11.2 Stochastic Calculus

- generalized Wiener process $\frac{ds^t}{dt} = \xi^t$
- colored Gaussian excitation $\Xi^{\omega\omega'} = \left\langle \xi^\omega \overline{\xi^{\omega'}} \right\rangle_{(\xi)} = 2\pi\delta(\omega - \omega')\, P_\xi(\omega)$
- bound power spectrum $\int_{-\infty}^{\infty} d\omega\, P_\xi(\omega) < \infty$

Fourier space:

$$\begin{aligned}
\xi^\omega &= \int_{-\infty}^{\infty} dt\, e^{i\omega t} \xi^t = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \frac{ds^t}{dt} = -\int_{-\infty}^{\infty} dt\, \frac{de^{i\omega t}}{dt}\, s^t = -i\omega \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, s^t \\
\Rightarrow \xi^\omega &= -i\omega\, s^\omega \\
\Rightarrow S^{\omega\omega'} &= \left\langle s^\omega \overline{s^{\omega'}} \right\rangle_{(s)} = \left\langle \frac{\xi^\omega}{-i\omega}\, \frac{\overline{\xi^{\omega'}}}{i\omega} \right\rangle_{(\xi)} = \frac{\Xi^{\omega\omega'}}{\omega^2} = 2\pi\delta(\omega - \omega') \underbrace{\frac{P_\xi(\omega)}{\omega^2}}_{=P_s(\omega)}
\end{aligned}$$

11.2.1 Stratonovich's Calculus

Transformed process: $f^t \equiv f(s^t),\ f : \mathbb{R} \mapsto \mathbb{R}$

$$\begin{aligned} \frac{df^t}{dt} &= \frac{df(s^t)}{ds^t}\frac{ds^t}{dt} = f'(s^t)\,\xi^t \\ f^t &= f(s^p + \int_p^t dt'\xi^{t'}) \end{aligned}$$

Calculation of the drift:

$$\begin{aligned} \langle \Delta f^t \rangle_{(\xi|s^t)} &= \langle f^{t+\Delta t} - f^t \rangle_{(\xi|s^t)} \\ &= \langle f(s^t + \int_t^{t+\Delta t} dt'\,\xi^{t'}) - f(s^t) \rangle_{(\xi|s^t)} \\ &\overset{\text{Taylor}}{=} f'(s^t)\,\langle \Delta s \rangle_{(\xi|s^t)} + \frac{1}{2} f''(s^t)\,\langle (\Delta s)^2 \rangle_{(\xi|s^t)} + \frac{1}{3!} f'''(s^t)\,\langle (\Delta s)^3 \rangle_{(\xi|s^t)} \\ &\quad + \frac{1}{4!} f''''(s^t)\,\langle (\Delta s)^4 \rangle_{(\xi|s^t)} + \mathcal{O}((\Delta s)^5) \end{aligned}$$

11.2.1 Stratonovich's Calculus

Required moments: $\Xi^{t' t''} = \delta(t' - t'')$

$$
\begin{aligned}
\langle \Delta s \rangle_{(\xi|s^t)} &= \int_t^{t+\Delta t} dt' \, \langle \xi^{t'} \rangle_{(\xi|s^t)} = 0 \\
\langle (\Delta s)^2 \rangle_{(\xi|s^t)} &= \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \, \langle \xi^{t'} \xi^{t''} \rangle_{(\xi|s^t)} = \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \Xi^{t' t''} = \Delta t \\
\langle (\Delta s)^3 \rangle_{(\xi|s^t)} &= \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \int_t^{t+\Delta t} dt''' \, \langle \xi^{t'} \xi^{t''} \xi^{t'''} \rangle_{(\xi|s^t)} = 0 \\
\langle (\Delta s)^4 \rangle_{(\xi|s^t)} &= \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \int_t^{t+\Delta t} dt''' \int_t^{t+\Delta t} dt'''' \, \langle \xi^{t'} \xi^{t''} \xi^{t'''} \xi^{t''''} \rangle_{(\xi|s^t)} \\
&= \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \int_t^{t+\Delta t} dt''' \int_t^{t+\Delta t} dt'''' (\Xi^{t' t''} \Xi^{t''' t''''} + \Xi^{t' t'''} \Xi^{t'' t''''} + \Xi^{t' t''''} \Xi^{t'' t'''}) \\
&= 3 \, (\Delta t)^2
\end{aligned}
$$

11.2.1 Stratonovich's Calculus

$$
\begin{aligned}
\Rightarrow \langle \Delta f^t \rangle_{(\xi|s^t)} &= f'(s^t)\, \langle \Delta s \rangle_{(\xi|s^t)} + \frac{1}{2} f''(s^t)\, \langle (\Delta s)^2 \rangle_{(\xi|s^t)} + \frac{1}{3!} f'''(s^t)\, \langle (\Delta s)^3 \rangle_{(\xi|s^t)} \\
&\quad + \frac{1}{4!} f''''(s^t)\, \langle (\Delta s)^4 \rangle_{(\xi|s^t)} + \mathcal{O}((\Delta s)^5) \\
&= \frac{1}{2} f''(s^t)\, \Delta t + \frac{3}{4!} f''''(s^t)\, (\Delta t)^2 + \mathcal{O}\left((\Delta t)^3\right) \\
\Rightarrow \left\langle \frac{df^t}{dt} \right\rangle_{(\xi|s^t)} &= \left\langle \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} \right\rangle_{(\xi|s^t)} = \frac{1}{2} f''(s^t)
\end{aligned}
$$

