9.3.4 Position Space Filter

Reconstructed signal in position space:

$$s^x = \int \frac{dk}{(2\pi)^u} s^k e^{-ikx}$$

Reconstructed mean in position space:

$$\begin{aligned}
m^x &= \int \frac{dk^u}{(2\pi)^u} e^{-ikx} m^k = \int \frac{dk^u}{(2\pi)^u} e^{-ikx} f(k)\, d^k \\
&= \int \frac{dk^u}{(2\pi)^u} e^{-ikx} f(k) \int dy^u\, e^{iky}\, d^y \\
&= \int dy^u \int \frac{dk^u}{(2\pi)^u} e^{-ik(x-y)} f(k)\, d^y \\
&= \int dy^u f(x-y)\, d^y = (f * d)^y
\end{aligned}$$

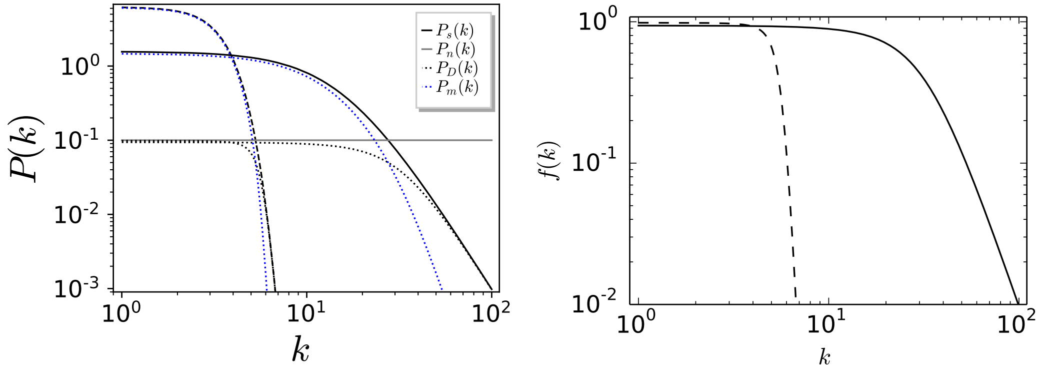
9.3.4 Position Space Filter

Fourier transformed spectral filter:

$$f(r) = \int \frac{dk^u}{(2\pi)^u} e^{-ikr} f(k) = \int \frac{dk^u}{(2\pi)^u} \frac{e^{-ikr}}{1 + {}^{P_n(k)}/_{P_s(k)}}$$

Power spectrum of the mean:

$$\begin{aligned} P_m(k) &= \frac{1}{V} \langle |m^k|^2 \rangle_{(d,s)} = \frac{1}{V} \langle |f(k)|^2 |d^k|^2 \rangle_{(d,s)} \\ &= \frac{1}{(1 + {}^{P_n}/_{P_s})^2} (P_s(k) + P_n(k)) \\ &= \frac{P_s^2(k)}{P_s(k) + P_n(k)} = \frac{P_s(k)}{1 + {}^{P_n(k)}/_{P_s(k)}} \end{aligned}$$



$P(k)$
10^0
10^{-1}
10^{-2}
10^{-3}
10^0
10^1
10^2
k
$P_s(k)$
$P_n(k)$
$P_D(k)$
$P_m(k)$
$f(k)$
10^0
10^{-1}
10^{-2}
10^0
10^1
10^2
k

9.3.5 Example: Large-Scale Signal

- white noise: $P_s(k) = \sigma_n^2$

$$\begin{aligned} N^{xy} &= \langle n^x n^y \rangle_{(n)} = \delta(x-y)\sigma_n^2 = C_n(x-y) \\ N^{kq} &= \int dx \int dy\, e^{ikx} \delta(x-y)\sigma_n^2 e^{-iqy} = \sigma_n^2 \int dx\, e^{i(k-q)x} = (2\pi)^u \delta(k-q)\sigma_n^2 \end{aligned}$$

