D)) \\
&= \widetilde{S_B}\,(D)
\end{aligned}
$$

$$
\Rightarrow \widetilde{F}\,(d,\,\beta,\,J) = \widetilde{U}\,(m_J,\,D_J) - \widetilde{T}\widetilde{S_B}\,(D_J) + J^\dagger m_J
$$

Legendre Transformation

$$F(J) = F(J_0) + \frac{\partial F}{\partial J}\bigg|_{J_0}^{\dagger} (J - J_0) + \ldots$$

$$G = F(J_0) - \frac{\partial F}{\partial J}\bigg|_{J_0}^{\dagger} J_0$$

F convex $\Rightarrow m_J = \frac{\partial F}{\partial J} \Rightarrow F$ can be reconstructed from $G(m)$

Gibbs Free Energy

$$
\begin{aligned}
G &= F - \frac{\partial F}{\partial J}^{\dagger} J \\
&= U - TS_B + J^{\dagger} m - J^{\dagger} m \\
\Rightarrow \widetilde{G}\left(d,\, \beta,\, m,\, D\right) &= \widetilde{U}\left(d,\, \beta,\, m,\, D\right) - T\, \widetilde{S}_B\left(D\right)
\end{aligned}
$$

Mean Field From Minimal Gibbs Free Energy:

$$\frac{\delta G(d,\, m,\, D)}{\delta m} = 0 \quad \Rightarrow \quad m = \langle s \rangle_{(s|d)} \Big|_{T=1}$$

Proof:

$$\begin{aligned}
\frac{\delta G}{\delta m} &= \frac{\delta}{\delta m}\Big(F(d,\, J(m)) - J^\dagger(m) m\Big) \\
&= \frac{\delta J(m)}{\delta m}^\dagger \frac{\delta F(d,\, J)}{\delta J} - \frac{\delta J^\dagger}{\delta m} m - J \\
&= \frac{\delta J(m)}{\delta m}^\dagger m - \frac{\delta J^\dagger}{\delta m} m - J \\
&= -J \stackrel{!}{=} 0
\end{aligned}$$

$$J = 0 \quad \Rightarrow \quad m = \frac{\partial F}{\partial J}\Big|_{J=0} = \langle s \rangle_{(s|d)}$$

Uncertainty Dispersion From Minimal Gibbs Free Energy:

$$\left(\frac{\delta^2 G}{\delta m \delta m^\dagger}\right)^{-1}\Bigg|_{m=\langle s\rangle_{(s|d)}} = \frac{-\delta^2 F}{\delta J \delta J^\dagger}\Bigg|_{J=0} = \beta D$$

Proof:

$$\begin{aligned}
\left(\frac{\delta^2 G}{\delta m \delta m^\dagger}\right)^{-1}\Bigg|_{m=\langle s\rangle_{(s|d)}} &= \left(-\frac{\delta J}{\delta m}\right)^{-1}\Bigg|_{m=\langle s\rangle_{(s|d)}} \\
&= -\left(\frac{\delta m(J)}{\delta J}\right)\Bigg|_{J=0} \\
&= -\frac{\delta^2 F(J)}{\delta J \delta J^\dagger}\Bigg|_{J=0} \\
&= \frac{1}{\beta}\frac{\delta^2}{\delta J \delta J^\dagger}\ln \mathcal{Z}(d,\, J,\, \beta) \\
&= \beta D
\end{aligned}$$