Itô's Calculus

Transformed process:

$$\begin{aligned} df &= f'(s^t)\, ds^t + \frac{1}{2} f''(s^t)\, dt \\ \frac{df^t}{dt} &= \frac{df(s^t)}{ds^t}\, \frac{ds^t}{dt} + \frac{1}{2} f''(s^t) = f'(s^t)\, \xi_t + \frac{1}{2} f''(s^t) \end{aligned}$$

Calculation of the drift:

$$\langle \frac{df^t}{dt} \rangle_{(\xi|s^t)} = \frac{1}{2} f''(s^t)$$

Itô's vs. Stratonovich's Calculus

Stratonovich picture	**Itô picture**
continuous	discrete
drift arises automatically	no drift without explicit drift term
coloured excitation spectrum	white excitation spectrum

11.3 Linear Stochastic Differential Equations

$$\sum_{n=0}^{N} a_n \frac{d^n s^t}{dt^n} = \xi,\ \mathcal{P}(\xi) = \mathcal{G}(\xi, \Xi)$$

$$\Xi^{\omega\omega'} = 2\pi\delta(\omega - \omega')P_\xi(\omega)$$

Fourier space: $$\int_{-\infty}^{\infty} dt\, e^{i\omega t} \sum_{n=0}^{N} a_n \frac{d^n s^t}{dt^n} = \xi^\omega$$

$$\sum_{n=0}^{N} a_n \int_{-\infty}^{\infty} dt\, e^{i\omega t} \frac{d^n}{dt^n} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega' t}\, s^{\omega'} = \xi^\omega$$

$$\sum_{n=0}^{N} a_n \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} (-i\omega')^n\, s^{\omega'} \int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega')t} = \xi^\omega$$

$$\sum_{n=0}^{N} a_n\, (-i\omega)^n\, s^\omega = \xi^\omega$$

11.3 Linear Stochastic Differential Equations

$$\sum_{n=0}^{N} a_n \left(-i\omega\right)^n s^\omega = \xi^\omega$$

$$\Rightarrow s^\omega = \left[\sum_{n=0}^{N} a_n \left(-i\omega\right)^n\right]^{-1} \xi^\omega$$

$$s = R\,\xi$$

$$\Rightarrow R^\omega_{\omega'} = 2\pi\delta(\omega-\omega')\left[\sum_{n=0}^{N} a_n \left(-i\omega\right)^n\right]^{-1}$$

$$\Rightarrow \mathcal{P}(s|R,\,\Xi) = \mathcal{G}(s,\,S),\ \text{with } S = R\,\Xi\,R^\dagger$$

$$S^{\omega\omega'} = 2\pi\delta(\omega-\omega')P_s(\omega)$$

$$P_s(\omega) = \frac{P_\xi(\omega)}{\left|\sum_{n=0}^{N} a_n \left(-i\omega\right)^n\right|^2} =: P_R(\omega)P_\xi(\omega)$$

11.3.1 Example: Wiener Process

$$\begin{aligned}
\sum_{n=0}^{N} a_n \frac{d^n s^t}{dt^n} &= \xi^\omega \\
P_R(\omega) &= \frac{P_\xi(\omega)}{\left|\sum_{n=0}^{N} a_n (-i\omega)^n\right|^2}
\end{aligned}$$

$$\begin{aligned}
\dot{s}(t) &= \xi^t \\
a_1 &= 1 \\
\Rightarrow P_R(\omega) &= \frac{1}{|a_1^2(-iw)^1|^2} = \frac{1}{\omega^2}
\end{aligned}$$

11.3.2 Example: Ornstein-Uhlenbeck Process

$$\sum_{n=0}^{N} a_n \frac{d^n s^t}{dt^n} = \xi^\omega$$

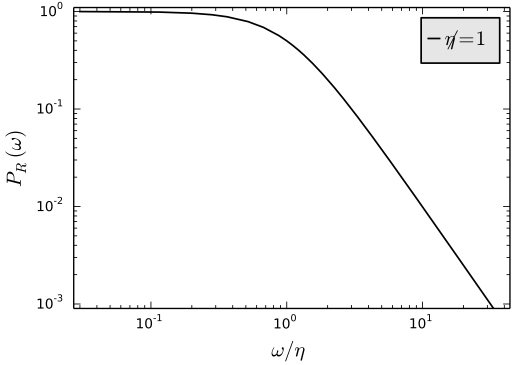
$$P_R(\omega) = \frac{P_\xi(\omega)}{\left|\sum_{n=0}^{N} a_n (-i\omega)^n\right|^2}$$

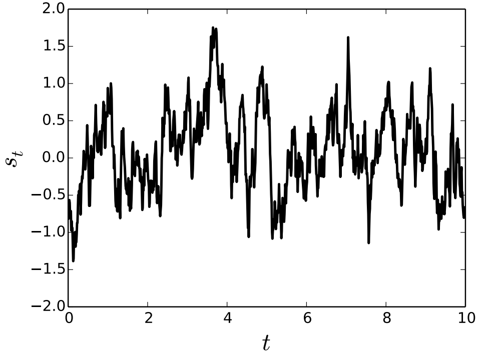
$$\begin{aligned}
\dot{s}_t + \eta\, s^t &= \xi^t \\
a_0 &= \eta \\
a_1 &= 1 \\
\Rightarrow P_R(\omega) &= |\eta - i\omega|^{-2} = (\eta^2 + \omega^2)^{-1}
\end{aligned}$$