- signal with a red signal spectrum: $P_s(k) = \sigma_s^2 \, (k/k_0)^{-2}$

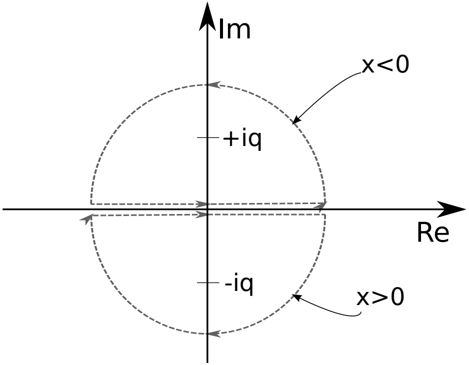
Spectral filter function:

$$f(k) = \frac{1}{1 + P_n(k)/P_s(k)} = \frac{1}{1 + \frac{\sigma_n^2}{\sigma_s^2 k_0^2} k^2} = \frac{1}{1 + q^{-2} k^2} = \frac{q^2}{k^2 + q^2}$$

9.3.5 Example: Large-Scale Signal

Position Space filter:

$$f(x) = \int \frac{dk}{2\pi} \frac{q^2}{q^2 + k^2} e^{-ikx} = \frac{q^2}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-ikx}}{(k + iq)(k - iq)}$$



Cauchy's Residue Theorem

$f(k)$: analytical function
γ: closed path in complex plane
$\{k_1, \ldots k_n\}$: singularities of f inside γ
$I(\gamma,\, k)$: winding number of the path with respect to a point k

Residuum:

$$\mathrm{Res}(f,\, k_l) \quad = \quad f(k)\,(k - k_l)|_{k=k_l}$$

Cauchy's Residue Theorem:

$$\oint_\gamma dk\, f(k) = 2\pi i \sum_{l=1}^{n} I(\gamma,\, k_l)\,\mathrm{Res}(f,\, k_l)$$

9.3.5 Example: Large-Scale Signal

For $x < 0$, $x = -|x|$:

$$f(x) = \int \frac{dk}{2\pi} \frac{q^2}{q^2 + k^2} e^{-ikx} = iq^2(+1) \left. \frac{e^{ik|x|}}{k + iq} \right|_{k=+iq} = \frac{iq^2 e^{i(iq)|x|}}{2iq} = \frac{q}{2} e^{-q|x|}$$

$$\begin{aligned}
f(x) &= \frac{q}{2} e^{-q\,|x|} = \frac{1}{2\,\lambda} e^{-|x|/\lambda} \\
q &= \frac{\sigma_s k_0}{\sigma_n} \\
\lambda &= \frac{1}{q} = \frac{\sigma_n}{\sigma_s k_0} \text{ correlation length of } f(x) \\
\lambda &= \int_0^\infty dx \frac{f(x)}{f(0)} = \int_0^\infty dx\, e^{-|x|/\lambda}
\end{aligned}$$

9.3.6 Deconvolution

- $\mathcal{P}(s, n) = \mathcal{G}(s, S)\mathcal{G}(n, N)$
- S, N homogeneous
- $R^x_y = b(x - y)$

$$d^y = \int dx\, b(y - x)s^x + n^y$$

Fourier space:

$$\begin{aligned}
d^k &= (b * s)^k + n^k = \int dy\, e^{iky} \left[\int dx\, b(y - x)\, s^x + n^y\right] \\
\Rightarrow (b * s)^k &= \int dx \int dy\, e^{iky} \int \frac{dk'}{(2\pi)^u} e^{-ik'(y-x)} b(k') \int \frac{dk''}{(2\pi)^u} e^{-ik''x} s^{k''} \\
&= \int \frac{dk'}{(2\pi)^u} \int \frac{dk''}{(2\pi)^u} \underbrace{\int dy\, e^{i(k-k')y}}_{(2\pi)^u \delta(k-k')} \underbrace{\int dx\, e^{i(k'-k'')x}}_{(2\pi)^u \delta(k'-k'')} b(k')\, s^{k''} \\
&= b(k)\, s^k
\end{aligned}$$

9.3.6 Deconvolution

Response:

$$R^{k}_{k'} = (2\pi)^u \delta(k - k')\, b(k)$$

Signal covariance:

$$S^{kk'} = (2\pi)^u \delta(k - k')\, P_s(k)$$

Noise covariance:

$$N_{kk'} = (2\pi)^u \delta(k - k')\, P_n(k)$$

Uncertainty covariance:

$$D^{kk'} = (2\pi)^u \delta(k - k')\, P_D(k)$$

9.3.6 Deconvolution

Calculation of the power spectrum:

$$\begin{aligned} D &= (S^{-1} + M)^{-1} \\ M &= R^{\dagger} N^{-1} R \end{aligned}$$

