9. Wiener Filter Theory

$$
\begin{aligned}
d &= R\,s + n \\
d^i &= R^i_x\, s^x + n^i \\
\mathcal{P}(n,s) &= \mathcal{G}(s,\,S)\,\mathcal{G}(n,\,N) \\
\mathcal{P}(s|d) &= \mathcal{G}(s-m,\,D) \\
m &= D\,j = D^{xy} j_y \\
D &= (S^{-1} + \underbrace{R^\dagger N^{-1} R}_{=M})^{-1} \\
j &= R^\dagger N^{-1} d \\
j_x &= \overline{R}^i_x \left(N^{-1}\right)_{ij} d^j
\end{aligned}
$$

9.1 Statistical Homogeneity

- $s,\, d,\, n:\ \mathbb{R}^u \to \mathbb{R},\, \mathbb{C}$
- complete data: $d = s + n$
- statistical homgogeneous signal:

$$S^{xy} \quad = \quad \langle s^x s^y \rangle_{(s)} = C_s(x-y)$$

- statistical homogeneous noise:

$$N^{xy} \quad = \quad \langle n^x n^y \rangle_{(n)} = C_n(x-y)$$

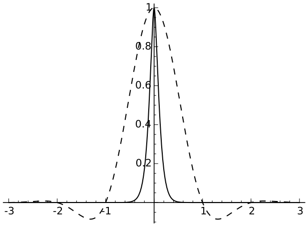

Figure: Possible correlation functions

9.2 Fourier Space

Fourier transform of a function $f : \mathbb{R}^{\mathrm{u}} \to \mathbb{C}$:

$$f(\not{k}) = \int dx\, e^{2\pi i \not{k} x} f(x)$$

$$f(x) = \int d\not{k}\, e^{-2\pi i \not{k} x} f(\not{k})$$

Absorb 2π-factor, $k = 2\pi\not{k}$:

$$f_k = f(k) = \int dx^u\, e^{ikx} f(x)$$

$$f_x = f(x) = \int \frac{dk^u}{(2\pi)^u}\, e^{-ikx} f(k)$$

Fourier Transformation Operator

Fourier transformation operator:

$$
\begin{aligned}
F_x^k &= e^{ikx} \text{ applied via scalar product } a^\dagger b = \int dx\, \overline{a}_x b_x \\
(F^{-1})_k^x &= e^{-ikx} \text{ applied via scalar product } a^\dagger b = \int \frac{dk}{(2\pi)^u}\, \overline{a}_k b_k \\
F^{-1} &= F^\dagger \;\Rightarrow F \text{ orthonormal transformation}
\end{aligned}
$$

9.3 Power Spectra

$$
\begin{aligned}
S^{xy} &= C_s(x-y) \text{ correlation translational invariant } = \text{ homogeneous} \\
S^{kk'} &= \left\langle s^k \overline{s}^{k'} \right\rangle_{(s)} = \left\langle (Fs)^k \, \overline{(Fs)}^{k'} \right\rangle_{(s)} = \left\langle (Fs)^k \, (Fs)^{\dagger k'} \right\rangle_{(s)} \\
&= \left\langle (Fs)^k \left(s^\dagger F^\dagger \right)^{k'} \right\rangle_{(s)} = \left(F \left\langle s\, s^\dagger \right\rangle_{(s)} F^\dagger \right)^{kk'} = \left(F\, S\, F^\dagger \right)^{kk'} \\
&= \left(F^k_x\, S^{xy}\ \ F^{\dagger}{}^{k'}_{y} \right) \Big| \text{Einstein sum} = \int dx\, e^{ikx} \int dy\, S^{xy}\, e^{-ik'y} \\
&= \int dx \int dy\, e^{i(kx-k'y)}\, C_s(\underbrace{x-y}_{=:r}) = \int dx \int dr\, e^{i(kx-k'\,(x-r))}\, C_s(r) \\
&= \int dx\, e^{i(k-k')\,x} \int dr\, e^{ik'\,r}\, C_s(r) \\
&= (2\pi)^u\ \delta(k-k')\, P_s(k)
\end{aligned}
$$

9.3.1 Units

- $[s^k] = \left[\int dx\, e^{ikx} s^x\right] = V\,[s^x]$
- $[C_s(r)] = [\langle s^x s^{x+r}\rangle] = [s^x]^2$
- $[P_s(k)] = [\int dr\, e^{ikr} C_s(r)] = V[s^x]^2 = \frac{[s^k]^2}{V}$
- $[\delta(k-k')] = \left[\frac{1}{k\text{–Volume}}\right] = V$

$$
\begin{aligned}
\Rightarrow P_s(k) &= \frac{\langle |s^k|^2\rangle}{V} \\
S^{kk'} &= (2\pi)^u \delta(k-k') P_s(k) = \langle s^k \bar{s}^{k'}\rangle_{(s)} = \mathbb{1}^{kk'} \frac{\langle |s^k|^2\rangle}{V} \\
\mathbb{1}^{kk'} &= (2\pi)^u \delta(k-k')
\end{aligned}
$$

9.3.2 Wiener-Khintchin Theorem

- statistical homogeneous signal s
- stationary auto-correlation $S^{xy} = \langle s^x \overline{s^y} \rangle_{(s)} = C_s(x-y)$

$\Rightarrow$ diagonal covariance matrix in Fourier space

$$
\begin{aligned}
S^{kk'} &= \langle s^k \overline{s^{k'}} \rangle_{(s)} = (2\pi)^u \delta(k-k')\, C_s(k) \\
P_s(k) &= \lim_{V \to \infty} \frac{1}{V} \langle | \int_V dx\, s^x e^{ikx} |^2 \rangle_{(s)} = C_s(k)
\end{aligned}
$$