White excitation: $P_\xi(\omega) = 1$

$$P_s(\omega) = P_R(\omega) = (\eta^2 + \omega^2)^{-1}$$

11.3.2 Example: Ornstein-Uhlenbeck Process





11.3.3 Example: Harmonic Oscillator

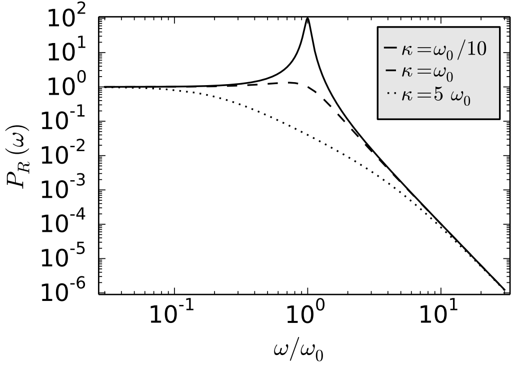
- κ: damping constant
- ω_0: eigenfrequency of the oscillator
- f : noise coupling constant

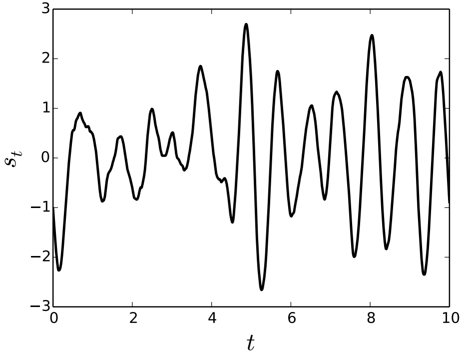
$$\ddot{s}_t + \kappa \dot{s}_t + \omega_0^2 \, s^t \quad = \quad f \, \xi^t$$

$$\begin{aligned} a_0 &= \omega_0^2 f^{-1} \\ a_1 &= \kappa f^{-1} \\ a_2 &= f^{-1} \end{aligned}$$

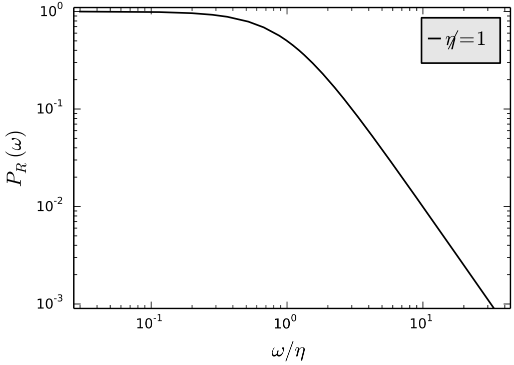
$$\Rightarrow \quad P_R(\omega) \quad = f^2 \, \left[\omega_0^4 + \left(\kappa^2 - 2\,\omega_0^2\right) \, \omega^2 + \omega^4\right]^{-1}$$