Fourier space:

$$\begin{aligned} M^{kq} &= \left(R^{\dagger} N^{-1} R\right)^{kq} \\ &= \left(R^{\dagger}\right)_{k}^{k'} \left(N^{-1}\right)_{k'q'} R_{q}^{q'} \\ &= \int \frac{dk'}{(2\pi)^u} \int \frac{dq'}{(2\pi)^u} (2\pi)^u \delta(k-k')\, \overline{b}(k) \left(N^{-1}\right)_{k'q'} (2\pi)^u \delta(q-q')\, b(q) \\ &= (2\pi)^u \delta(k-q) \underbrace{|b(k)|^2}_{P_R(k)} / P_n(k) \end{aligned}$$

9.3.6 Deconvolution

$$\Rightarrow P_D(k) \quad = (P_S^{-1}(k) + P_M(k))^{-1}$$
$$= \frac{P_s(k)}{1 + \frac{P_S(k) P_R(k)}{P_n(k)}}$$

Information source in Fourier space:

$$j_k \quad = \quad (R^\dagger N^{-1} d)_k = \frac{\overline{b(k)}\, d_k}{P_n(k)}$$

Signal mean in Fourier space:

$$m^k \quad = \quad (Dj)^k = \frac{(P_s/P_n)(k)\, \overline{b}^k}{1 + (P_s P_R/P_n)(k)}\, d^k = f(k) d^k$$

Fidelity Operator

$$f(k) = \frac{(P_s/P_n)(k)\,\overline{b}^k}{1+(P_sP_R/P_n)(k)} = \left(\frac{P_sP_R/P_n}{1+P_sP_R/P_n}\,\frac{\overline{b}}{P_R}\right)(k)$$

$$\text{Fidelity Operator: } Q = SR^{\dagger}N^{-1}R$$

$$P_Q(k) = \frac{P_sP_R}{P_n}(k)$$

$$\Rightarrow f(k) = \frac{P_Q(k)}{1+P_Q(k)}\,\frac{\overline{b}(k)}{P_R(k)} = \frac{P_Q(k)}{1+P_Q(k)}\,\frac{1}{b(k)}$$

$$= \frac{1}{b(k)}\begin{cases} 1 & \text{if } P_Q(k) \gg 1 \\ \underbrace{P_Q(k)}_{\ll 1} & \text{if } P_Q(k) \ll 1 \end{cases}$$

High fidelity regime (hifi): $P_Q(k) \gg 1$
Low fidelity regime (lofi): $P_Q(k) \ll 1$

Fidelity Operator

Signal mean:

$$
\begin{aligned}
m^k &= (f\,d)^k \\
&= \frac{P_Q(k)}{1+P_Q(k)} \, \frac{1}{b(k)} \left(b(k)\, s^k + n^k\right) \\
&= \frac{P_Q(k)}{1+P_Q(k)} \left(s^k + \frac{n^k}{b(k)}\right) \\
&= \begin{cases} s^k + \frac{n^k}{b(k)} & \text{if } P_Q(k) \gg 1 \\ P_Q(k)\left(s^k + \frac{n^k}{b(k)}\right) & \text{if } P_Q(k) \ll 1 \end{cases}
\end{aligned}
$$

Fidelity Operator

Signal mean power spectrum:

$$
\begin{aligned}
P_m(k) &= \frac{1}{V}\langle |m^k|^2\rangle_{(n,s)} \\
&= \frac{1}{V}\left(\frac{P_Q(k)}{1+P_Q(k)}\right)^2 \left(\langle |s^k|^2\rangle + \frac{\langle |n^k|^2\rangle}{|b(k)|^2}\right) \\
&= \frac{P_Q(k)^2}{(1+P_Q(k))^2}\left(P_s + \frac{P_n}{P_R}\right)(k) \\
&= \frac{P_Q(k)^2}{(1+P_Q(k))^2} P_s(k) \left(\frac{P_Q(k)+1}{P_Q(k)}\right) \\
&= \frac{P_Q}{1+P_Q}(k)\, P_s(k) \\
&= \begin{cases} P_s(k) & \text{if } P_Q(k) \gg 1 \\ P_Q(k)P_s(k) & \text{if } P_Q(k) \ll 1 \end{cases}
\end{aligned}
$$

