10 Matrix Algebra

Hermitian: $A = A^{\dagger}$
positive definitive: $A \geq 0 \ \rightarrow \ x^{\dagger}Ax \geq 0 \, \forall x \neq 0$
strictly positive definite: $A > 0 \ \rightarrow \ x^{\dagger}Ax > 0 \, \forall x \neq 0$

$$A > 0,\, B > 0 \quad \Rightarrow \quad A + B > 0$$

Eigensystem:
α_i: eigenvalues
a_i : orthonormal eigenvectors

$$\begin{aligned} Aa_i &= \alpha_i a_i \\ a_i^{\dagger} a_j &= \delta_{ij} \end{aligned}$$

$$\Rightarrow A = \sum_i \alpha_i a_i a_i^{\dagger},\; f(A) = \sum_i a_i a_i^{\dagger} f(\alpha_i)$$

Example 1

$f(x) = x^{\frac{1}{2}} \rightarrow f(A) = A^{\frac{1}{2}} = \sum_i \alpha_i^{\frac{1}{2}} a_i a_i^\dagger$

Proof:

$$
\begin{aligned}
A^{1/2}A^{1/2} &= \sum_i \alpha_i^{\frac{1}{2}} a_i a_i^\dagger \sum_j \alpha_j^{\frac{1}{2}} a_j a_j^\dagger \\
&= \sum_i \sum_j \sqrt{\alpha_i \alpha_j} a_i a_i^\dagger a_j a_j^\dagger \\
&= \sum_i \sum_j \sqrt{\alpha_i \alpha_j} a_i \delta_{ij} a_j^\dagger \\
&= \sum_i \alpha_i a_i a_i^\dagger \\
&= A
\end{aligned}
$$

Example 2

$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n f_n \rightarrow f(A) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n A^n$

Proof:

$$
\begin{aligned}
f(A) &= \sum_{n=0}^{\infty} \frac{1}{n!} f_n \left(\sum_i \alpha_i a_i a_i^\dagger \right)^n \\
&= \sum_{n=0}^{\infty} \frac{f_n}{n!} \left(\sum_{i_1} \ldots \sum_{i_n} \alpha_{i_1} \ldots \alpha_{i_n} a_{i_1} \underbrace{a_{i_1}^\dagger a_{i_2}}_{=\delta_{i_1,i_2}} \underbrace{a_{i_2}^\dagger a_{i_3}}_{=\delta_{i_2,i_3}} \ldots a_{i_n} a_{i_n}^\dagger \right) \\
&= \sum_{n=0}^{\infty} \frac{f_n}{n!} \sum_i \alpha_i^n a_i a_i^\dagger \\
&= \sum_i a_i a_i^\dagger \sum_{n=0}^{\infty} \frac{f_n}{n!} \alpha_i^n \\
&= \sum_i f(\alpha_i) a_i a_i^\dagger \; \square
\end{aligned}
$$

Example 3

$f(x) = x^{-1} \to f(A) = A^{-1} = \sum_i \alpha_i^{-1} a_i a_i^\dagger$

Proof:

$$\begin{aligned} f(A)A &= \sum_{ij} \alpha_i^{-1} \alpha_j a_i a_i^\dagger a_j a_j^\dagger \\ &= \sum_{ij} \alpha_i^{-1} \alpha_j a_i \delta_{ij} a_j^\dagger \\ &= \sum_i 1\, a_i a_i^\dagger \\ &= \mathbb{1} \\ \Rightarrow A^{-1} &= \frac{1}{A} \;\square \end{aligned}$$

Proof: Geometric Expansion of Modified Propagator

$$
\begin{aligned}
D' &= \left(D^{-1} - \Delta\right)^{-1} \\
&= \left[D^{-1/2}\left(\mathbb{1} - D^{1/2}\Delta\, D^{1/2}\right)\, D^{-1/2}\right]^{-1} \\
&= D^{1/2}\left(\mathbb{1} - D^{1/2}\Delta\, D^{1/2}\right)^{-1} D^{1/2} \\
&= D^{1/2}\left(\mathbb{1} - X\right)^{-1} D^{1/2} \\
&= D^{1/2}\left(\mathbb{1} + X + XX + \ldots\right)\, D^{1/2} \\
&= D + D\,\Delta\,D + D\,\Delta\,D\,\Delta\,D + \ldots
\end{aligned}
$$