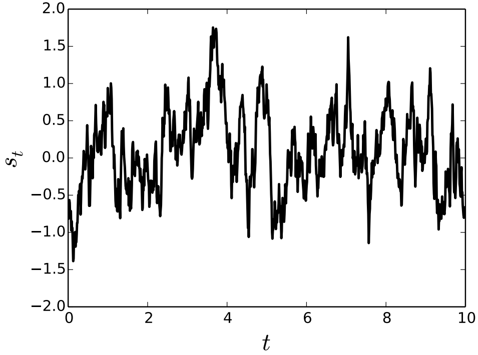
11.3.3 Example: Harmonic Oscillator





11.3.2 Example: Ornstein-Uhlenbeck Process

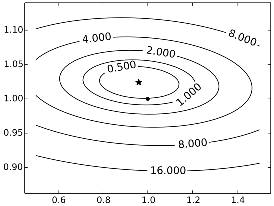


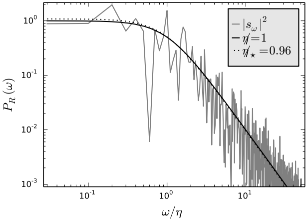


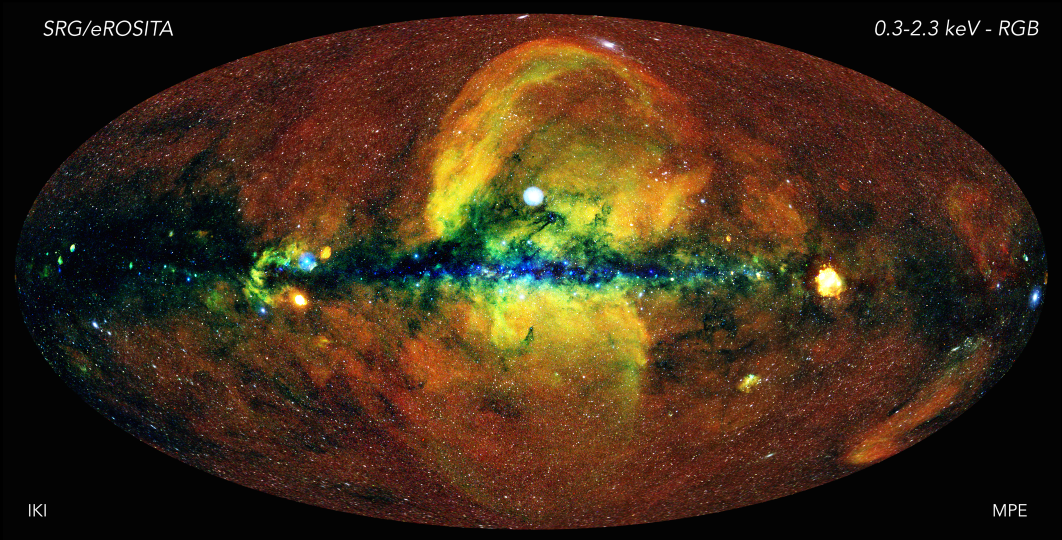
11.4 Parameter Determination

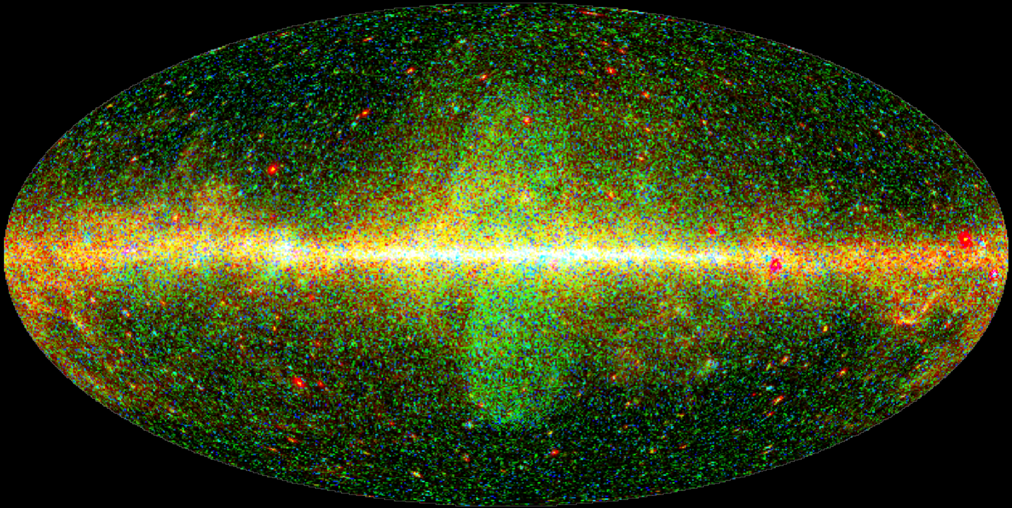
$$\mathcal{P}(a|s) \quad = \quad \frac{\mathcal{P}(s|a)\mathcal{P}(a)}{\mathcal{P}(s)} = \frac{e^{-\mathcal{H}(s,a)}}{\mathcal{Z}(s)}$$

$$\begin{aligned} \mathcal{H}(s,a) \quad &= \quad -\ln \mathcal{P}(s|a) - \ln \mathcal{P}(a) = \frac{1}{2}\left[s^{\dagger} S\, s + \ln|2\pi S|\right] + \mathcal{H}(a) \\ &\widehat{=} \quad \frac{1}{2}\int \frac{d\omega}{2\,\pi}\left[\frac{|s^{\omega}|^2}{P_s(\omega)} + \ln P_s(\omega)\right] \end{aligned}$$