9.3.7 Missing Data

Transparency/ Transfer operator:

$$T^x_y = \delta(x-y)P(x \notin \Omega|x,\,\Omega), \;\; P(x \notin \Omega|x,\,\Omega) = \begin{cases} 1 & \text{if } x \notin \Omega \\ 0 & \text{if } x \in \Omega \end{cases}$$

Modified data:

$$d'^x = R^x_{x'} T^{x'}_y s^y + n^x = R'^x_y s^y + n^x$$

Modified information source:

$$j'_x = (R'^\dagger)^{x'}_x (N^{-1})_{x'y} d'^y$$

9.3.7 Missing Data

Modified propagator:

$$
\begin{aligned}
D' &= (S^{-1} + R'^{\dagger} N^{-1} R')^{-1} \\
&= (S^{-1} + R^{\dagger} N^{-1} R - (R^{\dagger} N^{-1} R - R'^{\dagger} N^{-1} R'))^{-1} \\
&= (S^{-1} + R^{\dagger} N^{-1} R - \Delta)^{-1} \\
&= (S^{-1} + M - \Delta)^{-1}
\end{aligned}
$$

Blocking operator:

$$
B = \mathbb{1} - T \Rightarrow B_y^x = \delta(x-y)\, P(x \in \Omega | x,\, \Omega)
$$

9.3.7 Missing Data

For local $M = R^\dagger N^{-1} R = g(x)\,\delta(x-y)$:

- $R \propto \delta(x-y)$
- white noise

$$
\begin{aligned}
\Rightarrow \Delta \quad &= \quad R^\dagger N^{-1} R - R'^\dagger N^{-1} R' \\
&= \quad R^\dagger N^{-1} R - T^\dagger R^\dagger N^{-1} R\, T \\
&= \quad R^\dagger N^{-1} R - (\mathbb{1} - B)^\dagger R^\dagger N^{-1} R (\mathbb{1} - B) \\
&= \quad R^\dagger N^{-1} R - R^\dagger N^{-1} R - B^\dagger \underbrace{R^\dagger N^{-1} R}_{=M} B + B^\dagger M + M\, B \\
&= \quad -B^\dagger M\, B + B^\dagger M + M\, B \\
&= \quad B^\dagger M\, B
\end{aligned}
$$

9.3.7 Missing Data

Expansion of the modified propagator:

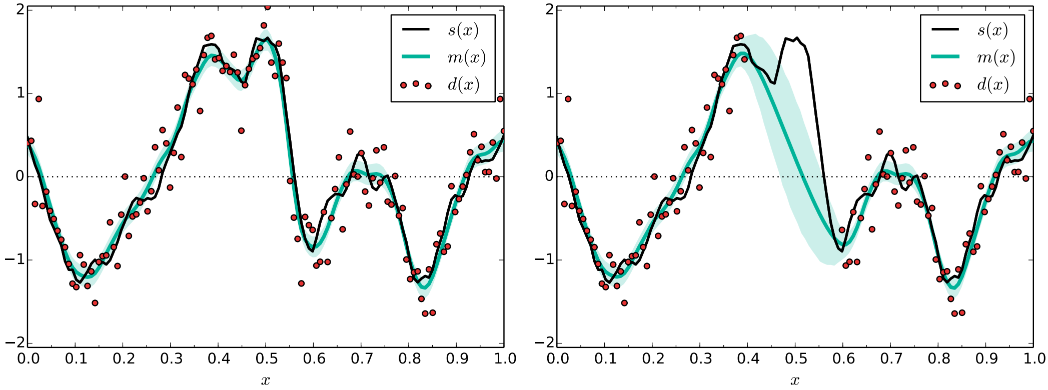
$$\begin{aligned} D' &= (S^{-1} + M - \Delta)^{-1} \\ &= (D^{-1} - \Delta)^{-1} \\ &= D\,(\mathbb{1} - \Delta D)^{-1} \\ &= D\,(\mathbb{1} + \Delta D + \Delta D \Delta D + \ldots) \\ &= D + D\Delta D + D\Delta D\Delta D + \ldots \end{aligned}$$

$$\begin{aligned} \Rightarrow D'^{xy} &= D^{xy} + D^{xx'}\Delta_{x'y'}D^{y'y} + \mathcal{O}(\Delta^2) \\ &\approx D^{xy} + D^{xz'}\,g(z')\,D^{y}_{z'} \end{aligned}$$

9.3.7 Missing Data

Reconstructed signal map:

$$m'^{x} \quad = \quad D'^{xy}\, j'_{y}$$



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