Assure convergence of geometric expansion: $X < \mathbb{1}$

$$
\begin{aligned}
X &= D^{1/2}\Delta\, D^{1/2} = D^{1/2} B^{\dagger} M\, B\, D^{1/2} \\
&\leq \left(S^{-1} + M\right)^{-1/2} M \left(S^{-1} + M\right)^{-1/2} \\
&< \left(S^{-1} + M\right)^{-1/2} \left(S^{-1} + M\right)\, \left(S^{-1} + M\right)^{-1/2} \\
&= \mathbb{1}
\end{aligned}
$$

□

11 Gaussian Processes

Markov property:

- $s : \mathbb{R} \mapsto \mathbb{R}^u$ (or $\mathbb{C}^u$)
- any future value is independent of past values if the present value is known:

$$f \geq t \geq p \Rightarrow \mathcal{P}(s^f | s^t, s^p) = \mathcal{P}(s^f | s^t)$$

- present isolates the future from the past:

$$f \geq t \geq p \Rightarrow \mathcal{P}(s^f, s^t | s^p) = \mathcal{P}(s^f | s^t)\, \mathcal{P}(s^t | s^p)$$

11.1.2 Wiener Process

$$\dot{s}^t = \frac{ds^t}{dt} = \sigma^t\,\xi^t,\ \text{with}\ \mathcal{P}(\xi) = \mathcal{G}(\xi,\ \mathbb{1})\ \text{and known}\ \sigma^t$$

s^p : known process values
s^f: process values of interest ($f > p$)

$$\begin{aligned} s^f &= s^p + \underbrace{\int_p^f dt\,\sigma^t}_{=L_t^f}\,\xi^t\ \text{for known}\ \xi \\ L_t^f &= \sigma^t\,\mathcal{P}(p \leq t \leq f | p,\, t,\, f) \\ (L^{-1})_f^t &= \delta(f-t)\,\frac{\partial}{\sigma^t \partial f} \end{aligned}$$

11.1.2 Wiener Process

Conditional probability:

$$
\begin{aligned}
\Rightarrow \mathcal{P}(s|s^p) &= \int \mathcal{D}\xi\, \mathcal{P}(s|\xi, s^p)\, \mathcal{P}(\xi|s^p) = \int \mathcal{D}\xi\, \delta\left[s - (s^p + L\xi)\right] \mathcal{G}(\xi, \mathbb{1}) \\
&= \int \mathcal{D}\xi\, \frac{\delta\left[\xi - L^{-1}(s - s^p)\right]}{|L|}\, \mathcal{G}(\xi, \mathbb{1}) = \frac{\mathcal{G}(L^{-1}(s-s^p), \mathbb{1})}{|L|} \\
&= \frac{\exp\left[-\frac{1}{2}(s-s^p)^\dagger \left(L^{-1}\right)^\dagger \mathbb{1} L^{-1}(s-s^p)\right]}{|2\pi\mathbb{1}|^{1/2}\,|L|} \\
&= \frac{\exp\left[-\frac{1}{2}(s-s^p)^\dagger \left(L^\dagger\right)^{-1} \mathbb{1}^{-1} L^{-1}(s-s^p)\right]}{|2\pi\mathbb{1}|^{1/2}\,|L|} \\
&= \frac{\exp\left[-\frac{1}{2}(s-s^p)^\dagger (L\mathbb{1}L^\dagger)^{-1}(s-s^p)\right]}{|2\pi\mathbb{1}|^{1/2}\,|L|} \\
&= \frac{\exp\left[-\frac{1}{2}(s-s^p)^\dagger (L\,L^\dagger)^{-1}(s-s^p)\right]}{|2\pi L\,L^\dagger|^{1/2}} = \mathcal{G}(s-s^p, LL^\dagger)
\end{aligned}
$$