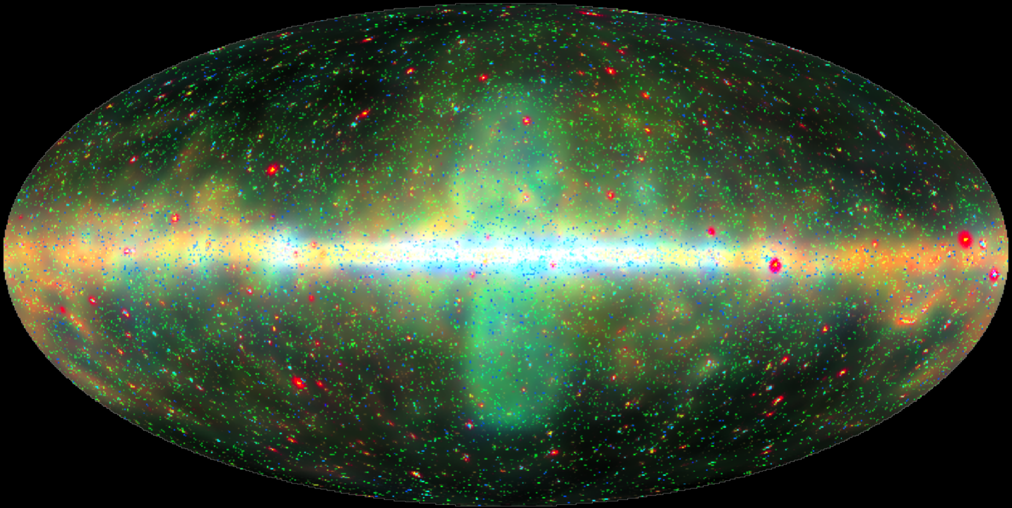




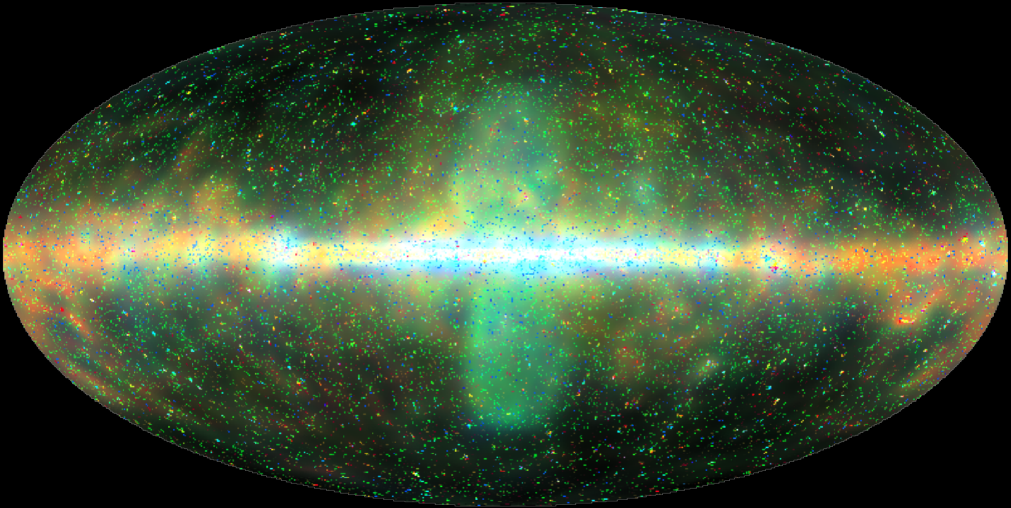

SRG/eROSITA
0.3-2.3 keV - RGB
IKI
MPE



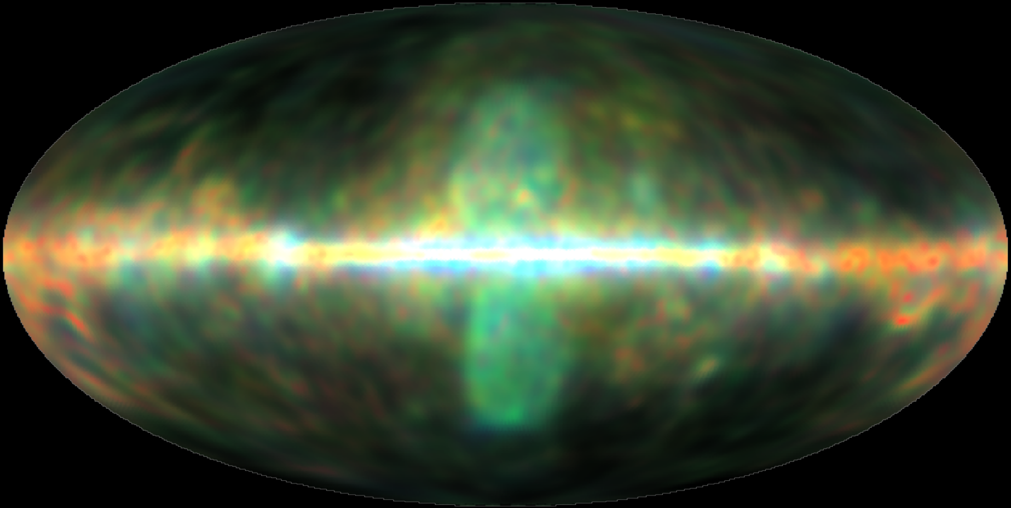
Selig et al. (2015)



Selig et al. (2015)

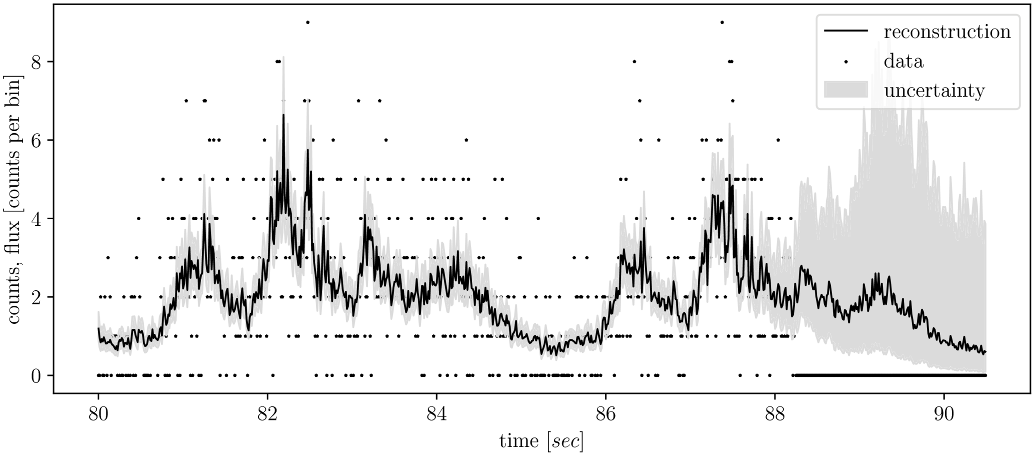


Selig et al. (2015)



Selig et al. (2015)