9.3.2 Wiener-Khintchin Theorem

- statistical homogeneous noise n
- stationary auto-correlation $N^{xy} = \langle n^x \overline{n^y} \rangle_{(n)} = C_n(x-y)$

$\Rightarrow$ diagonal covariance matrix in Fourier space

$$\begin{aligned} N^{kk'} &= \langle n^k \overline{n^{k'}} \rangle_{(s)} = (2\pi)^u \delta(k-k')\, C_n(k) \\ P_n(k) &= \lim_{V\to\infty} \frac{1}{V} \langle | \int_V dx\, n^x e^{ikx} |^2 \rangle_{(s)} = C_n(k) \end{aligned}$$

9.3.2 Wiener-Khintchin Theorem

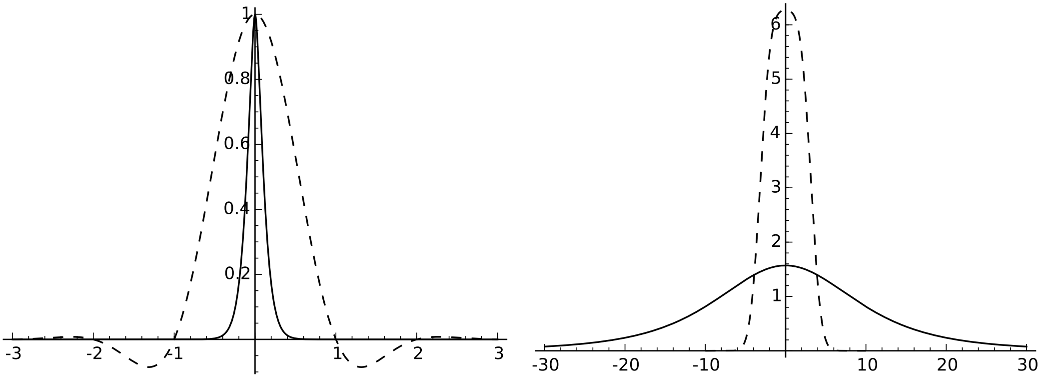

Figure: Possible correlation functions (left) and corresponding power spectra (right)

9.3.3 Fourier Space Filter

- Assume: $d = s + n$, $R = \mathbb{1}$, $\mathcal{P}(n,s) = \mathcal{G}(n,N)\,\mathcal{G}(s,S)$, $S^{xy} = C_s(x-y)$, $N^{xy} = C_n(x-y)$
- Gaussian posterior $\mathcal{P}(s|d) = \mathcal{G}(s-m, D)$
- mean $m = DN^{-1}d$
- variance $D = (S^{-1} + N^{-1})^{-1}$

Calculation of S^{-1} :

$$
\begin{aligned}
\mathbb{1}^k_q &= \left(S\,S^{-1}\right)^k_q \\
(2\pi)^u \delta(k-q) &= S^{kk'}\left(S^{-1}\right)_{k'q} = \int \frac{dk'}{(2\pi)^u} (2\pi)^u \delta(k-k') P_s(k) \left(S^{-1}\right)_{k'q} \\
&= P_s(k)\left(S^{-1}\right)_{kq} \\
\Rightarrow \left(S^{-1}\right)_{kq} &= \frac{(2\pi)^u \delta(k-q)}{P_s(k)}, \quad \left(N^{-1}\right)_{kq} = \frac{(2\pi)^u \delta(k-q)}{P_n(k)} \\
\Rightarrow M_{kq} &= (R^\dagger N^{-1} R)_{kq} = (N^{-1})_{kq}
\end{aligned}
$$

9.3.3 Fourier Space Filter

$$\Rightarrow D^{kq} = (S^{-1} + \underbrace{R^\dagger N^{-1} R}_{=M})^{-1\ kq}$$
$$= (2\pi)^u \delta(k-q) \left([P_s(k)]^{-1} + [P_n(k)]^{-1}\right)^{-1}$$

$$\Rightarrow j_k = (R^\dagger N^{-1} d)_k = (N^{-1} d)_k$$
$$= \int \frac{dk'}{(2\pi)^u} (2\pi)^u \delta(k'-k)[P_n(k)]^{-1} d_{k'}$$
$$= \frac{d_k}{P_n(k)}$$

9.3.3 Fourier Space Filter

$$\begin{aligned}
\Rightarrow m^k &= (Dj)^k = D^{kk'} j_{k'} \\
&= \int \frac{dk'}{(2\pi)^u} \frac{(2\pi)^u \delta(k-k')}{\frac{1}{P_s(k)} + \frac{1}{P_n(k)}} \frac{d^{k'}}{P_n(k')} \\
&= \underbrace{\frac{1}{1 + \frac{P_n(k)}{P_s(k)}}}_{f(k)=\text{filter function}} \qquad d^k
\end{aligned}$$

Filter Function

$\Rightarrow f(k)$ reweighs all Fourier modes of the data independently

$$f(k) = \frac{1}{1+\frac{P_n(k)}{P_s(k)}}$$

$$= \begin{cases} 1 & \text{if } P_s(k) \gg P_n(k) \Rightarrow \text{perfect pass through} \\ \underbrace{\frac{P_s(k)}{P_n(k)}}_{\ll 1} & \text{if } P_s(k) \ll P_n(k) \Rightarrow \text{signal-to-noise weighting} \end{cases}$$

$$\Rightarrow m^k = f(k)\, d^k = f(k)\,(s^k + n^k) = \left(\frac{s+n}{1+P_n/P_s}\right)^k$$

End