Uncertainty Dispersion of the Wiener Process

$$
\begin{aligned}
D^{tt'} &= \langle (s^t - d)\,(s^{t'} - d)\rangle_{(s|d)} \\
&= \langle (s^t - s^p)\,(s^{t'} - s^p)\rangle_{(s|s^p)} \\
&= L^t_{t''}\,\underbrace{\langle \xi^{t''}\,\xi^{t'''}\rangle_{(\xi)}}_{\mathbb{1}^{t''t'''}}\,L^{t'}_{t'''} \\
&= \int_{-\infty}^{\infty} dt''\, L^t_{t''} L^{t'}_{t''} \\
&= \int_{-\infty}^{\infty} dt''\, \sigma^{t''}\, P(p \le t'' \le t\,|\,p, t'', t)\, \sigma^{t''}\, P(p \le t'' \le t'\,|\,p, t'', t') \\
&= \int_{p}^{\min\{t,t'\}} dt''\, \left(\sigma^{t''}\right)^2
\end{aligned}
$$

11.1.3 Future Expectation

- Gaussian process $s : \mathbb{R} \mapsto \mathbb{R}$
- mean: $m = 0$
- prior: $S^{t_1 t_2} = \langle s^{t_1} s^{t_2} \rangle_{(s)}$
- data: $d = s^t$

Wiener filter formula:

$$\langle s^f \rangle_{(s^f|d)} = F_L d = \langle s^f d \rangle_{(s)} \langle d d \rangle_{(s)}^{-1} d = \langle s^f s^t \rangle_{(s)} \langle s^t s^t \rangle_{(s)}^{-1} s^t = \frac{S^{ft}}{S^{tt}} s^t$$

$f \geq t \geq p$

$$S^{fp} = \frac{S^{ft} S^{tp}}{S^{tt}}$$

11.1.3 Future Expectation

Proof:
Wick's theorem:

$$\langle s^f s^t s^t s^p \rangle_{(s)} = 2\, S^{ft} S^{tp} + S^{fp} S^{tt}$$

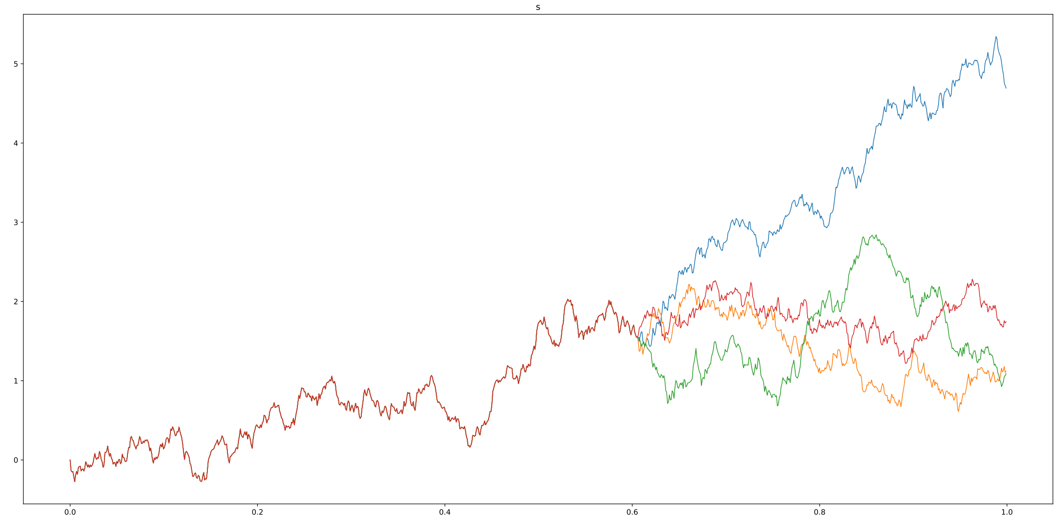
Markov property:

$$P(s^f, s^t, s^p) = P(s^f, s^p | s^t)\, P(s^t) = P(s^f | s^t)\, P(s^p | s^t)\, P(s^t)$$

$$\begin{aligned}
\Rightarrow \langle s^f s^t s^t s^p \rangle_{(s)} &= \int ds^t \int ds^f \int ds^p\, s^f\, \mathcal{P}(s^f | s^t)\, s^p\, \mathcal{P}(s^p | s^t)\, s^t s^t\, \mathcal{P}(s^t) \\
&= \langle \langle s^f \rangle_{(s^f|s^t)} \langle s^p \rangle_{(s^p|s^t)} s^t s^t \rangle_{(s^t)} = \langle S^{ft} \left(S^{tt}\right)^{-1} s^t S^{pt} \left(S^{tt}\right)^{-1} s^t s^t s^t \rangle_{(s^t)} \\
&= \frac{S^{ft} S^{pt}}{S^{tt} S^{tt}} \langle s^t s^t s^t s^t \rangle_{(s^t)} = \frac{S^{ft} S^{pt}}{S^{tt} S^{tt}}\, 3\, S^{tt} S^{tt} = 3\, S^{ft} S^{pt} \stackrel{!}{=} 2\, S^{ft} S^{tp} + S^{fp} S^{tt}
\end{aligned}$$

$$\Rightarrow S^{ft} S^{tp} = S^{fp} S^{tt} \quad \square$$

11.1.4 Example: Evolution of Stock Price



Evolution of a stock prize signal in case of a constant volatility σ.