Magnetar flare SGR 1900+14



Pumpe et al. (2018)

11.5 Lognormal Poisson Model

- event density: $\rho^x = \rho(x)$
- number of observed events: $d = (d^1, \ldots, d^n)$
- expected number of observed events: $\lambda = (\lambda^1, \ldots, \lambda^n)$
- exposure: $\kappa(x)$

$$\lambda^i = \int dx\, R^i(x)\, \rho(x) = R^i_x \varrho^x$$

Independent discrete events:

$$\mathcal{P}(d|\lambda) = \prod_{i=1}^{n} \frac{\left(\lambda^i\right)^{d^i} e^{-\lambda^i}}{d^i!}$$

$$\mathcal{H}(d|\lambda) = \sum_{i=1}^{n} [\lambda^i - d^i \ln \lambda^i + \ln(d^i!)]$$

11.5 Lognormal Poisson Model

Continuous case:

$$\begin{aligned}\lambda^x &= \int dy\, \delta(x-y)\kappa(y)\rho(y) \\ &= (\kappa\rho)^x\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathcal{H}(d|\rho) &\widehat{=} \int dx\, [\kappa(x)\,\rho(x) - d(x)\ln(\kappa(x)\rho(x))] \\ &= \kappa_x\rho^x - d_x \ln(\kappa\rho)^x \\ &= \kappa^\dagger\rho - d^\dagger \ln(\kappa\rho)\end{aligned}$$

11.5 Lognormal Poisson Model

Defining the prior:

- $\rho(x) > 0 \; \forall x$
- $\rho(x)$ varies on logarithmic scale

$$
\begin{aligned}
s^x &= \ln \frac{\rho^x}{\rho_0} \\
\rho^x &= \rho_0 e^{s^x} \text{ with } \rho_0 \text{ s.t. } \langle s \rangle_{(s)} = 0
\end{aligned}
$$

- $S = \langle s s^\dagger \rangle_{(s)}$
- Higher order corrections are ignored

11.5 Lognormal Poisson Model

Maximum Entropy principle with known 1st and 2nd moments:

$$\begin{aligned}\mathcal{P}(s) &= \mathcal{G}(s,\, S)\\ \mathcal{H}(s) &= \frac{1}{2}s^\dagger S^{-1}s + \frac{1}{2}\ln|2\pi S|\end{aligned}$$

Joint information Hamiltonian:

$$\begin{aligned}\mathcal{H}(d,s) &= \mathcal{H}(d|s) + \mathcal{H}(s)\\ &\widehat{=} \frac{1}{2}s^\dagger S^{-1}s + \underbrace{\kappa^\dagger\rho_0}_{=\kappa'\to\kappa}\, e^s - d^\dagger \ln(\kappa\rho_0 e^s)\\ &\widehat{=} \frac{1}{2}s^\dagger S^{-1}s + \kappa^\dagger e^s - d^\dagger s\end{aligned}$$

Free and Interaction Hamiltonian

Expansion of exponential function:

$$
\begin{aligned}
e^{s^x} &= 1 + s^x + \frac{1}{2}\left(s^x\right)^2 + \ldots \\
\kappa^\dagger e^s &= \int dx\, \kappa(x)\left(1 + s(x) + \frac{1}{2}\left(s(x)\right)^2 + \ldots\right) \\
\mathcal{H}(d,s) &\widehat{=} \frac{1}{2} s^\dagger S^{-1} s + \kappa^\dagger e^s - d^\dagger s \qquad\qquad \mid \hat{\kappa} = \mathrm{diag}(\kappa) \\
&\widehat{=} \underbrace{\frac{1}{2} s^\dagger \underbrace{\left(S^{-1} + \widehat{\kappa}\right)}_{=D^{-1}} s - \underbrace{(d-\kappa)^\dagger}_{=j^\dagger} s}_{\text{free Hamiltonian}} + \underbrace{\kappa^\dagger \underbrace{\left(e^s - 1 - s - \frac{s^2}{2}\right)}_{=\sum_{n=3}^{\infty} \frac{1}{n!} s^n}}_{\text{interaction Hamiltonian}} \\
&= \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + \sum_{n=3}^{\infty} \frac{\kappa^\dagger s^n}{n!}
\end{aligned}
$$

MAP Solution

$$
\begin{aligned}
\mathcal{H}(d,s) &\;\widehat{=}\; \frac{1}{2}s^\dagger S^{-1}s + \kappa^\dagger e^s - d^\dagger s \\
0 &\overset{!}{=} \frac{\partial \mathcal{H}(d,s)}{\partial s^x} \\
&= \frac{\partial}{\partial s^x}\left[\frac{1}{2}s^{x'}S^{-1}_{x'x''}s^{x''} + \kappa_{x'}e^{s^{x'}} - d_{x'}s^{x'}\right] \\
&= \frac{1}{2}S^{-1}_{xx''}s^{x''} + \frac{1}{2}s^{x'}S^{-1}_{x'x} + (\kappa e^s)_x - d_x \\
&= \left[\frac{1}{2}S^{-1}s + \frac{1}{2}(s^\dagger S^{-1})^\dagger + \kappa e^s - d\right]_x \\
&= \left[S^{-1}s + \kappa e^s - d\right]_x \\
\frac{\partial \mathcal{H}(d,s)}{\partial s} &= S^{-1}s - d + \kappa e^s \overset{!}{=} 0 \\
\Rightarrow m &= S(d - \kappa e^m)
\end{aligned}
$$