11.1.4 Example: Evolution of Stock Price

p_t: stock price at time t
q_t: evolution of corresponding stock market index
s_t: buy/sell signal

$$s^f \quad = \quad \ln \frac{p^f}{p^p} - \ln \frac{q^f}{q^p}$$

buy stock: $\langle s^f \rangle_{(s^f|s^t)} > s^t$ for $f > t > p$

sell stock: $\langle s^f \rangle_{(s^f|s^t)} < s^t$ for $f > t > p$

No arbitrage condition: signal s is a **martingale**

- $\langle s^f \rangle_{(s^f|s^t)} = s^t$ for all $f > t$
- s is Markov

11.1.4 Example: Evolution of Stock Price

- prior: $\mathcal{P}(s) = \mathcal{G}(s, S)$
- posterior: $\mathcal{P}(s|s^t) = \mathcal{G}(s - s^t, D) = \mathcal{G}(s - s^t|D_{(t)})$
- volatility: σ^t

$$\langle s^f \rangle_{(s^f|s^t)} \quad = \quad S^{ft} \left(S^{tt}\right)^{-1} s^t = s^t \Rightarrow S^{ft} = S^{tt} \text{ for all } f > t$$

$$\begin{aligned} \frac{d}{dt} S^{tt} &=: \left(\sigma^t\right)^2 \geq 0 \Rightarrow S^{ff} > S^{tt} \text{ for all } f > t \\ \Rightarrow S^{ab} &= \min\{S^{aa}, S^{bb}\} \text{ Wiener process} \\ \dot{s}_t &= \sigma^t \xi^t \text{ with } \mathcal{P}(\xi) = \mathcal{G}(\xi, \mathbb{1}) \end{aligned}$$

11.1.4 Example: Evolution of Stock Price

$$s^f = \ln\frac{p^f}{p^p} - \ln\frac{q^f}{q^p} \Rightarrow \frac{p^f}{p^p} = \frac{q^f}{q^p} e^{s^f} \Rightarrow \boxed{\left\langle \frac{p^f}{p^p} \right\rangle_{(s^f|s^t)} = \frac{q^f}{q^p} \left\langle e^{s^f} \right\rangle_{(s^f|s^t)} \geq \frac{q^f}{q^p} e^{s^t}}$$

Proof:

- $\langle s^f - s^t \rangle_{(s^f|s^t)} = 0$, $\langle e^{s^f} \rangle_{(s^f|s^t)} = e^{s^t} \langle e^{s^f - s^t} \rangle_{(s^f|s^t)}$
- $\mathcal{P}(s^f|s^t) = \mathcal{G}(s^f - s^t, D^{ff}_{(t)}) = \mathcal{G}(\Delta, \Sigma)$

$$\begin{aligned}
\langle e^{s^f} \rangle_{(s^f|s^t)} &= e^{s^t} \langle e^{s^f - s^t} \rangle_{(s^f|s^t)} = e^{s^t} \langle e^{\Delta} \rangle_{\mathcal{G}(\Delta,\Sigma)} = e^{s^t} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Delta^n \rangle_{\mathcal{G}(\Delta,\Sigma)} \\
&= e^{s^t} \sum_{n=0}^{\infty} \frac{1}{(2n)!} \langle \Delta^{2n} \rangle_{\mathcal{G}(\Delta,\Sigma)} = e^{s^t} \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{(2n)!}{2^n n!} \Sigma^n \\
&= e^{s^t} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\Sigma}{2} \right)^n = e^{s^t} e^{\frac{1}{2}\Sigma} = e^{s^t + \frac{1}{2}\Sigma} \geq e^{s^t} \quad \square
\end{aligned}$$

End