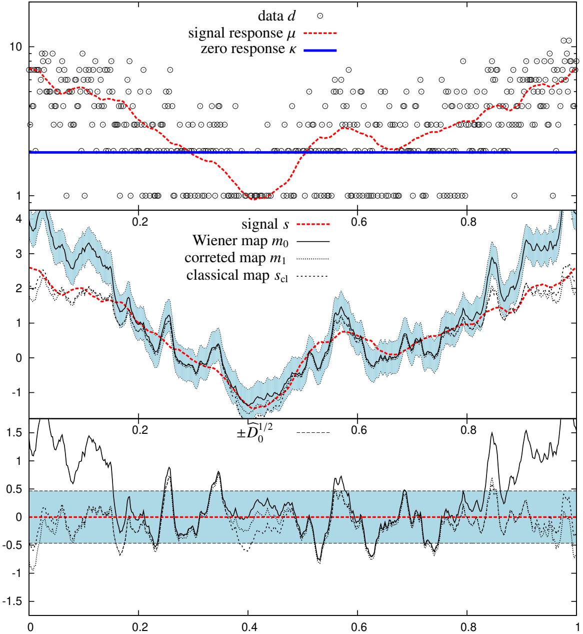
Numerical Stable MAP Solution

$$
\begin{aligned}
S^{-1} m &= d - \kappa e^{m} \\
\left(S^{-1} + \widehat{\kappa}\right) m &= d - \kappa \left(e^{m} - m\right) \\
D^{-1} m &= d - \kappa \left(e^{m} - m\right) \\
m &= D \left(d - \kappa(e^{m} - m)\right)
\end{aligned}
$$

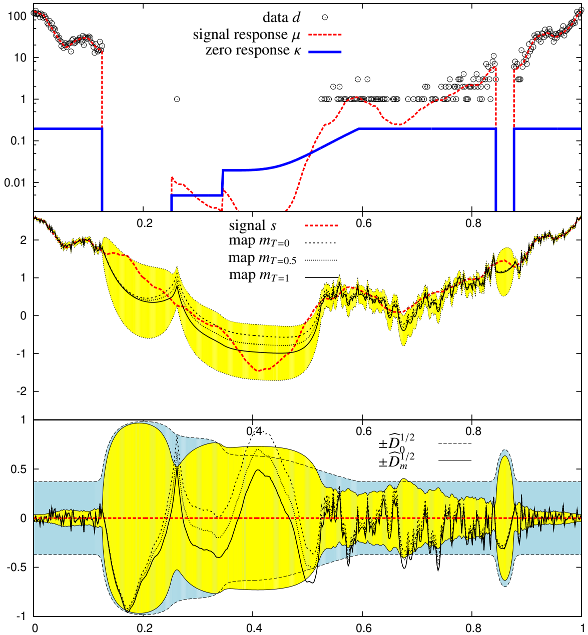
Comparison with Wiener filter $m = (S^{-1} + R^{\dagger} N^{-1} R)^{-1} R^{\dagger} N^{-1} d'$:

- $\widehat{\kappa} \sim R^{\dagger} N^{-1} R$
- $\kappa \sim R$
- $\mathbb{1} \sim R^{\dagger} N^{-1}$
- $\kappa \sim N$

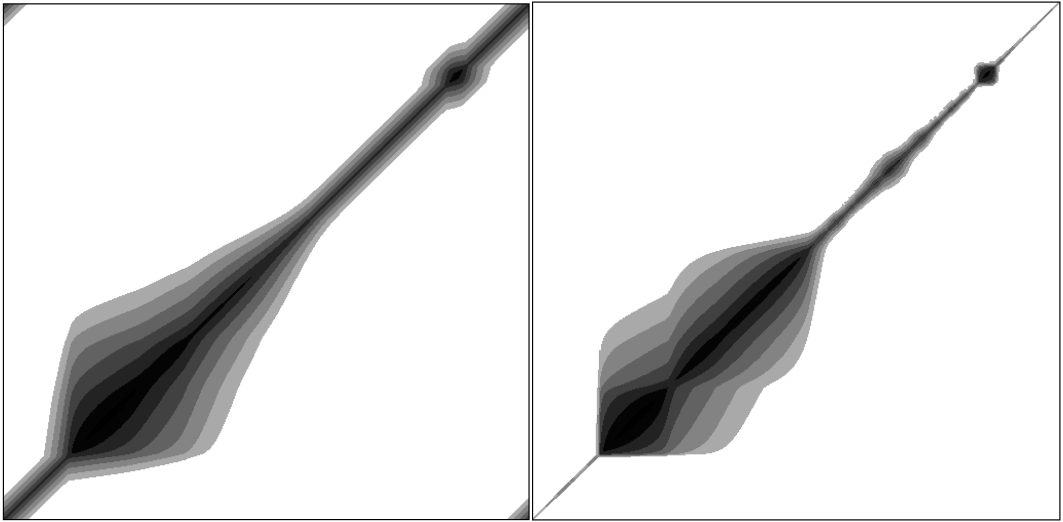
Classical Solution



Classical Solution



Uncertainty Covariance



Expansion around Classical Solution

$$
\begin{aligned}
\text{Hamiltonian: } \mathcal{H}(d, s) &\widehat{=} \frac{1}{2} s^\dagger S^{-1} s - d^\dagger s + \kappa^\dagger e^s \\
s &= m + \varphi \\
\mathcal{H}(d, \varphi | m) &= \mathcal{H}(d, s = m + \varphi) \\
&\widehat{=} \frac{1}{2} (m + \varphi)^\dagger S^{-1} (m + \varphi) + \kappa^\dagger e^m e^\varphi - d^\dagger (m + \varphi) \\
&\widehat{=} \frac{1}{2} \varphi^\dagger S^{-1} \varphi + m^\dagger S^{-1} \varphi + \kappa_m^\dagger e^\varphi - d^\dagger \varphi \\
&= \frac{1}{2} \varphi^\dagger S^{-1} \varphi - (d - S^{-1} m)^\dagger \varphi + \kappa_m^\dagger e^\varphi \\
&= \frac{1}{2} \varphi^\dagger S^{-1} \varphi - d_m^\dagger \varphi + \kappa_m^\dagger e^\varphi
\end{aligned}
$$

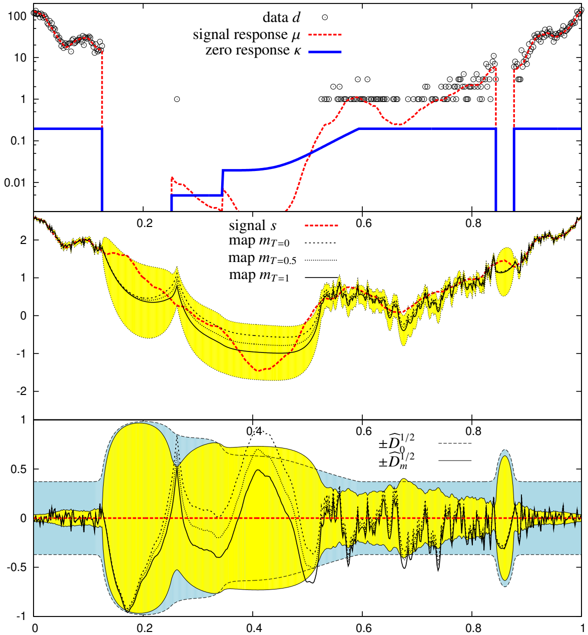
Expansion around Classical Solution

Shifted data vector:

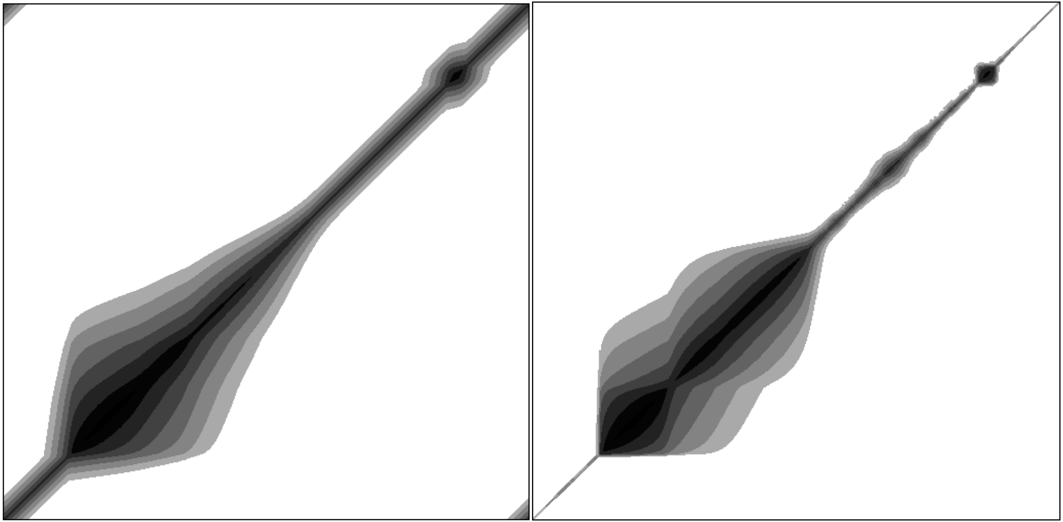
$$d_m \quad = d - S^{-1}m = d - S^{-1}S(d - \kappa e^m) = d - d + \kappa e^m = \kappa_m$$

$$\begin{aligned}
\Rightarrow \mathcal{H}(d,\,\varphi|m) \quad &= \quad \frac{1}{2}\varphi^\dagger(S^{-1} + \widehat{\kappa}_m)\varphi + \underbrace{(d_m - \kappa_m)^\dagger}_{=0}\varphi + \kappa_m^\dagger\left(e^\varphi - \varphi - \frac{\varphi^2}{2}\right) \\
&= \quad \frac{1}{2}\varphi^\dagger{\color{red}(S^{-1} + \widehat{\kappa}_m)}\varphi + \kappa_m^\dagger\left(e^\varphi - \varphi - \frac{\varphi^2}{2}\right) \\
&= \quad \frac{1}{2}\varphi^\dagger{\color{red}D_m^{-1}}\varphi + \kappa_m^\dagger\left(e^\varphi - \varphi - \frac{\varphi^2}{2}\right)
\end{aligned}$$

Classical Solution



Uncertainty Covariance



End